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VANISHING OF COHOMOLOGY WITH COEFFICIENTS IN A LOCALLY FREE SHEAF AND PSEUDOCONVEXITY

by Giuseppe VIGNA SURIA

INTRODUCTION

Cartan's Theorem B characterizes Stein spaces; a certain mathematical effort has been devoted to the study of a problem that can loosely be stated as follows: if we know that for a given analytic space X we have $H^p(X, \mathcal{S}) = 0$ for p ranging in a suitable set of integers N and \mathcal{S} belonging to a suitable class $\Xi(X)$ of sheaves on X , can we deduce that X is Stein, or, more generally, what kind of function theoretic or geometrical properties on X can we find?

A first elementary progress in this direction is that we can narrow N as much as possible if we allow $\Xi(X)$ to be very large: if $H^1(X, \mathcal{S}) = 0$ for all coherent sheaves on X ($N = \{1\}$, $\Xi(X) = \text{all coherent sheaves on } X$) then X is Stein; on the other hand a result of Coen [C] says that, in the case when X is an open subset D of a Stein space of dimension n , D is Stein if $H^p(D, \mathcal{O}) = 0$ for $p = 1, 2, \dots, n-1$ ($N = \{1, 2, \dots, n-1\}$, $\Xi(D) = \{\text{structure sheaf } \mathcal{O}\}$); a theorem due to Leiterer [L] makes a reasonably good compromise between the above facts by showing that an open subset D of a Stein manifold is Stein if $H^1(D, \mathcal{L}) = 0$ for every locally free sheaf \mathcal{L} on D ($N = \{1\}$, $\Xi(D) = \{\text{locally free sheaves}\}$, but actually Leiterer can make this class even smaller); if we further assume that D has a C^2 boundary then if $H^p(D, \mathcal{O}) = 0$ for $p > q$, where q is a fixed integer, D is q -pseudoconvex

$$(N = \{q+1, q+2, \dots, n-1\}, \Xi(D) = \{\mathcal{O}\}),$$

see [E-VS].

The goal of this paper is to give a contribution of the same type as above to our original vague problem.

Our basic assumption will be that D is an open subset of a Stein manifold M of dimension $n \geq 2$, but very often weaker hypotheses, such as requiring M to be only strongly holomorphically separated and/or allowing singularities, will be sufficient.

Using a rather elementary device, the Koszul complex of sheaves, we shall be able to prove that if D has C^2 boundary and $H^{q+1}(D, \mathcal{L}) = 0$ for \mathcal{L} ranging in a suitable class $\Xi_q(D)$ of locally free sheaves on D , then D is q -pseudoconvex. For $q = 0$ we shall not need the hypothesis on the boundary and M will be allowed to have singularities and even to be only holomorphically separated (in the latter case, on the other hand, we need to assume that D is relatively compact in M) so that if $H^1(D, \mathcal{L}) = 0$ for all \mathcal{L} in a suitable class $\Xi_0(D)$ of locally free sheaves, then D is Stein.

The locally free sheaves in these classes $\Xi_q(D)$ will also be investigated from a "topological" point of view and we shall see that they are associated to stably trivial vector bundles.

We say that an analytic space M is *strongly holomorphically separated* if, given any point $x \in M$, we can find $n = \dim M$ global holomorphic functions f_1, f_2, \dots, f_n on M defining x , i.e. we have

$$\{x\} = \{y \in M \text{ s.t. } f_1(y) = f_2(y) = \dots = f_n(y) = 0\};$$

this can surely be done if M is Stein [F-R] Satz 1 p. 91, but it is reasonable to conjecture that it is also true when holomorphic functions separate the points of M .

We shall prove and make some remarks on the following.

THEOREM 1. *Let M be a strongly holomorphically separated analytic manifold and D be an open subset with C^2 boundary of M ; if for every locally free sheaf \mathcal{L} on D we have $H^{q+1}(D, \mathcal{L}) = 0$ then D is q -pseudoconvex.*

Remark. If M is actually Stein the converse is also true: in this case in fact, if D is q -pseudoconvex then it is q -complete [VS] and therefore $H^p(D, \mathcal{S}) = 0$ for all $p > q$ and all coherent sheaves \mathcal{S} on D , [A-G] Corollary 1 p. 250.

