$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C})\,. \tag{1.2}$$

Each $V_m$ is an $SL_2(\mathbf{C})$ subrepresentation of $V$.

By $\mathfrak{sl}_2$ we denote the Lie algebra of $2 \times 2$ complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on $V$ gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on $V$.

$$L \cdot v = \frac{d}{dt}\Big|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2,\, v \in V\,. \tag{1.3}$$

Choose a basis $E_+$, $E_-$, $H$ of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of $V$.

$$E_+ = X\partial_Y\,, \qquad E_- = Y\partial_X\,, \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y\,. \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each $V_m$ is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers $m$, $n$ a representation $\tau$ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2,\, T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n),\, v \in V_m\,. \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

## 2. The Weyl algebra $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b\,. \tag{2.1}$$

The Euler operator $J$, which acts as scalar multiplication by $m$ on $V_m$, lies in $\mathscr{A}$.

$$J = X\partial_X + Y\partial_Y . \tag{2.2}$$

The next lemma assures us that $\mathscr{A}$ is large enough for the study of all spaces $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

LEMMA 2.3. *Let $U$ be a finite dimensional vector subspace of $V$ and let $T \in \mathrm{End}_{\mathbf{C}}(U)$. Then there exists an element of $\mathscr{A}$ whose restriction to $U$ equals $T$.*

*Proof:* The element $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^{N} (J-m)$ of $\mathscr{A}$ maps $X^a Y^b$ to a nonzero multiple of $X^c Y^d$ and kills all other monomials of degree at most $N$. But by enlarging $U$ we may assume that $\mathrm{End}_{\mathbf{C}}(U)$ is spanned by restrictions of elements of the form $S$. □

We use the inclusion of $\mathfrak{sl}_2$ in $\mathscr{A}$ to define a representation $\rho$ of $\mathfrak{sl}_2$ on $\mathscr{A}$.

$$\rho(L)a = [L, a] \qquad \text{for } L \in \mathfrak{sl}_2,\ a \in \mathscr{A} . \tag{2.4}$$

For integers $n$ let $\mathscr{A}^n$ be the set of $T$ in $\mathscr{A}$ such that $T(V_m) \subset V_{m+n}$ for all $m$.

This defines a grading of $\mathscr{A}$ which is preserved by the action of $\mathfrak{sl}_2$.

$$\mathscr{A} = \bigoplus_{n \in \mathbf{Z}} \mathscr{A}^n , \qquad \mathscr{A}^m \cdot \mathscr{A}^n \subset \mathscr{A}^{m+n} , \tag{2.5}$$

$$\rho(L)\mathscr{A}^n \subset \mathscr{A}^n \qquad \text{for all } L \in \mathfrak{sl}_2 . \tag{2.6}$$

The algebra $\mathscr{A}$ and representation $\rho$ have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

LEMMA 2.7. *For each $m, n$ the restriction map*

$$\mathrm{res}: \mathscr{A}^n \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$$

*is a surjective homomorphism of $\mathfrak{sl}_2$ representations.* □

The method of this paper is to deduce the decomposition of the representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ from the decomposition of the representation $\rho$ on $\mathscr{A}$ by means of Lemma 2.7.