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3. THE THEORY OF THE HIGHEST WEIGHT

Before decomposing the \mathfrak{sl}_2 -space \mathcal{A} we must review the finite dimensional representation theory of \mathfrak{sl}_2 .

The *weight vectors* of an \mathfrak{sl}_2 -representation W are the eigenvectors of H in W . The *weights* of W are the eigenvalues of its nonzero weight vectors.

Every finite dimensional \mathfrak{sl}_2 -module is spanned by its weight vectors. The weights of such a representation are all integers and are thus ordered by the usual order on \mathbf{R} . The largest of a finite set of integral weights is traditionally referred to as the *highest weight*.

Two finite dimensional irreducible \mathfrak{sl}_2 -representations are isomorphic if and only if they have the same highest weights, which are necessarily nonnegative.

The element $X^a Y^b$ of V is a weight vector of weight $a-b$. This shows that X^m is a vector of highest weight m in V_m and therefore that the V_m for $m \geq 0$ form a set of representatives of the equivalence classes of finite dimensional irreducible \mathfrak{sl}_2 -representations; which is precisely why we are studying them is this paper.

The last general fact which we will recall without proof is this: every finite dimensional representation of \mathfrak{sl}_2 is a direct sum of irreducible representations.

Given a representation W of \mathfrak{sl}_2 which is a sum of finite dimensional representations one often wishes to write it explicitly as a direct sum of irreducible representations, that is, of representations isomorphic to the V_m . A method for doing this is provided by the observation that the space of weight vectors of highest weight in V_m is the space annihilated by E_+ and is one dimensional. Thus for each $v \in W$ of weight m such that $E_+ v = 0$, there is a unique \mathfrak{sl}_2 -homomorphism from V_m to W taking X^m to v . The explicit decomposition of W therefore amounts to the determination of a basis consisting of weight vectors of the kernel of E_+ in W .

4. THE DECOMPOSITION OF \mathcal{A}

We apply the procedure of the last paragraph to the representation of \mathfrak{sl}_2 on \mathcal{A} . By definition of ρ the kernel of $\rho(E_+)$ is just the commutant of E_+ in \mathcal{A} .

Let \mathcal{B} be the subalgebra of \mathcal{A} generated by X , ∂_Y , and J .

PROPOSITION 4.1. \mathcal{B} is the commutant of E_+ in \mathcal{A} .

Proof: One easily verifies that E_+ commutes with X , ∂_Y , and J , which shows that \mathcal{B} is contained in the commutant of E_+ .

Let U be the \mathfrak{sl}_2 -subrepresentation of \mathcal{A} generated by \mathcal{B} . The considerations of Section 3 show that the inclusion of the commutant of E_+ in \mathcal{B} is equivalent to the assertion that U equals all of \mathcal{A} . We proceed to establish that equality.

The algebra \mathcal{B} is spanned as a vector space by the elements

$$J^a X^b (\partial_Y)^c \quad \text{with } a, b, c \geq 0. \quad (4.2)$$

We present two calculations.

$$\begin{aligned} & [E_-, J^a X^b (\partial_Y)^{c+1}] \\ &= - (b+c+1) J^a X^b (\partial_Y)^c \partial_X + b(J+1-b) J^a X^{b-1} (\partial_Y)^c \end{aligned} \quad (4.3)$$

$$\begin{aligned} & [E_-, J^a X^{b+1} (\partial_Y)^c] \\ &= (b+1) J^a X^b (\partial_Y)^c Y - c J^a X^b (\partial_Y)^{c-1} (b+1+X\partial_X) \end{aligned} \quad (4.4)$$

From (4.3) one concludes that $\mathcal{B} \cdot \partial_X \subset U$. From that and (4.4) one concludes that $\mathcal{B} \cdot Y \subset U$.

Because E_- commutes with ∂_X and Y , one has that

$$\rho(E_-)^n (\mathcal{B} \partial_X) = (\rho(E_-)^n \mathcal{B}) \cdot \partial_X$$

and that

$$\rho(E_-)^n (\mathcal{B} \cdot Y) = (\rho(E_-)^n \mathcal{B}) \cdot Y.$$

Because $V_m = \bigoplus_{n=0}^{\infty} E_-^n (CX^m)$ one knows that $U = \bigoplus_{n=0}^{\infty} \rho(E_-)^n \mathcal{B}$. And thus

$$U \cdot \partial_X \subset U, \quad U \cdot Y \subset U. \quad (4.5)$$

Iterating, we have

$$UY^d (\partial_X)^e \subset U \quad \text{for } d, e \geq 0. \quad (4.6)$$

But \mathcal{A} is generated as an algebra by X , Y , ∂_X , and ∂_Y and so (4.2) and (4.6) prove that $U = \mathcal{A}$. \square

COROLLARY 4.7. \mathcal{A}^0 is the subalgebra of \mathcal{A} generated by \mathfrak{sl}_2 and J .

Proof: \mathcal{A}^0 is the \mathfrak{sl}_2 -subrepresentation of \mathcal{A} generated by $\mathcal{A}^0 \cap \mathcal{B}$.

$\mathcal{A}^0 \cap \mathcal{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+^b$. \square

We remark that the subalgebra of \mathcal{A} generated by \mathfrak{sl}_2 is canonically isomorphic to the universal enveloping algebra of \mathfrak{sl}_2 . The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for \mathfrak{sl}_2 . Thus \mathcal{A}^0 is a little larger than the enveloping algebra of \mathfrak{sl}_2 .

For integers l, n define $\mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix}$ to be the set of $T \in \mathcal{B} \cap \mathcal{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of \mathcal{B} :

$$\mathcal{B} = \bigoplus \mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix}, \quad \mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix} \cdot \mathcal{B} \begin{pmatrix} n' \\ l' \end{pmatrix} \subset \mathcal{B} \begin{pmatrix} n+n' \\ l+l' \end{pmatrix}. \quad (4.8)$$

The generators of \mathcal{B} fit in as follows:

$$J \in \mathcal{B} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad X \in \mathcal{B} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \partial_Y \in \mathcal{B} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \quad (4.9)$$

PROPOSITION 4.10. i) $\mathcal{B} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{C}[J]$.

ii) $\mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix} \neq 0$ if and only if $l \geq 0, |n| \leq l$, and $l \equiv n \pmod{2}$. If these conditions are met, then

$$\mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}} (\partial_Y)^{\frac{l-n}{2}} \quad (4.11)$$

Proof: Immediate. \square

We note that the condition that $\mathcal{B} \begin{pmatrix} n \\ l \end{pmatrix} \neq (0)$ may be rephrased thus: $l \geq 0$ and n is a weight of V_l .

5. DECOMPOSITION OF $\text{Hom}(V_m, V_{m+n})$

THEOREM 5.1. Let l, m, n be integers with $l, m, m+n \geq 0$. There is an \mathfrak{sl}_2 -subrepresentation of $\text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ which is isomorphic to V_l if and only if $|n| \leq l, n \equiv l \pmod{2}$, and $m \geq \frac{l-n}{2}$.