The reason for assuming M strongly holomorphically separated is that, given $x \in M$ and global holomorphic functions f_1, f_2, \dots, f_n defining x we can construct the *Koszul complex of sheaves*

$$\mathcal{K}(\mathbf{f}, \mathcal{O}) = \{\mathcal{K}^p(\mathbf{f}, \mathcal{O}); d\}_{p \in \mathbb{Z}}$$

as follows: $\mathcal{K}^p(\mathbf{f}, \mathcal{O}) = \Lambda^p \mathcal{O}^n$ and $d: \mathcal{K}^p(\mathbf{f}, \mathcal{O}) \rightarrow \mathcal{K}^{p+1}(\mathbf{f}, \mathcal{O})$ is given by $d(\omega) = \mathbf{f} \wedge \omega$ where $\omega \in \mathcal{K}^p(\mathbf{f}, \mathcal{O})$ and $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathcal{O}^n = \Lambda^1 \mathcal{O}^n$.

The theorem will be an easy consequence of the following

LEMMA. Let \mathcal{L}_s denote $\text{Ker } d: \mathcal{K}^{n-s-1} \rightarrow \mathcal{K}^{n-s}$; the sequence

$$\xi_s \quad 0 \rightarrow \mathcal{L}_s \rightarrow \mathcal{K}^{n-s-1} \rightarrow \mathcal{L}_{s-1} \rightarrow 0$$

is a short exact sequence of analytic sheaves on $M - \{x\}$, for all s ; moreover \mathcal{L}_s is a locally free sheaf of rank $\binom{n-1}{n-s-2}$ on $M - \{x\}$; more precisely if $U_i = \{y \in M \text{ s.t. } f_i(y) \neq 0\}$ $i = 1, 2, \dots, n$, then

$$\mathcal{L}_{s|U_i} \simeq \Lambda^{n-s-2} \mathcal{O}_{|U_i}^{n-1}$$

for all $s \in \mathbb{Z}$. Furthermore $\mathcal{L}_{n-2} \simeq \mathcal{O}$ on M .

Proof. We shall find explicit isomorphisms of sheaves $\phi_i: \Lambda^{n-s-2} \mathcal{O}_{|U_i}^{n-1} \rightarrow \mathcal{L}_{s|U_i}$ and construct split exact sequences

$$0 \rightarrow \Lambda^{p-1} \mathcal{O}_{|U_i}^{n-1} \xrightarrow{\Phi_i} \Lambda^p \mathcal{O}_{|U_i}^n \xrightarrow{\Psi_i} \Lambda^p \mathcal{O}_{|U_i}^{n-1} \rightarrow 0$$

such that $\phi_i = \Phi_i$ and $\phi_i \circ \Psi_i = d$; an isomorphism $\mathcal{O} = \Lambda^0 \mathcal{O}^n \xrightarrow{d} \mathcal{L}_{n-2}$ will also be explicitly given; this clearly proves the lemma.

So let us fix $i = 1, 2, \dots, n$, let F_1, F_2, \dots, F_n be the formal symbols on which the exterior \mathcal{O} -algebra $\Lambda^* \mathcal{O}^n$ is constructed and think of $\Lambda^* \mathcal{O}^{n-1}$ as based on $F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_n$. An element $\omega \in \Lambda^p \mathcal{O}^n$ can be written uniquely as $\omega = F_i \wedge \mu + v$ where $\mu \in \Lambda^{p-1} \mathcal{O}^{n-1}$ and $v \in \Lambda^p \mathcal{O}^{n-1}$; we can define a sheaf homomorphism $h: \Lambda^p \mathcal{O}_{|U_i}^n \rightarrow \Lambda^{p-1} \mathcal{O}_{|U_i}^{n-1}$, depending on i , as follows:

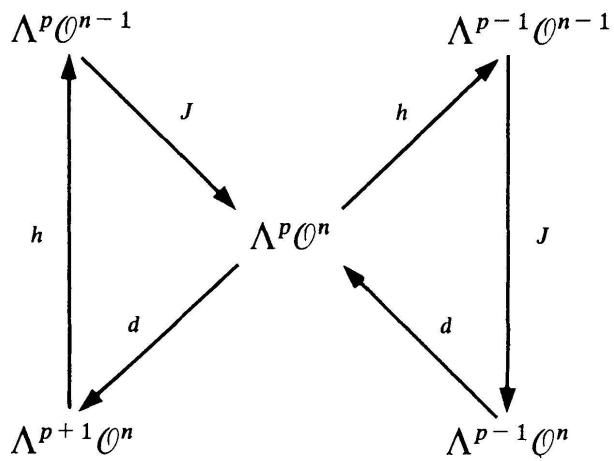
$$h(\omega) = h(F_i \wedge \mu + v) = f_i^{-1} \mu.$$

To simplify the notation we shall omit the restriction $|_{U_i}$.

Let

$$J: \Lambda^p \mathcal{O}^{n-1} \rightarrow \Lambda^p \mathcal{O}^n$$

denote the inclusion (depending on i if $p = 1, 2, \dots, n-1$), the sheaf homomorphisms constructed so far are indicated in the following diagram:



It is easy to check the following properties:

- 1) $d \circ J \circ h + J \circ h \circ d = id_{\Lambda^p \mathcal{O}^n}$,
- 2) $h \circ d \circ J = id_{\Lambda^{p-1} \mathcal{O}^{n-1}}$,
- 3) $d \circ J \circ h \circ d = d$;

we prove 1), the others are even easier to show; so take $\omega = F_i \wedge \mu + v \in \Lambda^p \mathcal{O}^n$ then

$$\begin{aligned}
 (d \circ J \circ h + J \circ h \circ d)(\omega) &= d(f_i^{-1} \mu) + J \circ h(\sum_k f_k F_k \wedge F_i \wedge \mu + \sum_k f_k F_k \wedge v) \\
 &= \sum_k f_k f_i^{-1} F_k \wedge \mu + J \circ h(-F_i \wedge \sum_{k \neq i} f_k F_k \mu) + v \\
 &= \sum_k f_k f_i^{-1} F_k \wedge \mu - \sum_{k \neq i} f_k f_i^{-1} F_k \wedge \mu + v = F_i \wedge \mu + v = \omega.
 \end{aligned}$$

Now define

$$\begin{aligned}
 \Phi_i: \Lambda^{p-1} \mathcal{O}^{n-1} &\rightarrow \Lambda^p \mathcal{O}^n \quad \text{by} \quad \Phi_i = d \circ J, \\
 \Psi_i: \Lambda^p \mathcal{O}^n &\rightarrow \Lambda^{p-1} \mathcal{O}^{n-1} \quad \text{by} \quad \Psi_i = h \circ d, \\
 \phi_i: \Lambda^p \mathcal{O}^{n-1} &\rightarrow \mathcal{L}_{n-p-2} \quad \text{by} \quad \phi_i = d \circ J.
 \end{aligned}$$

There is a list of facts to check, all following easily from 1), 2), and 3); they are: Φ_i is one to one, Ψ_i is onto, $\text{Ker } \Psi_i = \text{Im } \Phi_i$, $d \circ \phi_i = 0$, ϕ_i is one to one and onto and $\phi_i \circ \Psi_i = d$.

Moreover $J: \Lambda^0 \mathcal{O}^{n-1} \rightarrow \Lambda^0 \mathcal{O}^n$ is simply the identity so that $d: \Lambda^0 \mathcal{O}^n \rightarrow \mathcal{L}_{n-2}$ is an isomorphism on the whole of M .

