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3. The theory of the highest weight

Before decomposing the $\mathfrak{sl}_2$-space $\mathscr{A}$ we must review the finite dimensional representation theory of $\mathfrak{sl}_2$.

The *weight vectors* of an $\mathfrak{sl}_2$-representation W are the eigenvectors of H in W. The *weights* of W are the eigenvalues of its nonzero weight vectors.

Every finite dimensional $\mathfrak{sl}_2$-module is spanned by its weight vectors. The weights of such a representation are all integers and are thus ordered by the usual order on **R**. The largest of a finite set of integral weights is traditionally referred to as the *highest* weight.

Two finite dimensional irreducible $\mathfrak{sl}_2$-representations are isomorphic if and only if they have the same highest weights, which are necessarily nonnegative.

The element $X^a Y^b$ of V is a weight vector of weight a-b. This shows that X^m is a vector of highest weight m in V_m and therefore that the V_m for $m \geqslant 0$ form a set of representatives of the equivalence classes of finite dimensional irreducible $\mathfrak{sl}_2$-representations; which is precisely why we are studying them is this paper.

The last general fact which we will recall without proof is this: every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations.

Given a representation W of $\mathfrak{sl}_2$ which is a sum of finite dimensional representations one often wishes to write it explicitly as a direct sum of irreducible representations, that is, of representations isomorphic to the V_m. A method for doing this is provided by the observation that the space of weight vectors of highest weight in V_m is the space annihilated by E_+ and is one dimensional. Thus for each $v \in W$ of weight m such that $E_+ v = 0$, there is a unique $\mathfrak{sl}_2$-homomorphism from V_m to W taking X^m to v. The explicit decomposition of W therefore amounts to the determination of a basis consisting of weight vectors of the kernel of E_+ in W.

4. The decomposition of $\mathscr{A}$

We apply the procedure of the last paragraph to the representation of $\mathfrak{sl}_2$ on $\mathscr{A}$. By definition of ρ the kernel of $\rho(E_+)$ is just the commutant of E_+ in $\mathscr{A}$.

Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by X, ∂_Y, and J.

PROPOSITION 4.1. *$\mathscr{B}$ is the commutant of E_+ in $\mathscr{A}$.*

Proof: One easily verifies that E_+ commutes with X, ∂_Y, and J, which shows that $\mathscr{B}$ is contained in the commutant of E_+.

Let U be the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{B}$. The considerations of Section 3 show that the inclusion of the commutant of E_+ in $\mathscr{B}$ is equivalent to the assertion that U equals all of $\mathscr{A}$. We proceed to establish that equality.

The algebra $\mathscr{B}$ is spanned as a vector space by the elements

$$J^a X^b (\partial_Y)^c \qquad \text{with } a, b, c \geqslant 0 . \tag{4.2}$$

We present two calculations.

$$[E_-, J^a X^b (\partial_Y)^{c+1}] = -(b+c+1) J^a X^b (\partial_Y)^c \partial_X + b(J+1-b) J^a X^{b-1} (\partial_Y)^c \tag{4.3}$$

$$[E_-, J^a X^{b+1} (\partial_Y)^c] = (b+1) J^a X^b (\partial_Y)^c Y - c J^a X^b (\partial_Y)^{c-1} (b+1+X\partial_X) . \tag{4.4}$$

From (4.3) one concludes that $\mathscr{B} \cdot \partial_X \subset U$. From that and (4.4) one concludes that $\mathscr{B} \cdot Y \subset U$.

Because E_- commutes with ∂_X and Y, one has that

$$\rho(E_-)^n (\mathscr{B} \partial_X) = \big(\rho(E_-)^n \mathscr{B}\big) \cdot \partial_X$$

and that

$$\rho(E_-)^n (\mathscr{B} \cdot Y) = \big(\rho(E_-)^n \mathscr{B}\big) \cdot Y .$$

Because $V_m = \bigoplus_{n=0}^{\infty} E_-{}^n (\mathbf{C} X^m)$ one knows that $U = \bigoplus_{n=0}^{\infty} \rho(E_-)^n \mathscr{B}$. And thus

$$U \cdot \partial_X \subset U , \qquad U \cdot Y \subset U . \tag{4.5}$$

Iterating, we have

$$U Y^d (\partial_X)^e \subset U \qquad \text{for } d, e, \geqslant 0 . \tag{4.6}$$

But $\mathscr{A}$ is generated as an algebra by X, Y, ∂_X, and ∂_Y and so (4.2) and (4.6) prove that $U = \mathscr{A}$. □

COROLLARY 4.7. *$\mathscr{A}^0$ is the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ and J.*

Proof: $\mathscr{A}^0$ is the $\mathfrak{sl}_2$-subrepresentation of $\mathscr{A}$ generated by $\mathscr{A}^0 \cap \mathscr{B}$.

$\mathscr{A}^0 \cap \mathscr{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+{}^b$. □

We remark that the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ is canonically isomorphic to the universal enveloping algebra of $\mathfrak{sl}_2$. The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for $\mathfrak{sl}_2$. Thus $\mathscr{A}^0$ is a little larger than the enveloping algebra of $\mathfrak{sl}_2$.

For integers l, n define $\mathscr{B}\binom{n}{l}$ to be the set of $T \in \mathscr{B} \cap \mathscr{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of $\mathscr{B}$:

$$\mathscr{B} = \oplus \mathscr{B}\binom{n}{l}, \qquad \mathscr{B}\binom{n}{l} \cdot \mathscr{B}\binom{n'}{l'} \subset \mathscr{B}\binom{n+n'}{l+l'}. \tag{4.8}$$

The generators of $\mathscr{B}$ fit in as follows:

$$J \in \mathscr{B}\binom{0}{0}, \qquad X \in \mathscr{B}\binom{1}{1} \qquad \partial_Y \in \mathscr{B}\binom{-1}{1}. \tag{4.9}$$

PROPOSITION 4.10. *i)* $\mathscr{B}\binom{0}{0} = \mathbf{C}[J]$.

ii) $\mathscr{B}\binom{n}{l} \neq 0$ *if and only if* $l \geqslant 0, |n| \leqslant l$, *and* $l \equiv n \pmod 2$. *If these conditions are met, then*

$$\mathscr{B}\binom{n}{l} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}} \tag{4.11}$$

Proof: Immediate. □

We note that the condition that $\mathscr{B}\binom{n}{l} \neq (0)$ may be rephrased thus: $l \geqslant 0$ and n is a weight of V_l.

5. DECOMPOSITION OF $\mathrm{Hom}(V_m, V_{m+n})$

THEOREM 5.1. *Let* l, m, n *be integers with* $l, m, m+n \geqslant 0$. *There is an* $\mathfrak{sl}_2$*-subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *which is isomorphic to* V_l *if and only if* $|n| \leqslant l, n \equiv l \pmod 2$, *and* $m \geqslant \dfrac{l-n}{2}$.