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## VANISHING OF COHOMOLOGY WITH COEFFICIENTS IN A LOCALLY FREE SHEAF AND PSEUDOCONVEXITY

by Giuseppe VIGNA SURIA

### INTRODUCTION

Cartan's Theorem B characterizes Stein spaces; a certain mathematical effort has been devoted to the study of a problem that can loosely be stated as follows: if we know that for a given analytic space  $X$  we have  $H^p(X, \mathcal{S}) = 0$  for  $p$  ranging in a suitable set of integers  $N$  and  $\mathcal{S}$  belonging to a suitable class  $\Xi(X)$  of sheaves on  $X$ , can we deduce that  $X$  is Stein, or, more generally, what kind of function theoretic or geometrical properties on  $X$  can we find?

A first elementary progress in this direction is that we can narrow  $N$  as much as possible if we allow  $\Xi(X)$  to be very large: if  $H^1(X, \mathcal{S}) = 0$  for all coherent sheaves on  $X$  ( $N = \{1\}$ ,  $\Xi(X) =$  all coherent sheaves on  $X$ ) then  $X$  is Stein; on the other hand a result of Coen [C] says that, in the case when  $X$  is an open subset  $D$  of a Stein space of dimension  $n$ ,  $D$  is Stein if  $H^p(D, \mathcal{O}) = 0$  for  $p = 1, 2, \dots, n-1$  ( $N = \{1, 2, \dots, n-1\}$ ,  $\Xi(D) = \{\text{structure sheaf } \mathcal{O}\}$ ); a theorem due to Leiterer [L] makes a reasonably good compromise between the above facts by showing that an open subset  $D$  of a Stein manifold is Stein if  $H^1(D, \mathcal{L}) = 0$  for every locally free sheaf  $\mathcal{L}$  on  $D$  ( $N = \{1\}$ ,  $\Xi(D) = \{\text{locally free sheaves}\}$ , but actually Leiterer can make this class even smaller); if we further assume that  $D$  has a  $C^2$  boundary then if  $H^p(D, \mathcal{O}) = 0$  for  $p > q$ , where  $q$  is a fixed integer,  $D$  is  $q$ -pseudoconvex

$$(N = \{q+1, q+2, \dots, n-1\}, \Xi(D) = \{\mathcal{O}\}),$$

see [E-VS].

The goal of this paper is to give a contribution of the same type as above to our original vague problem.

Our basic assumption will be that  $D$  is an open subset of a Stein manifold  $M$  of dimension  $n \geq 2$ , but very often weaker hypotheses, such as requiring  $M$  to be only strongly holomorphically separated and/or allowing singularities, will be sufficient.

Using a rather elementary device, the Koszul complex of sheaves, we shall be able to prove that if  $D$  has  $C^2$  boundary and  $H^{q+1}(D, \mathcal{L}) = 0$  for  $\mathcal{L}$  ranging in a suitable class  $\Xi_q(D)$  of locally free sheaves on  $D$ , then  $D$  is  $q$ -pseudoconvex. For  $q = 0$  we shall not need the hypothesis on the boundary and  $M$  will be allowed to have singularities and even to be only holomorphically separated (in the latter case, on the other hand, we need to assume that  $D$  is relatively compact in  $M$ ) so that if  $H^1(D, \mathcal{L}) = 0$  for all  $\mathcal{L}$  in a suitable class  $\Xi_0(D)$  of locally free sheaves, then  $D$  is Stein.

The locally free sheaves in these classes  $\Xi_q(D)$  will also be investigated from a "topological" point of view and we shall see that they are associated to stably trivial vector bundles.

We say that an analytic space  $M$  is *strongly holomorphically separated* if, given any point  $x \in M$ , we can find  $n = \dim M$  global holomorphic functions  $f_1, f_2, \dots, f_n$  on  $M$  defining  $x$ , i.e. we have

$$\{x\} = \{y \in M \text{ s.t. } f_1(y) = f_2(y) = \dots = f_n(y) = 0\};$$

this can surely be done if  $M$  is Stein [F-R] Satz 1 p. 91, but it is reasonable to conjecture that it is also true when holomorphic functions separate the points of  $M$ .