The lemma is therefore proved. □

To proceed towards the proof of theorem 1 we must consider distinguished elements in $H^{s+1}(M - \{x\}, \mathcal{L}_s)$, the *test classes* introduced in [E-VS]; for this purpose we take the long exact cohomology sequence associated to ξ_s

$$\dots \rightarrow H^s(M - \{x\}, \mathcal{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(M - \{x\}, \mathcal{L}_s) \rightarrow \dots$$

and consider the elements $\alpha_s(x) \in H^{s+1}(M - \{x\}, \mathcal{L}_s)$ given inductively as follows: $\alpha_0(x) = \delta(F_1 \wedge F_2 \wedge \dots \wedge F_n)$, where $F_1 \wedge F_2 \wedge \dots \wedge F_n$ is in

$$H^0(M - \{x\}, \mathcal{L}_{-1}) = H^0(M - \{x\}, \mathcal{K}^n(\mathbf{f}, \mathcal{O}))$$

and $\alpha_s(x) = \delta(\alpha_{s-1}(x))$ if $s \geq 1$.

Proof of theorem 1. To say that D has C^2 boundary means that, given any point $x \in \partial D$ we can find a C^2 defining function $\varphi: U \rightarrow \mathbf{R}$, where U is an open neighbourhood of x , such that $D \cap U = \{y \in U \text{ s.t. } \varphi(y) < 0\}$ and $d\varphi(x) \neq 0$; in these conditions the number $n(x)$ of negative eigenvalues of the Levi form $\mathcal{L}(\varphi)(x)$ depends only on D and x and not on the choice of φ ; if D is not q -pseudoconvex there is a point $x \in \partial D$ such that $n(x) \geq q + 1$, but then, with a slight modification of the defining function φ [E-VS] 3.6 corollary 1, we can manage to obtain at least $q + 2$ negative eigenvalues for the complex Hessian $\mathcal{H}(\varphi)(x)$; following the argument of [A-G] proposition 12 p. 222, we can find a neighbourhood Q of x (as small as we want) such that

- a) $H^p(D \cap Q, \mathcal{O}) = 0$ for $p = 1, 2, \dots, q$ and
- b) the restriction $H^0(Q, \mathcal{O}) \rightarrow H^0(D \cap Q, \mathcal{O})$ is an isomorphism.

If we take global holomorphic functions f_1, f_2, \dots, f_n defining x and construct the Koszul complex $\mathcal{K}(\mathbf{f}, \mathcal{O})$ we find that $H^{q+1}(D, \mathcal{L}_q) = 0$ because \mathcal{L}_q is locally free on D ; therefore $\alpha_q(x)|_D \in H^{q+1}(D, \mathcal{L}_q)$ vanishes and, by further restricting, $\alpha_q(x)|_{D \cap Q}$ vanishes too; using the long exact sequence of cohomology

$$\dots \rightarrow H^s(D \cap Q, \mathcal{K}^{n-s-1}(\mathbf{f}, \mathcal{O})) \rightarrow H^s(D \cap Q, \mathcal{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(D \cap Q, \mathcal{L}_s) \rightarrow$$

associated to ξ_s for $s = 1, 2, \dots, q$ together with a) and the fact that $\mathcal{K}^{n-s-1}(\mathbf{f}, \mathcal{O})$ is a free sheaf we deduce that

$$0 = \alpha_q(x)|_{D \cap Q} = \alpha_{q-1}(x)|_{D \cap Q} = \dots = \alpha_0(x)|_{D \cap Q};$$

but then, taking $s = 0$, $\alpha_0(x)|_{D \cap Q} = 0$ means that the equation $\sum_i f_i g_i = 1$ has a solution $(g_1, g_2, \dots, g_n) \in H^0(D \cap Q, \mathcal{O})^n$. This would imply that at least one of the g_i 's does not extend to Q , contradicting b); therefore our original assumption that D is not q -pseudoconvex must be wrong and the theorem is proved. \square

This proof follows, with minor modifications, that of Proposition 2.1 of [E-VS], it is reported here mainly for reasons of clarity. It should also be remarked that if $H^p(D, \mathcal{O}) = 0$ for $p > q$ then $H^{q+1}(D, \mathcal{L}_q) = 0$ ([E-VS] Lemma 3.2, see introduction).

The theorem holds for $q = 0$ too, but in this case we can avoid the hypothesis on the boundary and also the smoothness of the ambient space M if we impose some mild conditions: more precisely we have the following

THEOREM 2. *Let M be a strongly holomorphically separated analytic space and D an open subset of M ; suppose that either M is Stein or that D is relatively compact in M . Then the following conditions are equivalent:*

- 1) D is Stein,
- 2) $H^1(D, \mathcal{L}) = 0$ for every locally free sheaf \mathcal{L} on D .

