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VANISHING OF COHOMOLOGY WITH COEFFICIENTS IN A LOCALLY FREE SHEAF AND PSEUDOCONVEXITY

by Giuseppe VIGNA SURIA

INTRODUCTION

Cartan's Theorem B characterizes Stein spaces; a certain mathematical effort has been devoted to the study of a problem that can loosely be stated as follows: if we know that for a given analytic space X we have $H^p(X, \mathscr{S}) = 0$ for p ranging in a suitable set of integers N and $\mathscr{S}$ belonging to a suitable class $\Xi(X)$ of sheaves on X, can we deduce that X is Stein, or, more generally, what kind of function theoretic or geometrical properties on X can we find?

A first elementary progress in this direction is that we can narrow N as much as possible if we allow $\Xi(X)$ to be very large: if $H^1(X, \mathscr{S}) = 0$ for all coherent sheaves on X ($N = \{1\}$, $\Xi(X) =$ all coherent sheaves on X) then X is Stein; on the other hand a result of Coen [C] says that, in the case when X is an open subset D of a Stein space of dimension n, D is Stein if $H^p(D, \mathcal{O}) = 0$ for $p = 1, 2, ..., n-1$ ($N = \{1, 2, ..., n-1\}$, $\Xi(D) = \{$structure sheaf $\mathcal{O}\}$); a theorem due to Leiterer [L] makes a reasonably good compromise between the above facts by showing that an open subset D of a Stein manifold is Stein if $H^1(D, \mathscr{L}) = 0$ for every locally free sheaf $\mathscr{L}$ on D ($N = \{1\}$, $\Xi(D) = \{$locally free sheaves$\}$, but actually Leiterer can make this class even smaller); if we further assume that D has a C^2 boundary then if $H^p(D, \mathcal{O}) = 0$ for $p > q$, where q is a fixed integer, D is q-pseudoconvex

$$(N = \{q+1, q+2, ..., n-1\}, \Xi(D) = \{\mathcal{O}\}),$$

see [E-VS].

The goal of this paper is to give a contribution of the same type as above to our original vague problem.

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Our basic assumption will be that D is an open subset of a Stein manifold M of dimension $n \geqslant 2$, but very often weaker hypotheses, such as requiring M to be only strongly holomorphically separated and/or allowing singularities, will be sufficient.

Using a rather elementary device, the Koszul complex of sheaves, we shall be able to prove that if D has C^2 boundary and $H^{q+1}(D, \mathscr{L}) = 0$ for $\mathscr{L}$ ranging in a suitable class $\Xi_q(D)$ of locally free sheaves on D, then D is q-pseudoconvex. For $q = 0$ we shall not need the hypothesis on the boundary and M will be allowed to have singularities and even to be only holomorphically separated (in the latter case, on the other hand, we need to assume that D is relatively compact in M) so that if $H^1(D, \mathscr{L}) = 0$ for all $\mathscr{L}$ in a suitable class $\Xi_0(D)$ of locally free sheaves, then D is Stein.

The locally free sheaves in these classes $\Xi_q(D)$ will also be investigated from a "topological" point of view and we shall see that they are associated to stably trivial vector bundles.

We say that an analytic space M is *strongly holomorphically separated* if, given any point $x \in M$, we can find $n = \dim M$ global holomorphic functions $f_1, f_2, \ldots, f_n$ on M *defining* x, i.e. we have

$$\{x\} = \{y \in M \text{ s.t. } f_1(y) = f_2(y) = \ldots = f_n(y) = 0\} ;$$

this can surely be done if M is Stein [F-R] Satz 1 p. 91, but it is reasonable to cojecturate that it is also true when holomorphic functions separate the points of M.

We shall prove and make some remarks on the following.

THEOREM 1. *Let M be a strongly holomorphically separated analytic manifold and D be an open subset with C^2 boundary of M; if for every locally free sheaf $\mathscr{L}$ on D we have $H^{q+1}(D, \mathscr{L}) = 0$ then D is q-pseudoconvex.*

Remark. If M is actually Stein the converse is also true: in this case in fact, if D is q-pseudoconvex then it is q-complete [VS] and therefore $H^p(D, \mathscr{S}) = 0$ for all $p > q$ and all coherent sheaves $\mathscr{S}$ on D, [A-G] Corollary 1 p. 250.