We shall prove and make some remarks on the following.

**THEOREM 1.** *Let  $M$  be a strongly holomorphically separated analytic manifold and  $D$  be an open subset with  $C^2$  boundary of  $M$ ; if for every locally free sheaf  $\mathcal{L}$  on  $D$  we have  $H^{q+1}(D, \mathcal{L}) = 0$  then  $D$  is  $q$ -pseudoconvex.*

*Remark.* If  $M$  is actually Stein the converse is also true: in this case in fact, if  $D$  is  $q$ -pseudoconvex then it is  $q$ -complete [VS] and therefore  $H^p(D, \mathcal{S}) = 0$  for all  $p > q$  and all coherent sheaves  $\mathcal{S}$  on  $D$ , [A-G] Corollary 1 p. 250.

The reason for assuming  $M$  strongly holomorphically separated is that, given  $x \in M$  and global holomorphic functions  $f_1, f_2, \dots, f_n$  defining  $x$  we can construct the *Koszul complex of sheaves*

$$\mathcal{K}(\mathbf{f}, \mathcal{O}) = \{\mathcal{K}^p(\mathbf{f}, \mathcal{O}); d\}_{p \in \mathbb{Z}}$$

as follows:  $\mathcal{K}^p(\mathbf{f}, \mathcal{O}) = \Lambda^p \mathcal{O}^n$  and  $d: \mathcal{K}^p(\mathbf{f}, \mathcal{O}) \rightarrow \mathcal{K}^{p+1}(\mathbf{f}, \mathcal{O})$  is given by  $d(\omega) = \mathbf{f} \wedge \omega$  where  $\omega \in \mathcal{K}^p(\mathbf{f}, \mathcal{O})$  and  $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathcal{O}^n = \Lambda^1 \mathcal{O}^n$ .

The theorem will be an easy consequence of the following

LEMMA. Let  $\mathcal{L}_s$  denote  $\text{Ker } d: \mathcal{K}^{n-s-1} \rightarrow \mathcal{K}^{n-s}$ ; the sequence

$$\xi_s \quad 0 \rightarrow \mathcal{L}_s \rightarrow \mathcal{K}^{n-s-1} \rightarrow \mathcal{L}_{s-1} \rightarrow 0$$

is a short exact sequence of analytic sheaves on  $M - \{x\}$ , for all  $s$ ; moreover  $\mathcal{L}_s$  is a locally free sheaf of rank  $\binom{n-1}{n-s-2}$  on  $M - \{x\}$ ; more precisely if  $U_i = \{y \in M \text{ s.t. } f_i(y) \neq 0\}$   $i = 1, 2, \dots, n$ , then

$$\mathcal{L}_s|_{U_i} \simeq \Lambda^{n-s-2} \mathcal{O}_{|U_i}^{n-1}$$

for all  $s \in \mathbb{Z}$ . Furthermore  $\mathcal{L}_{n-2} \simeq \mathcal{O}$  on  $M$ .

*Proof.* We shall find explicit isomorphisms of sheaves  $\phi_i: \Lambda^{n-s-2} \mathcal{O}_{|U_i}^{n-1} \rightarrow \mathcal{L}_s|_{U_i}$  and construct split exact sequences

$$0 \rightarrow \Lambda^{p-1} \mathcal{O}_{|U_i}^{n-1} \xrightarrow{\Phi_i} \Lambda^p \mathcal{O}_{|U_i}^n \xrightarrow{\Psi_i} \Lambda^p \mathcal{O}_{|U_i}^{n-1} \rightarrow 0$$

such that  $\phi_i = \Phi_i$  and  $\phi_i \circ \Psi_i = d$ ; an isomorphism  $\mathcal{O} = \Lambda^0 \mathcal{O}^n \xrightarrow{d} \mathcal{L}_{n-2}$  will also be explicitly given; this clearly proves the lemma.

