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Anhang: Appendix 2 SOME RELATIVES OF THE GAMMA FUNCTION

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Here the factor m^{1-s}/τ is never zero or infinite, while $A_s \pm B_s$ is zero or infinite only at certain integer values, as indicated in the table above.

The proof of Lemma 13 can now easily be completed as follows. If $s \leqslant 0$ is an integer, then $L(1-s, \chi) \neq 0$, so it follows that $L(s, \bar{\chi})$ equals zero if and only if $A_s \pm B_s$ is zero, as indicated in the table. □

APPENDIX 2

SOME RELATIVES OF THE GAMMA FUNCTION

This appendix will describe certain functions $\gamma_1(x), \gamma_2(x), \ldots$ which satisfy a modified form of the Kubert identities, with a polynomial correction term. (See (22) below.) They are defined as partial derivatives of the Hurwitz function by the formula

$$\gamma_{1-t}(x) = \partial\zeta_t(x)/\partial t\,. \tag{18}$$

We will show that γ_1 is related to the classical gamma function via Lerch's identity

$$\gamma_1(x) = \log\left(\Gamma(x)/\sqrt{2\pi}\right). \tag{19}$$

(Compare [27, p. 60].) As a bonus, we will give a self-contained exposition of the basic properties of the gamma function, based on formulas (18) and (19).

Let us begin with equation (18), which defines $\gamma_s(x)$ as an analytic function of both variables for all $s \neq 0$ and all $x > 0$. Recall that the Hurwitz function $\zeta_t(x) = x^{-t} + (x+1)^{-t} + \ldots$ (analytically extended in t for $t \neq 1$) satisfies

$$\zeta_t(x+1) = \zeta_t(x) - x^{-t}\,.$$

Differentiating with respect to t, and then substituting $t = 1 - s$, we obtain

$$\gamma_s(x+1) = \gamma_s(x) + x^{s-1}\log x\,. \tag{20}$$

In particular,

$$\gamma_1(x+1) = \gamma_1(x) + \log x\,.$$

Note that

$$\zeta_t'(x) = -t\zeta_{t+1}(x)$$

hence

$$\zeta_t''(x) = t(t+1)\zeta_{t+2}(x)\,,$$

where the prime stands for the derivative with respect to x. By analytic continuation, this last equation holds also at $t = 0$. Differentiating with respect to t at $t = 0$, we obtain

$$\gamma_1''(x) = \zeta_2(x)\,. \tag{21}$$

In particular, it follows that $\gamma_1''(x) > 0$ *for all* $x > 0$.

Let us define the gamma function as follows. (Compare Artin [1].)

LEMMA 15 (Bohr and Mollerup). *There is one and only one twice continuously differentiable function* $\Gamma(x) > 0$ *for* $x > 0$ *which satisfies*

$$\Gamma(x+1) = x\Gamma(x)\,, \qquad \Gamma(1) = 1\,, \qquad \text{and} \qquad (\log \Gamma(x))'' \geqslant 0\,.$$

Proof. Evidently it suffices to show that there is one and, up to an additive constant, only one C^2-function

$$f(x) = \log \Gamma(x) + c$$

for $x > 0$ which satisfies the two conditions

$$f(x+1) = f(x) + \log x$$

and

$$f''(x) \geqslant 0\,.$$

Existence is clear, since the equation $\gamma_1(x)$ satisfies both of these conditions. To prove uniqueness, let us differentiate twice to obtain

$$f''(x+1) = f''(x) - 1/x^2\,,$$

hence

$$f''(x+n+1) = f''(x) - x^{-2} - (x+1)^{-2} - \ldots - (x+n)^{-2} \geqslant 0\,.$$

Taking the limit as $n \to \infty$, it follows that

$$f''(x) \geqslant \zeta_2(x)\,.$$

On the other hand, note that the difference $f(x) - \gamma_1(x)$ is periodic, of period 1. Hence its second derivative $f''(x) - \zeta_2(x)$ is periodic, and has average $\int_0^1 (f''(x) - \zeta_2(x))dx$ equal to zero. Clearly it follows that $f''(x) = \zeta_2(x)$ everywhere. Integrating twice, we see that

$$f(x) = \gamma_1(x) + ax + b\,.$$

Subtracting the corresponding equation for $f(x+1)$, we see that $a = 0$, which completes the proof. $\square$

This argument shows that

$$\gamma_1(x) = \log(\Gamma(x)/C)$$

for some constant C, whose precise value will be computed later.