The reason for assuming M strongly holomorphically separated is that, given $x \in M$ and global holomorphic functions $f_1, f_2, \ldots, f_n$ defining x we can construct the *Koszul complex of sheaves*

$$\mathscr{K}(\mathbf{f}, \mathcal{O}) = \{\mathscr{K}^p(\mathbf{f}, \mathcal{O}); d\}_{p \in Z}$$

as follows: $\mathscr{K}^p(\mathbf{f}, \mathcal{O}) = \Lambda^p \mathcal{O}^n$ and $d\colon \mathscr{K}^p(\mathbf{f}, \mathcal{O}) \to \mathscr{K}^{p+1}(\mathbf{f}, \mathcal{O})$ is given by $d(\omega) = \mathbf{f} \wedge \omega$ where $\omega \in \mathscr{K}^p(\mathbf{f}, \mathcal{O})$ and $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \mathcal{O}^n = \Lambda^1 \mathcal{O}^n$.

The theorem will be an easy consequence of the following

LEMMA. *Let* $\mathscr{L}_s$ *denote* $\operatorname{Ker} d\colon \mathscr{K}^{n-s-1} \to \mathscr{K}^{n-s}$; *the sequence*

$$\xi_s \qquad 0 \to \mathscr{L}_s \to \mathscr{K}^{n-s-1} \to \mathscr{L}_{s-1} \to 0$$

is a short exact sequence of analytic sheaves on $M - \{x\}$, *for all* s; *moreover* $\mathscr{L}_s$ *is a locally free sheaf of rank* $\binom{n-1}{n-s-2}$ *on* $M - \{x\}$; *more precisely if* $U_i = \{y \in M \text{ s.t. } f_i(y) \neq 0\}$ $i = 1, 2, \ldots, n$, *then*

$$\mathscr{L}_{s|U_i} \simeq \Lambda^{n-s-2} \mathcal{O}^{n-1}_{|U_i}$$

for all $s \in Z$. *Furthermore* $\mathscr{L}_{n-2} \simeq \mathcal{O}$ *on* M.

Proof. We shall find explicit isomorphisms of sheaves $\phi_i\colon \Lambda^{n-s-2} \mathcal{O}^{n-1}_{|U_i} \to \mathscr{L}_{s|U_i}$ and construct split exact sequences

$$0 \to \Lambda^{p-1} \mathcal{O}^{n-1}_{|U_i} \xrightarrow{\Phi_i} \Lambda^p \mathcal{O}^n_{|U_i} \xrightarrow{\Psi_i} \Lambda^p \mathcal{O}^{n-1}_{|U_i} \to 0$$

such that $\phi_i = \Phi_i$ and $\phi_i \circ \Psi_i = d$; an isomorphism $\mathcal{O} = \Lambda^0 \mathcal{O}^n \xrightarrow{d} \mathscr{L}_{n-2}$ will also be explicitely given; this clearly proves the lemma.

So let us fix $i = 1, 2, \ldots, n$, let $F_1, F_2, \ldots, F_n$ be the formal symbols on which the exterior $\mathcal{O}$-algebra $\Lambda^{\cdot} \mathcal{O}^n$ is constructed and think of $\Lambda^{\cdot} \mathcal{O}^{n-1}$ as based on $F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_n$. An element $\omega \in \Lambda^p \mathcal{O}^n$ can be written uniquely as $\omega = F_i \wedge \mu + \nu$ where $\mu \in \Lambda^{p-1} \mathcal{O}^{n-1}$ and $\nu \in \Lambda^p \mathcal{O}^{n-1}$; we can define a sheaf homomorphism $h\colon \Lambda^p \mathcal{O}^n_{|U_i} \to \Lambda^{p-1} \mathcal{O}^{n-1}_{|U_i}$, depending on i, as follows:

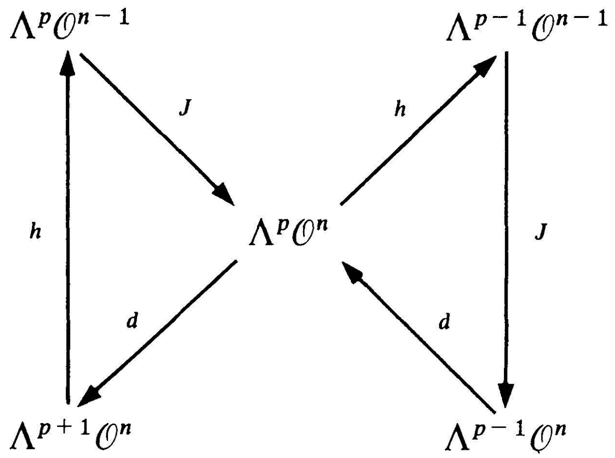
$$h(\omega) = h(F_i \wedge \mu + \nu) = f_i^{-1} \mu .$$

To simplify the notation we shall omit the restriction $_{|U_i}$.

Let

$$J\colon \Lambda^p \mathcal{O}^{n-1} \to \Lambda^p \mathcal{O}^n$$

denote the inclusion (depending on i if $p = 1, 2, \ldots, n-1$), the sheaf homomorphisms constructed so far are indicated in the following diagram:



It is easy to check the following properties:

1) $d \circ J \circ h + J \circ h \circ d = id_{\Lambda^p \mathcal{O}^n}$,

2) $h \circ d \circ J = id_{\Lambda^p \mathcal{O}^{n-1}}$,

3) $d \circ J \circ h \circ d = d$;

we prove 1), the others are even easier to show; so take $\omega = F_i \wedge \mu + \nu \in \Lambda^p\mathcal{O}^n$ then

$$\begin{aligned}(d \circ J \circ h + J \circ h \circ d)(\omega) &= d(f_i^{-1}\mu) + J \circ h(\Sigma_k f_k F_k \wedge F_i \wedge \mu + \Sigma_k f_k F_k \wedge \nu) \\ &= \Sigma_k f_k f_i^{-1} F_k \wedge \mu + J \circ h(-F_i \wedge \Sigma_{k \neq i} f_k F_k \mu) + \nu \\ &= \Sigma_k f_k f_i^{-1} F_k \wedge \mu - \Sigma_{k \neq i} f_k f_i^{-1} F_k \wedge \mu + \nu = F_i \wedge \mu + \nu = \omega .\end{aligned}$$

Now define

$$\Phi_i : \Lambda^{p-1}\mathcal{O}^{n-1} \to \Lambda^p\mathcal{O}^n \quad \text{by} \quad \Phi_i = d \circ J ,$$
$$\Psi_i : \Lambda^p\mathcal{O}^n \to \Lambda^p\mathcal{O}^{n-1} \quad \text{by} \quad \Psi_i = h \circ d ,$$
$$\phi_i : \Lambda^p\mathcal{O}^{n-1} \to \mathcal{L}_{n-p-2} \quad \text{by} \quad \phi_i = d \circ J .$$

There is a list of facts to check, all following easily from 1), 2), and 3); they are: Φ_i is one to one, Ψ_i is onto, Ker Ψ_i = Im Φ_i, $d \circ \phi_i = 0$, ϕ_i is one to one and onto and $\phi_i \circ \Psi_i = d$.

Moreover $J : \Lambda^0\mathcal{O}^{n-1} \to \Lambda^0\mathcal{O}^n$ is simply the identity so that $d : \Lambda^0\mathcal{O}^n \to \mathcal{L}_{n-2}$ is an isomorphism on the whole of M.

The lemma is therefore proved. □

To proceed towards the proof of theorem 1 we must consider distinguished elements in $H^{s+1}(M - \{x\}, \mathcal{L}_s)$, the *test classes* introduced in [E-VS]; for this purpose we take the long exact cohomology sequence associated to ξ_s

$$\ldots \to H^s(M - \{x\}, \mathcal{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(M - \{x\}, \mathcal{L}_s) \to \ldots$$

and consider the elements $\alpha_s(x) \in H^{s+1}(M-\{x\}, \mathscr{L}_s)$ given inductively as follows: $\alpha_0(x) = \delta(F_1 \wedge F_2 \wedge \ldots \wedge F_n)$, where $F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is in

$$H^0(M-\{x\}, \mathscr{L}_{-1}) = H^0(M-\{x\}, \mathscr{K}^n(\mathbf{f}, \mathcal{O}))$$

$$\text{and} \quad \alpha_s(x) = \delta(\alpha_{s-1}(x)) \quad \text{if} \quad s \geqslant 1 .$$