So let us fix  $i = 1, 2, \dots, n$ , let  $F_1, F_2, \dots, F_n$  be the formal symbols on which the exterior  $\mathcal{O}$ -algebra  $\Lambda^* \mathcal{O}^n$  is constructed and think of  $\Lambda^* \mathcal{O}^{n-1}$  as based on  $F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_n$ . An element  $\omega \in \Lambda^p \mathcal{O}^n$  can be written uniquely as  $\omega = F_i \wedge \mu + \nu$  where  $\mu \in \Lambda^{p-1} \mathcal{O}^{n-1}$  and  $\nu \in \Lambda^p \mathcal{O}^{n-1}$ ; we can define a sheaf homomorphism  $h: \Lambda^p \mathcal{O}_{|U_i}^n \rightarrow \Lambda^{p-1} \mathcal{O}_{|U_i}^{n-1}$ , depending on  $i$ , as follows:

$$h(\omega) = h(F_i \wedge \mu + \nu) = f_i^{-1} \mu.$$

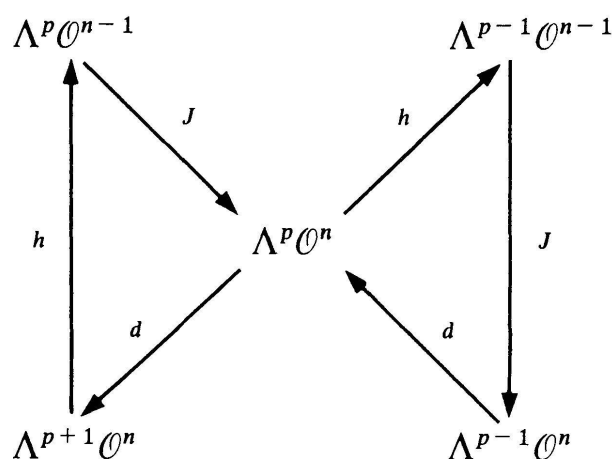
To simplify the notation we shall omit the restriction  $|_{U_i}$ .

Let

$$J: \Lambda^p \mathcal{O}^{n-1} \rightarrow \Lambda^p \mathcal{O}^n$$

denote the inclusion (depending on  $i$  if  $p = 1, 2, \dots, n-1$ ), the sheaf homomorphisms constructed so far are indicated in the following diagram:





It is easy to check the following properties:

- 1)  $d \circ J \circ h + J \circ h \circ d = id_{\Lambda^p \mathcal{O}^n}$ ,
- 2)  $h \circ d \circ J = id_{\Lambda^p \mathcal{O}^{n-1}}$ ,
- 3)  $d \circ J \circ h \circ d = d$ ;

we prove 1), the others are even easier to show; so take  $\omega = F_i \wedge \mu + \nu \in \Lambda^p \mathcal{O}^n$  then

$$\begin{aligned}
 (d \circ J \circ h + J \circ h \circ d)(\omega) &= d(f_i^{-1} \mu) + J \circ h(\sum_k f_k F_k \wedge F_i \wedge \mu + \sum_k f_k F_k \wedge \nu) \\
 &= \sum_k f_k f_i^{-1} F_k \wedge \mu + J \circ h(-F_i \wedge \sum_{k \neq i} f_k F_k \mu) + \nu \\
 &= \sum_k f_k f_i^{-1} F_k \wedge \mu - \sum_{k \neq i} f_k f_i^{-1} F_k \wedge \mu + \nu = F_i \wedge \mu + \nu = \omega.
 \end{aligned}$$

Now define

$$\begin{aligned}
 \Phi_i: \Lambda^{p-1} \mathcal{O}^{n-1} &\rightarrow \Lambda^p \mathcal{O}^n & \text{by } \Phi_i &= d \circ J, \\
 \Psi_i: \Lambda^p \mathcal{O}^n &\rightarrow \Lambda^p \mathcal{O}^{n-1} & \text{by } \Psi_i &= h \circ d, \\
 \phi_i: \Lambda^p \mathcal{O}^{n-1} &\rightarrow \mathcal{L}_{n-p-2} & \text{by } \phi_i &= d \circ J.
 \end{aligned}$$