Proof. 1) \Rightarrow 2) by Cartan's Theorem B.

2) \Rightarrow 1) We must show that D is holomorphically convex, and this will be done by proving that, given any discrete sequence $\{x_n\}$ in D we can find a holomorphic function g on D such that $\overline{\lim}_{n \rightarrow \infty} |g(x_n)| = \infty$.

In either of our hypotheses we can suppose that $\{x_n\}$ converges to a point $x \in \partial D$; construct the Koszul complex $\mathcal{K}(\mathbf{f}, \mathcal{O})$ starting from this point. Since \mathcal{L}_0 is locally free (of rank $n-1$ on $M-x$ and on D) we get $H^1(D, \mathcal{L}_0) = 0$ so that $\alpha_0(x)|_D = 0$; as before this means that the equation $\sum_1^n f_i g_i = 1$ has a solution (g_1, \dots, g_n) in $H^0(D, \mathcal{O})^n$. For one at least of the g_i 's we must have $\overline{\lim}_{n \rightarrow \infty} |g_i(x_n)| = \infty$. □

As in the remark above if $H^p(D, \mathcal{O}) = 0$ for $p = 1, 2, \dots, n-1$ then $H^1(D, \mathcal{L}_0) = 0$, so that we obtain Coen's result [C] under slightly different hypotheses.

The proofs of theorems 1 and 2 should have persuaded the reader that a deeper investigation of these powerful sheaves \mathcal{L}_q is worth a little effort. So far we have discovered that \mathcal{L}_q is a locally free sheaf of rank $\binom{n-1}{n-q-2}$ on $M - \{x\}$. Let us call E_q the corresponding holomorphic vector bundle; E_q is a subbundle of the trivial bundle $(M - \{x\}) \times \mathbf{C}^N$, $N = \binom{n}{n-q-1}$.

We shall now discuss the topological properties of these bundles E_q ; in the literature trivial and stably trivial bundles are considered [A], [L].

Definition. Let X be a topological space (respectively an analytic space) and E a topological (analytic) vector bundle on X with complex fibre; we say that E is *stably trivial in the category $\mathbf{Top}(X)$* of topological vector bundles on X (respectively *in the category $\mathbf{An}(X)$* of analytic vector bundles on X) or, more quickly, that E is *topologically stably trivial (analytically stably trivial)* if there

exists a trivial vector bundle F in $\mathfrak{Top}(X)$ (resp. in $\mathfrak{Un}(X)$) s.t. $E \oplus F$ is trivial in $\mathfrak{Top}(X)$ (in $\mathfrak{Un}(X)$).

From what follows it will be apparent that from our point of view it is irrelevant to make any difference between the categories of topological and C^∞ vector bundles (and those in between), provided the mathematical background allows the definitions. This should be kept in mind while reading the rest of this paper.

PROPOSITION. *The bundles E_q are stably trivial in $\mathfrak{Top}(M - \{x\})$.*

Proof. Call \mathcal{T}_q the sheaf of germs of continuous sections of E_q and \mathcal{C} the sheaf of germs of continuous \mathbf{C} -valued functions on M .

First of all we notice that if X is any open subset of $M - \{x\}$ we have $H^p(X, \mathcal{T}_q) = 0$ for all $p > 0$ and q , because \mathcal{T}_q is a sheaf of \mathcal{C} -modules and thus fine.

Therefore the sequences

$$0 \rightarrow \Gamma(X, \mathcal{T}_q) \rightarrow \Gamma(X, \Lambda^{n-q-1} \mathcal{C}^n) \xrightarrow{d} \Gamma(X, \mathcal{T}_{q-1}) \rightarrow 0$$

are all exact. Since $\mathcal{T}_{n-2} \simeq \mathcal{C}$ our assertion will follow if we can split these exact sequences.