Proof of theorem 1. To say that D has C^2 boundary means that, given any point $x \in \partial D$ we can find a C^2 *defining function* $\varphi : U \to \mathbf{R}$, where U is an open neighbourhood of x, such that $D \cap U = \{y \in U \text{ s.t. } \varphi(y) < 0\}$ and $d\varphi(x) \neq 0$; in these conditions the number $n(x)$ of negative eigenvalues of the Levi form $\mathscr{L}(\varphi)(x)$ depends only on D and x and not on the choice of φ; if D is not q-pseudoconvex there is a point $x \in \partial D$ such that $n(x) \geqslant q + 1$, but then, with a slight modification of the defining function φ [E-VS] 3.6 corollary 1, we can manage to obtain at least $q + 2$ negative eigenvalues for the complex Hessian $\mathscr{H}(\varphi)(x)$; following the argument of [A-G] proposition 12 p. 222, we can find a neighbourhood Q of x (as small as we want) such that

a) $H^p(D \cap Q, \mathcal{O}) = 0$ for $p = 1, 2, \ldots, q$ and

b) the restriction $H^0(Q, \mathcal{O}) \to H^0(D \cap Q, \mathcal{O})$ is an isomorphism.

If we take global holomorphic functions $f_1, f_2, \ldots, f_n$ defining x and construct the Koszul complex $\mathscr{K}(\mathbf{f}, \mathcal{O})$ we find that $H^{q+1}(D, \mathscr{L}_q) = 0$ because $\mathscr{L}_q$ is locally free on D; therefore $\alpha_q(x)_{|D} \in H^{q+1}(D, \mathscr{L}_q)$ vanishes and, by further restricting, $\alpha_q(x)_{|D \cap Q}$ vanishes too; using the long exact sequence of cohomology

$$\ldots \to H^s(D \cap Q, \mathscr{K}^{n-s-1}(\mathbf{f}, \mathcal{O})) \to H^s(D \cap Q, \mathscr{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(D \cap Q, \mathscr{L}_s) \to$$

associated to ξ_s for $s = 1, 2, \ldots, q$ together with a) and the fact that $\mathscr{K}^{n-s-1}(\mathbf{f}, \mathcal{O})$ is a free sheaf we deduce that

$$0 = \alpha_q(x)_{|D \cap Q} = \alpha_{q-1}(x)_{|D \cap Q} = \ldots = \alpha_0(x)_{|D \cap Q};$$

but then, taking $s = 0$, $\alpha_0(x)_{|D \cap Q} = 0$ means that the equation $\Sigma_1^n f_i g_i = 1$ has a solution $(g_1, g_2, \ldots, g_n) \in H^0(D \cap Q, \mathcal{O})^n$. This would imply that at least one of the g_i's does not extend to Q, contradicting b); therefore our original assumption that D is not q-pseudoconvex must be wrong and the theorem is proved. □

This proof follows, with minor modifications, that of Proposition 2.1 of [E-VS], it is reported here mainly for reasons of clarity. It should also be remarked that if $H^p(D, \mathcal{O}) = 0$ for $p > q$ then $H^{q+1}(D, \mathscr{L}_q) = 0$ ([E-VS] Lemma 3.2, see introduction).

The theorem holds for $q = 0$ too, but in this case we can avoid the hypothesis on the boundary and also the smoothness of the ambient space M if we impose some mild conditions: more precisely we have the following

THEOREM 2. *Let M be a strongly holomorphically separated analytic space and D an open subset of M; suppose that either M is Stein or that D is relatively compact in M. Then the following conditions are equivalent:*
1) D is Stein,
2) $H^1(D, \mathscr{L}) = 0$ for every locally free sheaf $\mathscr{L}$ on D.

Proof. *1) ⇒ 2)* by Cartan's Theorem B.
2) ⇒ 1) We must show that D is holomorphically convex, and this will be done by proving that, given any discrete sequence $\{x_n\}$ in D we can find a holomorphic fonction g on D such that $\overline{\lim_{n\to\infty}} | g(x_n) | = \infty$.