There is a list of facts to check, all following easily from 1), 2), and 3); they are:  $\Phi_i$  is one to one,  $\Psi_i$  is onto,  $\text{Ker } \Psi_i = \text{Im } \Phi_i$ ,  $d \circ \phi_i = 0$ ,  $\phi_i$  is one to one and onto and  $\phi_i \circ \Psi_i = d$ .

Moreover  $J: \Lambda^0 \mathcal{O}^{n-1} \rightarrow \Lambda^0 \mathcal{O}^n$  is simply the identity so that  $d: \Lambda^0 \mathcal{O}^n \rightarrow \mathcal{L}_{n-2}$  is an isomorphism on the whole of  $M$ .

The lemma is therefore proved.  $\square$

To proceed towards the proof of theorem 1 we must consider distinguished elements in  $H^{s+1}(M - \{x\}, \mathcal{L}_s)$ , the *test classes* introduced in [E-VS]; for this purpose we take the long exact cohomology sequence associated to  $\xi_s$

$$\dots \rightarrow H^s(M - \{x\}, \mathcal{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(M - \{x\}, \mathcal{L}_s) \rightarrow \dots$$

and consider the elements  $\alpha_s(x) \in H^{s+1}(M - \{x\}, \mathcal{L}_s)$  given inductively as follows:  $\alpha_0(x) = \delta(F_1 \wedge F_2 \wedge \dots \wedge F_n)$ , where  $F_1 \wedge F_2 \wedge \dots \wedge F_n$  is in

$$H^0(M - \{x\}, \mathcal{L}_{-1}) = H^0(M - \{x\}, \mathcal{K}^n(\mathbf{f}, \mathcal{O}))$$

$$\text{and } \alpha_s(x) = \delta(\alpha_{s-1}(x)) \text{ if } s \geq 1.$$

*Proof of theorem 1.* To say that  $D$  has  $C^2$  boundary means that, given any point  $x \in \partial D$  we can find a  $C^2$  defining function  $\varphi: U \rightarrow \mathbf{R}$ , where  $U$  is an open neighbourhood of  $x$ , such that  $D \cap U = \{y \in U \text{ s.t. } \varphi(y) < 0\}$  and  $d\varphi(x) \neq 0$ ; in these conditions the number  $n(x)$  of negative eigenvalues of the Levi form  $\mathcal{L}(\varphi)(x)$  depends only on  $D$  and  $x$  and not on the choice of  $\varphi$ ; if  $D$  is not  $q$ -pseudoconvex there is a point  $x \in \partial D$  such that  $n(x) \geq q + 1$ , but then, with a slight modification of the defining function  $\varphi$  [E-VS] 3.6 corollary 1, we can manage to obtain at least  $q + 2$  negative eigenvalues for the complex Hessian  $\mathcal{H}(\varphi)(x)$ ; following the argument of [A-G] proposition 12 p. 222, we can find a neighbourhood  $Q$  of  $x$  (as small as we want) such that

a)  $H^p(D \cap Q, \mathcal{O}) = 0$  for  $p = 1, 2, \dots, q$  and

b) the restriction  $H^0(Q, \mathcal{O}) \rightarrow H^0(D \cap Q, \mathcal{O})$  is an isomorphism.

