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3. The theory of the highest weight

Before decomposing the $\mathfrak{sl}_2$-space $\mathscr{A}$ we must review the finite dimensional representation theory of $\mathfrak{sl}_2$.

The *weight vectors* of an $\mathfrak{sl}_2$-representation W are the eigenvectors of H in W. The *weights* of W are the eigenvalues of its nonzero weight vectors.

Every finite dimensional $\mathfrak{sl}_2$-module is spanned by its weight vectors. The weights of such a representation are all integers and are thus ordered by the usual order on $\mathbf{R}$. The largest of a finite set of integral weights is traditionally referred to as the *highest* weight.

Two finite dimensional irreducible $\mathfrak{sl}_2$-representations are isomorphic if and only if they have the same highest weights, which are necessarily nonnegative.

The element $X^a Y^b$ of V is a weight vector of weight a-b. This shows that X^m is a vector of highest weight m in V_m and therefore that the V_m for $m \geqslant 0$ form a set of representatives of the equivalence classes of finite dimensional irreducible $\mathfrak{sl}_2$-representations; which is precisely why we are studying them is this paper.

The last general fact which we will recall without proof is this: every finite dimensional representation of $\mathfrak{sl}_2$ is a direct sum of irreducible representations.

Given a representation W of $\mathfrak{sl}_2$ which is a sum of finite dimensional representations one often wishes to write it explicitly as a direct sum of irreducible representations, that is, of representations isomorphic to the V_m. A method for doing this is provided by the observation that the space of weight vectors of highest weight in V_m is the space annihilated by E_+ and is one dimensional. Thus for each $v \in W$ of weight m such that $E_+ v = 0$, there is a unique $\mathfrak{sl}_2$-homomorphism from V_m to W taking X^m to v. The explicit decomposition of W therefore amounts to the determination of a basis consisting of weight vectors of the kernel of E_+ in W.

4. The decomposition of $\mathscr{A}$

We apply the procedure of the last paragraph to the representation of $\mathfrak{sl}_2$ on $\mathscr{A}$. By definition of ρ the kernel of $\rho(E_+)$ is just the commutant of E_+ in $\mathscr{A}$.

Let $\mathscr{B}$ be the subalgebra of $\mathscr{A}$ generated by X, ∂_Y, and J.