In either of our hypotheses we can suppose that $\{x_n\}$ converges to a point $x \in \partial D$; construct the Koszul complex $\mathscr{K}(\mathbf{f}, \mathscr{O})$ starting from this point. Since $\mathscr{L}_0$ is locally free (of rank $n-1$ on $M-x$ and on D) we get $H^1(D, \mathscr{L}_0) = 0$ so that $\alpha_0(x)_{|D} = 0$; as before this means that the equation $\Sigma_1^n f_i g_i = 1$ has a solution $(g_1, ..., g_n)$ in $H^0(D, \mathscr{O})^n$. For one at least of the g_i's we must have $\overline{\lim_{n\to\infty}} | g_i(x_n) | = \infty$. □

As in the remark above if $H^p(D, \mathscr{O}) = 0$ for $p = 1, 2, ..., n-1$ then $H^1(D, \mathscr{L}_0) = 0$, so that we obtain Coen's result [C] under slightly different hypotheses.

The proofs of theorems 1 and 2 should have persuaded the reader that a deeper investigation of these powerful sheaves $\mathscr{L}_q$ is worth a little effort. So far we have discovered that $\mathscr{L}_q$ is a locally free sheaf of rank $\binom{n-1}{n-q-2}$ on $M-\{x\}$. Let us call E_q the corresponding holomorphic vector bundle; E_q is a subbundle of the trivial bundle $(M-\{x\}) \times \mathbf{C}^N$, $N = \binom{n}{n-q-1}$.

We shall now discuss the topological properties of these bundles E_q; in the literature trivial and stably trivial bundles are considered [A], [L].

Definition. Let X be a topological space (respectively an analytic space) and E a topological (analytic) vector bundle on X with complex fibre; we say that E is *stably trivial in the category* $\mathfrak{Top}(X)$ of topological vector bundles on X (respectively *in the category* $\mathfrak{An}(X)$ of analytic vector bundles on X) or, more quickly, that E *is topologically stably trivial (analytically stably trivial)* if there exists a trivial vector bundle F in $\mathfrak{Top}(X)$ (resp. in $\mathfrak{An}(X)$) s.t. $E \oplus F$ is trivial in $\mathfrak{Top}(X)$ (in $\mathfrak{An}(X)$).

From what follows it will be apparent that from our point of view it is irrelevant to make any difference between the categories of topological and C^∞ vector bundles (and those in between), provided the mathematical background allows the definitions. This should be kept in mind while reading the rest of this paper.

PROPOSITION. *The bundles* E_q *are stably trivial in* $\mathfrak{Top}(M-\{x\})$.

Proof. Call $\mathscr{T}_q$ the sheaf of germs of continuous sections of E_q and $\mathscr{C}$ the sheaf of germs of continuous $\mathbf{C}$-valued functions on M.

First of all we notice that if X is any open subset of $M-\{x\}$ we have $H^p(X, \mathscr{T}_q) = 0$ for all $p > 0$ and q, because $\mathscr{T}_q$ is a sheaf of $\mathscr{C}$-modules and thus fine.

Therefore the sequences

$$0 \to \Gamma(X, \mathscr{T}_q) \to \Gamma(X, \Lambda^{n-q-1}\mathscr{C}^n) \xrightarrow{d} \Gamma(X, \mathscr{T}_{q-1}) \to 0$$

are all exact. Since $\mathscr{T}_{n-2} \simeq \mathscr{C}$ our assertion will follow if we can split these exact sequences.