If we take global holomorphic functions  $f_1, f_2, \dots, f_n$  defining  $x$  and construct the Koszul complex  $\mathcal{K}(\mathbf{f}, \mathcal{O})$  we find that  $H^{q+1}(D, \mathcal{L}_q) = 0$  because  $\mathcal{L}_q$  is locally free on  $D$ ; therefore  $\alpha_q(x)|_D \in H^{q+1}(D, \mathcal{L}_q)$  vanishes and, by further restricting,  $\alpha_q(x)|_{D \cap Q}$  vanishes too; using the long exact sequence of cohomology

$$\dots \rightarrow H^s(D \cap Q, \mathcal{K}^{n-s-1}(\mathbf{f}, \mathcal{O})) \rightarrow H^s(D \cap Q, \mathcal{L}_{s-1}) \xrightarrow{\delta} H^{s+1}(D \cap Q, \mathcal{L}_s) \rightarrow$$

associated to  $\xi_s$  for  $s = 1, 2, \dots, q$  together with a) and the fact that  $\mathcal{K}^{n-s-1}(\mathbf{f}, \mathcal{O})$  is a free sheaf we deduce that

$$0 = \alpha_q(x)|_{D \cap Q} = \alpha_{q-1}(x)|_{D \cap Q} = \dots = \alpha_0(x)|_{D \cap Q};$$

but then, taking  $s = 0$ ,  $\alpha_0(x)|_{D \cap Q} = 0$  means that the equation  $\sum_1^n f_i g_i = 1$  has a solution  $(g_1, g_2, \dots, g_n) \in H^0(D \cap Q, \mathcal{O})^n$ . This would imply that at least one of the  $g_i$ 's does not extend to  $Q$ , contradicting b); therefore our original assumption that  $D$  is not  $q$ -pseudoconvex must be wrong and the theorem is proved.  $\square$

This proof follows, with minor modifications, that of Proposition 2.1 of [E-VS], it is reported here mainly for reasons of clarity. It should also be remarked that if  $H^p(D, \mathcal{O}) = 0$  for  $p > q$  then  $H^{q+1}(D, \mathcal{L}_q) = 0$  ([E-VS] Lemma 3.2, see introduction).

The theorem holds for  $q = 0$  too, but in this case we can avoid the hypothesis on the boundary and also the smoothness of the ambient space  $M$  if we impose some mild conditions: more precisely we have the following

**THEOREM 2.** *Let  $M$  be a strongly holomorphically separated analytic space and  $D$  an open subset of  $M$ ; suppose that either  $M$  is Stein or that  $D$  is relatively compact in  $M$ . Then the following conditions are equivalent:*

- 1)  $D$  is Stein,
- 2)  $H^1(D, \mathcal{L}) = 0$  for every locally free sheaf  $\mathcal{L}$  on  $D$ .

*Proof.* 1)  $\Rightarrow$  2) by Cartan's Theorem B.

2)  $\Rightarrow$  1) We must show that  $D$  is holomorphically convex, and this will be done by proving that, given any discrete sequence  $\{x_n\}$  in  $D$  we can find a holomorphic function  $g$  on  $D$  such that  $\lim_{n \rightarrow \infty} |g(x_n)| = \infty$ .

In either of our hypotheses we can suppose that  $\{x_n\}$  converges to a point  $x \in \partial D$ ; construct the Koszul complex  $\mathcal{K}(\mathbf{f}, \mathcal{O})$  starting from this point. Since  $\mathcal{L}_0$  is locally free (of rank  $n-1$  on  $M-x$  and on  $D$ ) we get  $H^1(D, \mathcal{L}_0) = 0$  so that  $\alpha_0(x)|_D = 0$ ; as before this means that the equation  $\sum_1^n f_i g_i = 1$  has a solution  $(g_1, \dots, g_n)$  in  $H^0(D, \mathcal{O})^n$ . For one at least of the  $g_i$ 's we must have  $\lim_{n \rightarrow \infty} |g_i(x_n)| = \infty$ .  $\square$

As in the remark above if  $H^p(D, \mathcal{O}) = 0$  for  $p = 1, 2, \dots, n-1$  then  $H^1(D, \mathcal{L}_0) = 0$ , so that we obtain Coen's result [C] under slightly different hypotheses.

