Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 29 (1983)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE CLEBSCH-GORDAN FORMULAS
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Kapitel: 1. SOME REPRESENTATIONS OF sl_2
DOI: https://doi.org/10.5169/seals-52986

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L'Enseignement Mathématique, t. 29 (1983), p. 339-346

THE CLEBSCH-GORDAN FORMULAS

by Daniel FLATH

0. INTRODUCTION

The explicit decomposition of tensor products of irreducible representations is of fundamental importance in many applications of representation theory. For finite dimensional representations of the Lie algebra $\mathfrak{sl}_2$ definitive results are contained in the famous Clebsch-Gordan formulas which are constantly and routinely used by physicists in applying the quantum theory of angular momentum. We give in this article a presentation and derivation of equivalent results, Theorems 5.1 and 5.4.

We shall base a study of the representations $\mathrm{Hom}(V, W)$ (rather than $V \otimes W$) for irreducible $\mathfrak{sl}_2$-representations V and W on the analysis of a Weyl algebra $\mathscr{A}$ of polynomial differential operators in two variables. This point of view is one developed in a recent attack on the Clebsch-Gordan problem for $\mathfrak{sl}_3$ [2].

The usefulness of the Weyl algebra in the resolution of the Clebsch-Gordan problem is well-known. For years physicists have worked with it under the name "boson calculus" [1]. One mathematical reference is [3]. Nothing in the present article is new except possibly the arrangement of the proofs which has been made with the benefit of experience gained working with $\mathfrak{sl}_3$. It seems to me that this arrangement has a naturalness and simplicity to recommend it.

I would like to thank L. C. Biedenharn for interesting discussions on the subject of this paper.

1. SOME REPRESENTATIONS OF $\mathfrak{sl}_2$

Let $V = \mathbf{C}[X, Y]$, the vector space of polynomials in two variables X and Y. For integers m let V_m be the subspace of homogeneous polynomials of degree m, with $V_m = (0)$ for negative m.

Let $SL_2(\mathbf{C})$ act linearly on V as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C}) . \tag{1.2}$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V.

By $\mathfrak{sl}_2$ we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on V.

$$L \cdot v = \frac{d}{dt}\bigg|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2, v \in V . \tag{1.3}$$

Choose a basis E_+, E_-, H of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of V.

$$E_+ = X\partial_Y , \qquad E_- = Y\partial_X , \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y . \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2, T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m . \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. The Weyl algebra $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b . \tag{2.1}$$