To this purpose we observe that, given any element $\sigma \in \Lambda^1\mathscr{C}^n$, and $\omega \in \Lambda^p\mathscr{C}^n$ we can define the *contraction of* ω *along* σ, denoted by $\sigma \vee \omega \in \Lambda^{p-1}\mathscr{C}^n$, by imposing $\mathscr{C}$-linearity to the contraction given on generators as follows:

$$F_i \vee (F_{i_1} \wedge F_{i_2} \wedge \dots \wedge F_{i_p}) = \begin{cases} 0 \text{ if } i \notin (i_1, i_2, \dots, i_p) \\ (-1)^k F_{i_1} \wedge \dots \wedge F_{i_{k-1}} \wedge f_{i_{k+1}} \wedge \dots \wedge F_{i_p} \text{ if } i = i_k . \end{cases}$$

It is easily seen that if $\omega \in \Lambda^p\mathscr{C}^n$, $\nu \in \Lambda^r\mathscr{C}^n$ and $\sigma \in \Lambda^1\mathscr{C}^n$ we have the relation

$$\sigma \vee (\omega \wedge \nu) = (\sigma \vee \omega) \wedge \nu + (-1)^p \omega \wedge (\sigma \vee \nu) .$$

Let us consider the distinguished element

$$\sigma = \Sigma_1^n \frac{\bar{f}_i}{|f|^2} F_i \in \Gamma(M-\{x\}, \Lambda^1\mathscr{C}^n)$$

where $\bar{f}_i$ is the complex conjugate of f_i and $|f|^2 = \Sigma_1^n f_i \bar{f}_i$.

Since $\sigma \vee (\Sigma_1^n f_i F_i) = 1$ it follows from the relation above that if we define $\tilde{\sigma}: \Lambda^p\mathscr{C}^n \to \Lambda^{p-1}\mathscr{C}^n$ by $\tilde{\sigma}(\omega) = \sigma \vee \omega$, we get

$$d \circ \tilde{\sigma} + \tilde{\sigma} \circ d = id_{\Lambda^p\mathscr{C}^n} .$$

Therefore $\tilde{\sigma}_{|\mathrm{T}_{q-1}}: \Gamma(X, \mathscr{T}_{q-1}) \to \Gamma(X, \Lambda^{n-q-1}\mathscr{C}^n)$ gives the desired topological splitting. □

It should be remarked that no such *analytic* splitting is available in general: for suppose that M is an analytic manifold (or a Cohen-Macauley space, see remark later) then, if $n \geqslant 3$, $H^1(U-\{x\}, \mathscr{O}) = 0$ for any Stein neighbourhood U of x, and if E_o was stably trivial in $\mathfrak{An}(M-\{x\})$ we would get immediately $H^1(U-\{x\}, \mathscr{L}_0) = 0$, which, as in the proof of theorem 2 would contradict the Riemann Removable Singularities Theorem.

In the light of the above remarks a more efficient though more technical version of our results can be given; first of all, if X is an analytic space of dimension n, let us call $\Xi_q(X)$ the class of locally free analytic sheaves on X which are associated to a topologically stabily trivial analytic subbundle of rank $\binom{n-1}{n-q-2}$ of the product bundle $X \times \mathbf{C}^N$, $N = \binom{n}{n-q-1}$; since our sheaves $\mathscr{L}_q$ are in $\Xi_q(D)$ the above results can be restated as

THEOREM 1'. *Let M be a strongly holomorphically separated analytic manifold of dimension n, D an open subset with C^2 boundary of M and suppose that $H^{q+1}(D, \mathscr{L}) = 0$ for every locally free sheaf in $\Xi_q(X)$, then D is q-pseudoconvex. If M is Stein the converse is also true.* □

Since we know [G] that on a Stein space every topologically trivial analytic vector bundle is also analytically trivial, our second theorem can be improved as follows (this is also an improvement of Leiterer's theorem [L]):

THEOREM 2'. *Let M be a strongly holomorphically separated analytic space of dimension n, D an open subset of M and suppose that M is Stein or that D is relatively compact in M. The following conditions are equivalent:*

1) D is Stein,

2) $H^1(D, \mathscr{O}) = 0$ and every analytic vector bundle on D which is trivial in $\mathfrak{Top}(D)$ is also trivial in $\mathfrak{An}(D)$,

3) $H^1(D, \mathscr{O}) = 0$ and every sheaf in $\Xi_0(D)$ is associated to a vector bundle which is stably trivial in $\mathfrak{An}(D)$,

4) $H^1(D, \mathscr{L}) = 0$ for every sheaf $\mathscr{L}$ in $\Xi_0(D)$,

5) However we choose functions $f_1, f_2, ..., f_n$ in $\Gamma(M, \mathscr{O})$ with no common zero on D the equation $\Sigma_1^n f_i g_i = 1$ has a solution $(g_1, g_2, ..., g_n)$ in $\Gamma(D, \mathscr{O})^n$.