The proofs of theorems 1 and 2 should have persuaded the reader that a deeper investigation of these powerful sheaves  $\mathcal{L}_q$  is worth a little effort. So far we have discovered that  $\mathcal{L}_q$  is a locally free sheaf of rank  $\binom{n-1}{n-q-2}$  on  $M - \{x\}$ . Let us call  $E_q$  the corresponding holomorphic vector bundle;  $E_q$  is a subbundle of the trivial bundle  $(M - \{x\}) \times \mathbf{C}^N$ ,  $N = \binom{n}{n-q-1}$ .

We shall now discuss the topological properties of these bundles  $E_q$ ; in the literature trivial and stably trivial bundles are considered [A], [L].

**Definition.** Let  $X$  be a topological space (respectively an analytic space) and  $E$  a topological (analytic) vector bundle on  $X$  with complex fibre; we say that  $E$  is *stably trivial* in the category  $\mathfrak{Top}(X)$  of topological vector bundles on  $X$  (respectively in the category  $\mathfrak{An}(X)$  of analytic vector bundles on  $X$ ) or, more quickly, that  $E$  is *topologically stably trivial* (*analytically stably trivial*) if there

exists a trivial vector bundle  $F$  in  $\mathfrak{Top}(X)$  (resp. in  $\mathfrak{Un}(X)$ ) s.t.  $E \oplus F$  is trivial in  $\mathfrak{Top}(X)$  (in  $\mathfrak{Un}(X)$ ).

From what follows it will be apparent that from our point of view it is irrelevant to make any difference between the categories of topological and  $C^\infty$  vector bundles (and those in between), provided the mathematical background allows the definitions. This should be kept in mind while reading the rest of this paper.

**PROPOSITION.** *The bundles  $E_q$  are stably trivial in  $\mathfrak{Top}(M - \{x\})$ .*

*Proof.* Call  $\mathcal{T}_q$  the sheaf of germs of continuous sections of  $E_q$  and  $\mathcal{C}$  the sheaf of germs of continuous  $\mathbf{C}$ -valued functions on  $M$ .

First of all we notice that if  $X$  is any open subset of  $M - \{x\}$  we have  $H^p(X, \mathcal{T}_q) = 0$  for all  $p > 0$  and  $q$ , because  $\mathcal{T}_q$  is a sheaf of  $\mathcal{C}$ -modules and thus fine.

Therefore the sequences

$$0 \rightarrow \Gamma(X, \mathcal{T}_q) \rightarrow \Gamma(X, \Lambda^{n-q-1}\mathcal{C}^n) \xrightarrow{d} \Gamma(X, \mathcal{T}_{q-1}) \rightarrow 0$$

are all exact. Since  $\mathcal{T}_{n-2} \simeq \mathcal{C}$  our assertion will follow if we can split these exact sequences.

To this purpose we observe that, given any element  $\sigma \in \Lambda^1\mathcal{C}^n$ , and  $\omega \in \Lambda^p\mathcal{C}^n$  we can define the *contraction of  $\omega$  along  $\sigma$* , denoted by  $\sigma \vee \omega \in \Lambda^{p-1}\mathcal{C}^n$ , by imposing  $\mathcal{C}$ -linearity to the contraction given on generators as follows:

$$F_i \vee (F_{i_1} \wedge F_{i_2} \wedge \dots \wedge F_{i_p}) = \begin{cases} 0 & \text{if } i \notin (i_1, i_2, \dots, i_p) \\ (-1)^k F_{i_1} \wedge \dots \wedge F_{i_{k-1}} \wedge F_{i_{k+1}} \wedge \dots \wedge F_{i_p} & \text{if } i = i_k. \end{cases}$$

It is easily seen that if  $\omega \in \Lambda^p\mathcal{C}^n$ ,  $v \in \Lambda^r\mathcal{C}^n$  and  $\sigma \in \Lambda^1\mathcal{C}^n$  we have the relation

$$\sigma \vee (\omega \wedge v) = (\sigma \vee \omega) \wedge v + (-1)^p \omega \wedge (\sigma \vee v).$$

Let us consider the distinguished element

$$\sigma = \sum_1^n \frac{\bar{f}_i}{|f|^2} F_i \in \Gamma(M - \{x\}, \Lambda^1\mathcal{C}^n)$$

where  $\bar{f}_i$  is the complex conjugate of  $f_i$  and  $|f|^2 = \sum_1^n f_i \bar{f}_i$ .

