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THE CLEBSCH-GORDAN FORMULAS

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0. INTRODUCTION

The explicit decomposition of tensor products of irreducible representations is of fundamental importance in many applications of representation theory. For finite dimensional representations of the Lie algebra $\mathfrak{sl}_2$ definitive results are contained in the famous Clebsch-Gordan formulas which are constantly and routinely used by physicists in applying the quantum theory of angular momentum. We give in this article a presentation and derivation of equivalent results, Theorems 5.1 and 5.4.

We shall base a study of the representations $\mathrm{Hom}(V, W)$ (rather than $V \otimes W$) for irreducible $\mathfrak{sl}_2$-representations V and W on the analysis of a Weyl algebra $\mathscr{A}$ of polynomial differential operators in two variables. This point of view is one developed in a recent attack on the Clebsch-Gordan problem for $\mathfrak{sl}_3$ [2].

The usefulness of the Weyl algebra in the resolution of the Clebsch-Gordan problem is well-known. For years physicists have worked with it under the name "boson calculus" [1]. One mathematical reference is [3]. Nothing in the present article is new except possibly the arrangement of the proofs which has been made with the benefit of experience gained working with $\mathfrak{sl}_3$. It seems to me that this arrangement has a naturalness and simplicity to recommend it.

I would like to thank L. C. Biedenharn for interesting discussions on the subject of this paper.

1. SOME REPRESENTATIONS OF $\mathfrak{sl}_2$

Let $V = \mathbf{C}[X, Y]$, the vector space of polynomials in two variables X and Y. For integers m let V_m be the subspace of homogeneous polynomials of degree m, with $V_m = (0)$ for negative m.

Let $SL_2(\mathbf{C})$ act linearly on V as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = aX + cY \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot Y = bX + dY \tag{1.1}$$

$$g \cdot X^a Y^b = (g \cdot X)^a (g \cdot Y)^b \qquad \text{for } g \in SL_2(\mathbf{C}). \tag{1.2}$$

Each V_m is an $SL_2(\mathbf{C})$ subrepresentation of V.

By $\mathfrak{sl}_2$ we denote the Lie algebra of 2×2 complex matrices with trace 0. The representation of $SL_2(\mathbf{C})$ on V gives rise, through differentiation, to a representation of $\mathfrak{sl}_2$ on V.

$$L \cdot v = \left.\frac{d}{dt}\right|_{t=0} \exp(tL) \cdot v \qquad \text{for } L \in \mathfrak{sl}_2, v \in V. \tag{1.3}$$

Choose a basis E_+, E_-, H of $\mathfrak{sl}_2$ as follows:

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4}$$

An easy calculation establishes the following equalities of linear endomorphisms of V.

$$E_+ = X\partial_Y, \qquad E_- = Y\partial_X, \tag{1.5}$$

$$H = X\partial_X - Y\partial_Y. \tag{1.6}$$

From (1.5) and (1.6) one easily deduces that each V_m is an *irreducible* representation of $\mathfrak{sl}_2$ (and of $SL_2(\mathbf{C})$).

We define for integers m, n a representation τ of $\mathfrak{sl}_2$ on $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$ by means of formula (1.7).

$$(\tau(L) \cdot T)v = L(Tv) - T(Lv)$$
$$\text{for } L \in \mathfrak{sl}_2, T \in \mathrm{Hom}_{\mathbf{C}}(V_m, V_n), v \in V_m. \tag{1.7}$$

The principal result of this article is the explicit decomposition of the $\mathfrak{sl}_2$-representations $\mathrm{Hom}_{\mathbf{C}}(V_m, V_n)$.

2. The Weyl algebra $\mathscr{A}$

Let $\mathscr{A}$ be the subalgebra of $\mathrm{End}_{\mathbf{C}}(V)$ consisting of polynomial differential operators on $V = \mathbf{C}[X, Y]$. The algebra $\mathscr{A}$ is spanned by the elements

$$D(i, j, a, b) = X^i Y^j \partial_X{}^a \partial_Y{}^b. \tag{2.1}$$