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## THE CLEBSCH-GORDAN FORMULAS

by Daniel FLATH

### 0. INTRODUCTION

The explicit decomposition of tensor products of irreducible representations is of fundamental importance in many applications of representation theory. For finite dimensional representations of the Lie algebra  $\mathfrak{sl}_2$  definitive results are contained in the famous Clebsch-Gordan formulas which are constantly and routinely used by physicists in applying the quantum theory of angular momentum. We give in this article a presentation and derivation of equivalent results, Theorems 5.1 and 5.4.

We shall base a study of the representations  $\text{Hom}(V, W)$  (rather than  $V \otimes W$ ) for irreducible  $\mathfrak{sl}_2$ -representations  $V$  and  $W$  on the analysis of a Weyl algebra  $\mathcal{A}$  of polynomial differential operators in two variables. This point of view is one developed in a recent attack on the Clebsch-Gordan problem for  $\mathfrak{sl}_3$  [2].

The usefulness of the Weyl algebra in the resolution of the Clebsch-Gordan problem is well-known. For years physicists have worked with it under the name "boson calculus" [1]. One mathematical reference is [3]. Nothing in the present article is new except possibly the arrangement of the proofs which has been made with the benefit of experience gained working with  $\mathfrak{sl}_3$ . It seems to me that this arrangement has a naturalness and simplicity to recommend it.

I would like to thank L. C. Biedenharn for interesting discussions on the subject of this paper.

### 1. SOME REPRESENTATIONS OF $\mathfrak{sl}_2$

Let  $V = \mathbf{C}[X, Y]$ , the vector space of polynomials in two variables  $X$  and  $Y$ . For integers  $m$  let  $V_m$  be the subspace of homogeneous polynomials of degree  $m$ , with  $V_m = (0)$  for negative  $m$ .

Let  $SL_2(\mathbf{C})$  act linearly on  $V$  as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \quad (1.1)$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \quad \text{for } g \in SL_2(\mathbf{C}). \quad (1.2)$$

Each  $V_m$  is an  $SL_2(\mathbf{C})$  subrepresentation of  $V$ .

By  $\mathfrak{sl}_2$  we denote the Lie algebra of  $2 \times 2$  complex matrices with trace 0. The representation of  $SL_2(\mathbf{C})$  on  $V$  gives rise, through differentiation, to a representation of  $\mathfrak{sl}_2$  on  $V$ .

$$L \cdot v = \frac{d}{dt} \Big|_{t=0} \exp(tL) \cdot v \quad \text{for } L \in \mathfrak{sl}_2, v \in V. \quad (1.3)$$

Choose a basis  $E_+, E_-, H$  of  $\mathfrak{sl}_2$  as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.4)$$

An easy calculation establishes the following equalities of linear endomorphisms of  $V$ .

$$E_+ = X\partial_Y, \quad E_- = Y\partial_X, \quad (1.5)$$

$$H = X\partial_X - Y\partial_Y. \quad (1.6)$$

From (1.5) and (1.6) one easily deduces that each  $V_m$  is an *irreducible* representation of  $\mathfrak{sl}_2$  (and of  $SL_2(\mathbf{C})$ ).

We define for integers  $m, n$  a representation  $\tau$  of  $\mathfrak{sl}_2$  on  $\text{Hom}_{\mathbf{C}}(V_m, V_n)$  by means of formula (1.7).

$$\begin{aligned} (\tau(L) \cdot T)v &= L(Tv) - T(Lv) \\ \text{for } L \in \mathfrak{sl}_2, T \in \text{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \end{aligned} \quad (1.7)$$

The principal result of this article is the explicit decomposition of the  $\mathfrak{sl}_2$ -representations  $\text{Hom}_{\mathbf{C}}(V_m, V_n)$ .

## 2. THE WEYL ALGEBRA $\mathcal{A}$

Let  $\mathcal{A}$  be the subalgebra of  $\text{End}_{\mathbf{C}}(V)$  consisting of polynomial differential operators on  $V = \mathbf{C}[X, Y]$ . The algebra  $\mathcal{A}$  is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X^a \partial_Y^b. \quad (2.1)$$