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**Autor:** Flath, Daniel  
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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \quad (1.1)$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \quad \text{for } g \in SL_2(\mathbf{C}). \quad (1.2)$$

Each  $V_m$  is an  $SL_2(\mathbf{C})$  subrepresentation of  $V$ .

By  $\mathfrak{sl}_2$  we denote the Lie algebra of  $2 \times 2$  complex matrices with trace 0. The representation of  $SL_2(\mathbf{C})$  on  $V$  gives rise, through differentiation, to a representation of  $\mathfrak{sl}_2$  on  $V$ .

$$L \cdot v = \frac{d}{dt} \Big|_{t=0} \exp(tL) \cdot v \quad \text{for } L \in \mathfrak{sl}_2, v \in V. \quad (1.3)$$

Choose a basis  $E_+, E_-, H$  of  $\mathfrak{sl}_2$  as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

An easy calculation establishes the following equalities of linear endomorphisms of  $V$ .

$$E_+ = X\partial_Y, \quad E_- = Y\partial_X, \quad (1.5)$$

$$H = X\partial_X - Y\partial_Y. \quad (1.6)$$

From (1.5) and (1.6) one easily deduces that each  $V_m$  is an *irreducible* representation of  $\mathfrak{sl}_2$  (and of  $SL_2(\mathbf{C})$ ).

We define for integers  $m, n$  a representation  $\tau$  of  $\mathfrak{sl}_2$  on  $\text{Hom}_{\mathbf{C}}(V_m, V_n)$  by means of formula (1.7).

$$\begin{aligned} (\tau(L) \cdot T)v &= L(Tv) - T(Lv) \\ \text{for } L \in \mathfrak{sl}_2, T \in \text{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \end{aligned} \quad (1.7)$$

The principal result of this article is the explicit decomposition of the  $\mathfrak{sl}_2$ -representations  $\text{Hom}_{\mathbf{C}}(V_m, V_n)$ .

## 2. THE WEYL ALGEBRA $\mathcal{A}$

Let  $\mathcal{A}$  be the subalgebra of  $\text{End}_{\mathbf{C}}(V)$  consisting of polynomial differential operators on  $V = \mathbf{C}[X, Y]$ . The algebra  $\mathcal{A}$  is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X^a \partial_Y^b. \quad (2.1)$$

The Euler operator  $J$ , which acts as scalar multiplication by  $m$  on  $V_m$ , lies in  $\mathcal{A}$ .

$$J = X\partial_X + Y\partial_Y. \quad (2.2)$$

The next lemma assures us that  $\mathcal{A}$  is large enough for the study of all spaces  $\text{Hom}_{\mathbb{C}}(V_m, V_n)$ .

**LEMMA 2.3.** *Let  $U$  be a finite dimensional vector subspace of  $V$  and let  $T \in \text{End}_{\mathbb{C}}(U)$ . Then there exists an element of  $\mathcal{A}$  whose restriction to  $U$  equals  $T$ .*

*Proof:* The element  $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^N (J-m)$  of  $\mathcal{A}$  maps  $X^a Y^b$  to a nonzero multiple of  $X^c Y^d$  and kills all other monomials of degree at most  $N$ . But by enlarging  $U$  we may assume that  $\text{End}_{\mathbb{C}}(U)$  is spanned by restrictions of elements of the form  $S$ .  $\square$

We use the inclusion of  $\mathfrak{sl}_2$  in  $\mathcal{A}$  to define a representation  $\rho$  of  $\mathfrak{sl}_2$  on  $\mathcal{A}$ .

$$\rho(L)a = [L, a] \quad \text{for } L \in \mathfrak{sl}_2, a \in \mathcal{A}. \quad (2.4)$$

For integers  $n$  let  $\mathcal{A}^n$  be the set of  $T$  in  $\mathcal{A}$  such that  $T(V_m) \subset V_{m+n}$  for all  $m$ .

This defines a grading of  $\mathcal{A}$  which is preserved by the action of  $\mathfrak{sl}_2$ .

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n, \quad \mathcal{A}^m \cdot \mathcal{A}^n \subset \mathcal{A}^{m+n}, \quad (2.5)$$

$$\rho(L)\mathcal{A}^n \subset \mathcal{A}^n \quad \text{for all } L \in \mathfrak{sl}_2. \quad (2.6)$$

The algebra  $\mathcal{A}$  and representation  $\rho$  have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

**LEMMA 2.7.** *For each  $m, n$  the restriction map*

$$\text{res}: \mathcal{A}^n \rightarrow \text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$$

*is a surjective homomorphism of  $\mathfrak{sl}_2$  representations.*  $\square$

The method of this paper is to deduce the decomposition of the representations  $\text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$  from the decomposition of the representation  $\rho$  on  $\mathcal{A}$  by means of Lemma 2.7.