To this purpose we observe that, given any element $\sigma \in \Lambda^1 \mathcal{C}^n$, and $\omega \in \Lambda^p \mathcal{C}^n$ we can define the *contraction of ω along σ* , denoted by $\sigma \lrcorner \omega \in \Lambda^{p-1} \mathcal{C}^n$, by imposing \mathcal{C} -linearity to the contraction given on generators as follows:

$$F_i \lrcorner (F_{i_1} \wedge F_{i_2} \wedge \dots \wedge F_{i_p}) = \begin{cases} 0 & \text{if } i \notin (i_1, i_2, \dots, i_p) \\ (-1)^k F_{i_1} \wedge \dots \wedge F_{i_{k-1}} \wedge f_{i_{k+1}} \wedge \dots \wedge F_{i_p} & \text{if } i = i_k. \end{cases}$$

It is easily seen that if $\omega \in \Lambda^p \mathcal{C}^n$, $v \in \Lambda^r \mathcal{C}^n$ and $\sigma \in \Lambda^1 \mathcal{C}^n$ we have the relation

$$\sigma \lrcorner (\omega \wedge v) = (\sigma \lrcorner \omega) \wedge v + (-1)^p \omega \wedge (\sigma \lrcorner v).$$

Let us consider the distinguished element

$$\sigma = \sum_1^n \frac{\bar{f}_i}{|f|^2} F_i \in \Gamma(M - \{x\}, \Lambda^1 \mathcal{C}^n)$$

where \bar{f}_i is the complex conjugate of f_i and $|f|^2 = \sum_1^n f_i \bar{f}_i$.

Since $\sigma \lrcorner (\sum_1^n f_i F_i) = 1$ it follows from the relation above that if we define $\tilde{\sigma}: \Lambda^p \mathcal{C}^n \rightarrow \Lambda^{p-1} \mathcal{C}^n$ by $\tilde{\sigma}(\omega) = \sigma \lrcorner \omega$, we get

$$d \circ \tilde{\sigma} + \tilde{\sigma} \circ d = id_{\Lambda^p \mathcal{C}^n}.$$

Therefore $\tilde{\sigma}_{|T_{q-1}}: \Gamma(X, \mathcal{T}_{q-1}) \rightarrow \Gamma(X, \Lambda^{n-q-1} \mathcal{C}^n)$ gives the desired topological splitting. \square

It should be remarked that no such *analytic* splitting is available in general: for suppose that M is an analytic manifold (or a Cohen-Macauley space, see remark later) then, if $n \geq 3$, $H^1(U - \{x\}, \mathcal{O}) = 0$ for any Stein neighbourhood U of x , and if E_0 was stably trivial in $\mathfrak{A}n(M - \{x\})$ we would get immediately $H^1(U - \{x\}, \mathcal{L}_0) = 0$, which, as in the proof of theorem 2 would contradict the Riemann Removable Singularities Theorem.

In the light of the above remarks a more efficient though more technical version of our results can be given; first of all, if X is an analytic space of dimension n , let us call $\Xi_q(X)$ the class of locally free analytic sheaves on X which are associated to a topologically stably trivial analytic subbundle of rank $\binom{n-1}{n-q-2}$ of the product bundle $X \times \mathbf{C}^N$, $N = \binom{n}{n-q-1}$; since our sheaves \mathcal{L}_q are in $\Xi_q(D)$ the above results can be restated as

THEOREM 1'. *Let M be a strongly holomorphically separated analytic manifold of dimension n , D an open subset with C^2 boundary of M and suppose that $H^{q+1}(D, \mathcal{L}) = 0$ for every locally free sheaf in $\Xi_q(X)$, then D is q -pseudoconvex. If M is Stein the converse is also true.* \square

Since we know [G] that on a Stein space every topologically trivial analytic vector bundle is also analytically trivial, our second theorem can be improved as follows (this is also an improvement of Leiterer's theorem [L]):

THEOREM 2'. *Let M be a strongly holomorphically separated analytic space of dimension n , D an open subset of M and suppose that M is Stein or that D is relatively compact in M . The following conditions are equivalent:*

