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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C}). \tag{1.2}$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V.

By $\mathfrak{sl}_2$ we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on V.

$$L \cdot v = \left.\frac{d}{dt}\right|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2,\ v \in V. \tag{1.3}$$

Choose a basis E_+, E_-, H of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of V.

$$E_+ = X\partial_Y, \qquad E_- = Y\partial_X, \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y. \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2,\ T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n),\ v \in V_m. \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. The Weyl algebra $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b. \tag{2.1}$$

The Euler operator J, which acts as scalar multiplication by m on V_m, lies in $\mathscr{A}$.

$$J = X\partial_X + Y\partial_Y. \tag{2.2}$$

The next lemma assures us that $\mathscr{A}$ is large enough for the study of all spaces $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

LEMMA 2.3. *Let U be a finite dimensional vector subspace of V and let $T \in \mathrm{End}_{\mathbf{C}}(U)$. Then there exists an element of $\mathscr{A}$ whose restriction to U equals T.*

Proof: The element $S = X^c Y^d (\partial_X)^a (\partial_Y)^b \prod_{\substack{m=0 \\ m \neq a+b}}^{N} (J-m)$ of $\mathscr{A}$ maps $X^a Y^b$ to a nonzero multiple of $X^c Y^d$ and kills all other monomials of degree at most N. But by enlarging U we may assume that $\mathrm{End}_{\mathbf{C}}(U)$ is spanned by restrictions of elements of the form S. □

We use the inclusion of $\mathfrak{sl}_2$ in $\mathscr{A}$ to define a representation ρ of $\mathfrak{sl}_2$ on $\mathscr{A}$.

$$\rho(L)a = [L, a] \qquad \text{for } L \in \mathfrak{sl}_2,\ a \in \mathscr{A}. \tag{2.4}$$

For integers n let $\mathscr{A}^n$ be the set of T in $\mathscr{A}$ such that $T(V_m) \subset V_{m+n}$ for all m.

This defines a grading of $\mathscr{A}$ which is preserved by the action of $\mathfrak{sl}_2$.

$$\mathscr{A} = \bigoplus_{n \in \mathbf{Z}} \mathscr{A}^n, \qquad \mathscr{A}^m \cdot \mathscr{A}^n \subset \mathscr{A}^{m+n}, \tag{2.5}$$

$$\rho(L)\mathscr{A}^n \subset \mathscr{A}^n \qquad \text{for all } L \in \mathfrak{sl}_2. \tag{2.6}$$

The algebra $\mathscr{A}$ and representation ρ have been defined just so that the next lemma, which is an immediate consequence of Lemma 2.3, will be true.

LEMMA 2.7. *For each m, n the restriction map*

$$\mathrm{res}: \mathscr{A}^n \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$$

is a surjective homomorphism of $\mathfrak{sl}_2$ representations. □

The method of this paper is to deduce the decomposition of the representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ from the decomposition of the representation ρ on $\mathscr{A}$ by means of Lemma 2.7.