Proof. Scheme: *1)* $\Rightarrow$ *2)* $\Rightarrow$ *3)* $\Rightarrow$ *4)* $\Rightarrow$ *1)*; *1)* $\Rightarrow$ *5)* $\Rightarrow$ *1)*. *1)* $\Rightarrow$ *2)* by the above mentioned theorem of Grauert [G] and Cartan's Theorem B; *2)* $\Rightarrow$ *3)*, *3)* $\Rightarrow$ *4)* and *1)* $\Rightarrow$ *5)* are trivial; *4)* $\Rightarrow$ *1)* and *5)* $\Rightarrow$ *1)* as in theorem 2. □

Comparing this theorem with Leiterer's one we see that there is no need to embed M in $\mathbf{C}^{2n+1}$ and use Bott's periodicity theorem.

As a final remark we observe that if M is a manifold or, more in general a Cohen-Macauley space, the sheaves $\mathscr{L}_q$, though locally free on $M-\{x\}$ have no chance to be locally free on M for $s = 0, 1, \ldots, n-3$; in fact we have $\operatorname{codh}_x \mathscr{L}_q = n - q - 2$, where codh_x indicates the homological codimension at x.

This can be shown as follows (without too many details since it is of a rather marginal importance for our purposes).

Claim: the sequence of sheaves

$$\xi \quad 0 \to \mathscr{K}^0(\mathbf{f}, \mathscr{O}) \xrightarrow{d} \mathscr{K}^1(\mathbf{f}, \mathscr{O}) \xrightarrow{d} \ldots \xrightarrow{d} \mathscr{K}^n(\mathbf{f}, \mathscr{O}) \xrightarrow{\pi} \mathscr{O}/(\mathbf{f}) \to 0$$

is exact on M, where π denotes the projection from $\mathscr{K}^n(\mathbf{f}, \mathscr{O}) \simeq \mathscr{O}$ to the quotient sheaf $\mathscr{O}/(\mathbf{f}) = \mathscr{O}/(f_1, f_2, \ldots, f_n)$; if we prove this we are done, because we know [A-G] prop. 4, page 200, that the last sheaf of ξ has homological codimension surely $\geqslant n$, and so ξ is a free resolution of $\mathscr{O}/(\mathbf{f})$ of minimal length.

The sequence ξ is surely exact on $M-\{x\}$ (lemma), so let U be a Stein neighbourhood of X and take an acyclic resolution $(\mathscr{F}^{\cdot}, \delta)$ of $\mathscr{O}$ chosen in such a way that the double complex

$$K^{\cdot\cdot} = \{K_{p,r} = \Gamma(U-\{x\}, \mathscr{K}^p(\mathbf{f}, \mathscr{F}^r); d, \delta\} \text{ is anticommutative.}$$

It is rather easy to see that the rows of $K^{\cdot\cdot}$ are exact, so that we obtain a degenerate spectral sequence

$$E^2_{p,r} \Rightarrow 0$$

where

$$E^2_{p,r} = \frac{\operatorname{Ker} d \colon H^r(U-\{x\}, \mathscr{K}^p(\mathbf{f}, \mathscr{O})) \to H^r(U-\{x\}, \mathscr{K}^{p+1}(\mathbf{f}, \mathscr{O}))}{\operatorname{Im} d \colon H^r(U-\{x\}, \mathscr{K}^{p-1}(\mathbf{f}, \mathscr{O})) \to H^r(U-\{x\}, \mathscr{K}^p(\mathbf{f}, \mathscr{O}))}$$

Since a) $H^r(U-\{x\}, \mathscr{O}) = 0$ for $r = 1, 2, \ldots, n-2$ we obtain $E^2_{p,0} = 0$ for $p = 0, 1, \ldots, n-1$: but we can replace $U-\{x\}$ with U in the expression of $E^2_{p,r}$ by the Riemann Removable Singularities Theorem (which, together with a) characterizes Cohen-Macauley spaces ([S-T] theorem 1.14).

So our sequence ξ is exact except, perhaps, at the last two places, where it is exact for trivial reasons.