- 1) D is Stein,
- 2) $H^1(D, \mathcal{O}) = 0$ and every analytic vector bundle on D which is trivial in $\mathfrak{A}n(D)$ is also trivial in $\mathfrak{A}n(D)$,
- 3) $H^1(D, \mathcal{O}) = 0$ and every sheaf in $\Xi_0(D)$ is associated to a vector bundle which is stably trivial in $\mathfrak{A}n(D)$,
- 4) $H^1(D, \mathcal{L}) = 0$ for every sheaf \mathcal{L} in $\Xi_0(D)$,
- 5) However we choose functions f_1, f_2, \dots, f_n in $\Gamma(M, \mathcal{O})$ with no common zero on D the equation $\sum_1^n f_i g_i = 1$ has a solution (g_1, g_2, \dots, g_n) in $\Gamma(D, \mathcal{O})^n$.

Proof. Scheme: $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$; $1) \Rightarrow 5) \Rightarrow 1)$. $1) \Rightarrow 2)$ by the above mentioned theorem of Grauert [G] and Cartan's Theorem B; $2) \Rightarrow 3)$, $3) \Rightarrow 4)$ and $1) \Rightarrow 5)$ are trivial; $4) \Rightarrow 1)$ and $5) \Rightarrow 1)$ as in theorem 2. \square

Comparing this theorem with Leiterer's one we see that there is no need to embed M in \mathbf{C}^{2n+1} and use Bott's periodicity theorem.

As a final remark we observe that if M is a manifold or, more in general a Cohen-Macauley space, the sheaves \mathcal{L}_q , though locally free on $M - \{x\}$ have no chance to be locally free on M for $s = 0, 1, \dots, n-3$; in fact we have $\text{codh}_x \mathcal{L}_q = n - q - 2$, where codh_x indicates the homological codimension at x .

This can be shown as follows (without too many details since it is of a rather marginal importance for our purposes).

Claim: the sequence of sheaves

$$\xi: 0 \rightarrow \mathcal{K}^0(\mathbf{f}, \mathcal{O}) \xrightarrow{d} \mathcal{K}^1(\mathbf{f}, \mathcal{O}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{K}^n(\mathbf{f}, \mathcal{O}) \xrightarrow{\pi} \mathcal{O}/(\mathbf{f}) \rightarrow 0$$

is exact on M , where π denotes the projection from $\mathcal{K}^n(\mathbf{f}, \mathcal{O}) \simeq \mathcal{O}$ to the quotient sheaf $\mathcal{O}/(\mathbf{f}) = \mathcal{O}/(f_1, f_2, \dots, f_n)$; if we prove this we are done, because we know [A-G] prop. 4, page 200, that the last sheaf of ξ has homological codimension surely $\geq n$, and so ξ is a free resolution of $\mathcal{O}/(\mathbf{f})$ of minimal length.

The sequence ξ is surely exact on $M - \{x\}$ (lemma), so let U be a Stein neighbourhood of X and take an acyclic resolution $(\mathcal{F}^\bullet, \delta)$ of \mathcal{O} chosen in such a way that the double complex

$$K^{\bullet\bullet} = \{K_{p,r} = \Gamma(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{F}^r); d, \delta)\} \text{ is anticommutative.}$$

It is rather easy to see that the rows of $K^{\bullet\bullet}$ are exact, so that we obtain a degenerate spectral sequence

$$E_{p,r}^2 = 0$$

where

$$E_{p,r}^2 = \frac{\text{Ker } d: H^r(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{O})) \rightarrow H^r(U - \{x\}, \mathcal{K}^{p+1}(\mathbf{f}, \mathcal{O}))}{\text{Im } d: H^r(U - \{x\}, \mathcal{K}^{p-1}(\mathbf{f}, \mathcal{O})) \rightarrow H^r(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{O}))}$$

Since a) $H^r(U - \{x\}, \mathcal{O}) = 0$ for $r = 1, 2, \dots, n-2$ we obtain $E_{p,0}^2 = 0$ for $p = 0, 1, \dots, n-1$: but we can replace $U - \{x\}$ with U in the expression of $E_{p,r}^2$ by the Riemann Removable Singularities Theorem (which, together with a) characterizes Cohen-Macauley spaces ([S-T] theorem 1.14).

So our sequence ξ is exact except, perhaps, at the last two places, where it is exact for trivial reasons.