Since  $\sigma \vee (\sum_1^n f_i F_i) = 1$  it follows from the relation above that if we define  $\tilde{\sigma}: \Lambda^p\mathcal{C}^n \rightarrow \Lambda^{p-1}\mathcal{C}^n$  by  $\tilde{\sigma}(\omega) = \sigma \vee \omega$ , we get

$$d \circ \tilde{\sigma} + \tilde{\sigma} \circ d = id_{\Lambda^p\mathcal{C}^n}.$$

Therefore  $\tilde{\sigma}|_{T_{q-1}}: \Gamma(X, \mathcal{T}_{q-1}) \rightarrow \Gamma(X, \Lambda^{n-q-1}\mathcal{C}^n)$  gives the desired topological splitting.  $\square$

It should be remarked that no such *analytic* splitting is available in general: for suppose that  $M$  is an analytic manifold (or a Cohen-Macaulay space, see remark later) then, if  $n \geq 3$ ,  $H^1(U - \{x\}, \mathcal{O}) = 0$  for any Stein neighbourhood  $U$  of  $x$ , and if  $E_o$  was stably trivial in  $\mathfrak{A}n(M - \{x\})$  we would get immediately  $H^1(U - \{x\}, \mathcal{L}_0) = 0$ , which, as in the proof of theorem 2 would contradict the Riemann Removable Singularities Theorem.

In the light of the above remarks a more efficient though more technical version of our results can be given; first of all, if  $X$  is an analytic space of dimension  $n$ , let us call  $\Xi_q(X)$  the class of locally free analytic sheaves on  $X$  which are associated to a topologically stably trivial analytic subbundle of rank  $\binom{n-1}{n-q-2}$  of the product bundle  $X \times \mathbb{C}^N$ ,  $N = \binom{n}{n-q-1}$ ; since our sheaves  $\mathcal{L}_q$  are in  $\Xi_q(D)$  the above results can be restated as

**THEOREM 1'.** *Let  $M$  be a strongly holomorphically separated analytic manifold of dimension  $n$ ,  $D$  an open subset with  $C^2$  boundary of  $M$  and suppose that  $H^{q+1}(D, \mathcal{L}) = 0$  for every locally free sheaf in  $\Xi_q(X)$ , then  $D$  is  $q$ -pseudoconvex. If  $M$  is Stein the converse is also true.*  $\square$

Since we know [G] that on a Stein space every topologically trivial analytic vector bundle is also analytically trivial, our second theorem can be improved as follows (this is also an improvement of Leiterer's theorem [L]):

**THEOREM 2'.** *Let  $M$  be a strongly holomorphically separated analytic space of dimension  $n$ ,  $D$  an open subset of  $M$  and suppose that  $M$  is Stein or that  $D$  is relatively compact in  $M$ . The following conditions are equivalent:*

- 1)  $D$  is Stein,
- 2)  $H^1(D, \mathcal{O}) = 0$  and every analytic vector bundle on  $D$  which is trivial in  $\text{Top}(D)$  is also trivial in  $\mathfrak{A}n(D)$ ,
- 3)  $H^1(D, \mathcal{O}) = 0$  and every sheaf in  $\Xi_0(D)$  is associated to a vector bundle which is stably trivial in  $\mathfrak{A}n(D)$ ,
- 4)  $H^1(D, \mathcal{L}) = 0$  for every sheaf  $\mathcal{L}$  in  $\Xi_0(D)$ ,
- 5) However we choose functions  $f_1, f_2, \dots, f_n$  in  $\Gamma(M, \mathcal{O})$  with no common zero on  $D$  the equation  $\sum_1^n f_i g_i = 1$  has a solution  $(g_1, g_2, \dots, g_n)$  in  $\Gamma(D, \mathcal{O})^n$ .

