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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \quad (1.1)$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \quad \text{for } g \in SL_2(\mathbf{C}). \quad (1.2)$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V .

By \mathfrak{sl}_2 we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of \mathfrak{sl}_2 on V .

$$L \cdot v = \left. \frac{d}{dt} \right|_{t=0} \exp(tL) \cdot v \quad \text{for } L \in \mathfrak{sl}_2, v \in V. \quad (1.3)$$

Choose a basis E_+ , E_- , H of \mathfrak{sl}_2 as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

An easy calculation establishes the following equalities of linear endomorphisms of V .

$$E_+ = X\partial_Y, \quad E_- = Y\partial_X, \quad (1.5)$$

$$H = X\partial_X - Y\partial_Y. \quad (1.6)$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of \mathfrak{sl}_2 (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of \mathfrak{sl}_2 on $\text{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$\begin{aligned} (\tau(L) \cdot T)v &= L(Tv) - T(Lv) \\ \text{for } L \in \mathfrak{sl}_2, T \in \text{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \end{aligned} \quad (1.7)$$

The principal result of this article is the explicit decomposition of the \mathfrak{sl}_2 -representations $\text{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. THE WEYL ALGEBRA \mathcal{A}

Let \mathcal{A} be the subalgebra of $\text{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra \mathcal{A} is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X^a \partial_Y^b. \quad (2.1)$$

The Euler operator J , which acts as scalar multiplication by m on V_m , lies in \mathcal{A} .

$$J = X\partial_X + Y\partial_Y. \tag{2.2}$$

The next lemma assures us that \mathcal{A} is large enough for the study of all spaces $\text{Hom}_{\mathbb{C}}(V_m, V_n)$.

LEMMA 2.3. *Let U be a finite dimensional vector subspace of V and let $T \in \text{End}_{\mathbb{C}}(U)$. Then there exists an element of \mathcal{A} whose restriction to U equals T .*

Proof: The element $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^N (J-m)$ of \mathcal{A} maps $X^a Y^b$ to a nonzero multiple of $X^c Y^d$ and kills all other monomials of degree at most N . But by enlarging U we may assume that $\text{End}_{\mathbb{C}}(U)$ is spanned by restrictions of elements of the form S . □

We use the inclusion of \mathfrak{sl}_2 in \mathcal{A} to define a representation ρ of \mathfrak{sl}_2 on \mathcal{A} .

$$\rho(L)a = [L, a] \quad \text{for } L \in \mathfrak{sl}_2, a \in \mathcal{A}. \tag{2.4}$$

For integers n let \mathcal{A}^n be the set of T in \mathcal{A} such that $T(V_m) \subset V_{m+n}$ for all m .

This defines a grading of \mathcal{A} which is preserved by the action of \mathfrak{sl}_2 .

$$\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}^n, \quad \mathcal{A}^m \cdot \mathcal{A}^n \subset \mathcal{A}^{m+n}, \tag{2.5}$$

$$\rho(L)\mathcal{A}^n \subset \mathcal{A}^n \quad \text{for all } L \in \mathfrak{sl}_2. \tag{2.6}$$

The algebra \mathcal{A} and representation ρ have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

LEMMA 2.7. *For each m, n the restriction map*

$$\text{res}: \mathcal{A}^n \rightarrow \text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$$

is a surjective homomorphism of \mathfrak{sl}_2 representations. □

The method of this paper is to deduce the decomposition of the representations $\text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$ from the decomposition of the representation ρ on \mathcal{A} by means of Lemma 2.7.