*Proof.* Scheme:  $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$ ;  $1) \Rightarrow 5) \Rightarrow 1)$ .  $1) \Rightarrow 2)$  by the above mentioned theorem of Grauert [G] and Cartan's Theorem B;  $2) \Rightarrow 3)$ ,  $3) \Rightarrow 4)$  and  $1) \Rightarrow 5)$  are trivial;  $4) \Rightarrow 1)$  and  $5) \Rightarrow 1)$  as in theorem 2.  $\square$

Comparing this theorem with Leiterer's one we see that there is no need to embed  $M$  in  $\mathbb{C}^{2n+1}$  and use Bott's periodicity theorem.

As a final remark we observe that if  $M$  is a manifold or, more in general a Cohen-Macaulay space, the sheaves  $\mathcal{L}_q$ , though locally free on  $M - \{x\}$  have no chance to be locally free on  $M$  for  $s = 0, 1, \dots, n - 3$ ; in fact we have  $\text{codh}_x \mathcal{L}_q = n - q - 2$ , where  $\text{codh}_x$  indicates the homological codimension at  $x$ .

This can be shown as follows (without too many details since it is of a rather marginal importance for our purposes).

Claim: the sequence of sheaves

$$\xi \quad 0 \rightarrow \mathcal{K}^0(\mathbf{f}, \mathcal{O}) \xrightarrow{d} \mathcal{K}^1(\mathbf{f}, \mathcal{O}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{K}^n(\mathbf{f}, \mathcal{O}) \xrightarrow{\pi} \mathcal{O}/(\mathbf{f}) \rightarrow 0$$

is exact on  $M$ , where  $\pi$  denotes the projection from  $\mathcal{K}^n(\mathbf{f}, \mathcal{O}) \simeq \mathcal{O}$  to the quotient sheaf  $\mathcal{O}/(\mathbf{f}) = \mathcal{O}/(f_1, f_2, \dots, f_n)$ ; if we prove this we are done, because we know [A-G] prop. 4, page 200, that the last sheaf of  $\xi$  has homological codimension surely  $\geq n$ , and so  $\xi$  is a free resolution of  $\mathcal{O}/(\mathbf{f})$  of minimal length.

The sequence  $\xi$  is surely exact on  $M - \{x\}$  (lemma), so let  $U$  be a Stein neighbourhood of  $X$  and take an acyclic resolution  $(\mathcal{F}^\bullet, \delta)$  of  $\mathcal{O}$  chosen in such a way that the double complex

$$K'' = \{K_{p,r} = \Gamma(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{F}^r)); d, \delta\} \text{ is anticommutative.}$$

It is rather easy to see that the rows of  $K''$  are exact, so that we obtain a degenerate spectral sequence

$$E_{p,r}^2 \Rightarrow 0$$

where

$$E_{p,r}^2 = \frac{\text{Ker } d: H^r(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{O})) \rightarrow H^r(U - \{x\}, \mathcal{K}^{p+1}(\mathbf{f}, \mathcal{O}))}{\text{Im } d: H^r(U - \{x\}, \mathcal{K}^{p-1}(\mathbf{f}, \mathcal{O})) \rightarrow H^r(U - \{x\}, \mathcal{K}^p(\mathbf{f}, \mathcal{O}))}$$

Since a)  $H^r(U - \{x\}, \mathcal{O}) = 0$  for  $r = 1, 2, \dots, n - 2$  we obtain  $E_{p,0}^2 = 0$  for  $p = 0, 1, \dots, n - 1$ : but we can replace  $U - \{x\}$  with  $U$  in the expression of  $E_{p,r}^2$  by the Riemann Removable Singularities Theorem (which, together with a) characterizes Cohen-Macaulay spaces ([S-T] theorem 1.14).

So our sequence  $\xi$  is exact except, perhaps, at the last two places, where it is exact for trivial reasons.

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