Remark: The customary definition of the gamma function is the expression

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

which was used in §2 and Appendix 1. Here is an outline proof that this expression does indeed satisfy the conditions of Lemma 15. Integration by parts shows that $\Gamma(x+1) = x\Gamma(x)$. Note that a twice differentiable positive function satisfies $(\log f(x))'' \geqslant 0$ if and only if the matrix

$$\begin{bmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{bmatrix}$$

is positive semi-definite, for all x. But the collection of all 2×2 positive semi-definite matrices forms a convex cone. It follows that the sum $f(x) + g(x)$ of any two functions which satisfy this condition will also satisfy it. Similarly the integral

$$\begin{bmatrix} \Gamma(x) & \Gamma'(x) \\ \Gamma'(x) & \Gamma''(x) \end{bmatrix} = \int_0^\infty \begin{bmatrix} 1 & \log t \\ \log t & (\log t)^2 \end{bmatrix} e^{-t}t^{x-1}dt$$

is a positive semi-definite matrix. Hence $(\log \Gamma(x))'' \geqslant 0$ as required. □

Now consider the Kubert identity

$$m^t\zeta_t(x) = \sum_0^{m-1} \zeta_t((x+k)/m).$$

If we differentiate both sides with respect to t, then substitute $t = 1 - s$ and $\zeta_t = -\beta_s/s$, we obtain

$$\gamma_s(x) = (\log m)\beta_s(x)/s + m^{s-1}\sum_0^m \gamma_s((x+k)/m). \tag{22}$$

Thus γ_s *satisfies* the Kubert identity $(*_s)$, *except for a correction term involving the Bernoulli polynomial* $\beta_s(x)$, *for* $s = 1, 2, 3, \ldots$.

If we work modulo the logarithms of positive rational numbers, then the function

$$\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/\mathbf{Q} \log \mathbf{Q}^+$$

induced by γ_s actually satisfies $(*_s)$. *It seems natural to conjecture that this is a universal Kubert function on* $\mathbf{Q}/\mathbf{Z}$ *for integers* $s \geqslant 1$.

For $s = 1$, the "even" part of this conjecture can easily be proved using Bass' theorem, together with the classical identity

$$\gamma_1(x) + \gamma_1(1-x) + \log(2 \sin \pi x) = 0$$

for $0 < x < 1$, which is proved below, and the fact that $\gamma_1(1) = \log(1/\sqrt{2\pi})$ where π is transcendental. For the odd part of γ_1, Rohrlich has conjectured universality even if we work modulo the logarithms of *all* algebraic numbers. See [17, p. 66].

In the case $s = 1$, formula (22) takes the form

$$\gamma_1(x) = (\log m)\left(x - \frac{1}{2}\right) + \textstyle\sum_0^{m-1} \gamma_1((x+k)/m). \tag{23}$$

Hence the derivative $\gamma_1'(x) = \Gamma'(x)/\Gamma(x)$ satisfies

$$\gamma_1'(x) = \log m + m^{-1} \textstyle\sum_0^{m-1} \gamma_1'((x+k)/m). \tag{24}$$

Note that $\gamma_1'(x+1) = \gamma_1'(x) + 1/x \equiv \gamma_1'(x) \bmod \mathbf{Q}$, if x is positive and rational. *We may conjecture that* γ_1' *induces a universal function* $\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/(\mathbf{Q} + \mathbf{Q} \log \mathbf{Q}^+)$ *satisfying* $(*_0)$. (It can be shown that $\gamma_1'(1)$ is equal to the negative of Euler's constant. Thus even at $x = 1$ the number theoretic properties of $\gamma_1'(x)$ are not known.)

As a typical application of (23), taking $x = 1$ we obtain the equation

$$\gamma_1(1/m) + \gamma_1(2/m) + \ldots + \gamma_1((m-1)/m) = \log(1/\sqrt{m}).$$

In particular, $\gamma_1(1/2) = \log(1/\sqrt{2})$.

As a further application of (23), we will prove the classical formula

$$\gamma_1(x) + \gamma_1(1-x) + \log(2 \sin \pi x) = 0 \tag{25}$$

for $0 < x < 1$. If we add (23) to the corresponding formula for $\gamma_1(1-x)$, then the correction terms cancel out. Hence the sum $\gamma_1(x) + \gamma_1(1-x)$ satisfies the Kubert identities $(*_1)$ in their original form. By Theorem 1, this implies that

$$\gamma_1(x) + \gamma_1(1-x) = c \log(2 \sin \pi x)$$

for some constant c. One way to evaluate c would be to differentiate twice:

$$\zeta_2(x) + \zeta_2(1-x) = -c\pi^2/\sin^2 \pi x,$$

and to note that both $\zeta_2(x)$ and $\pi^2/\sin^2 \pi x$ are asymptotic to $1/x^2$ as $x \to 0$. (Compare Appendix 1.) Another would be to substitute $x = 1/2$, noting that $\gamma_1(1/2) = -\dfrac{1}{2} \log 2$ while $\log(2 \sin \pi/2) = \log 2$. Using either method, one finds that $c = -1$, proving equation (25). $\square$

Next let us prove Lerch's identity (19). We showed during the proof of Lemma 15 that $\gamma_1(x) = \log(\Gamma(x)/C)$ for some constant $C > 0$. Exponentiating (25), we obtain

$$\frac{\Gamma(x)}{C}\frac{\Gamma(1-x)}{C} 2 \sin \pi x = 1 .$$

Since

$$\Gamma(x) \sim x^{-1}, \qquad \Gamma(1-x) \sim 1, \qquad \text{and} \qquad 2 \sin \pi x \sim 2\pi x$$

as $x \to 0$, it follows that $C = \sqrt{2\pi}$, as required. □

This argument also proves the classical *Euler functional equation*

$$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x .$$

Taking $x = 1/2$, it proves that $\Gamma(1/2) = \sqrt{\pi}$.

Similarly, exponentiating (23), we obtain the classical *Gauss multiplication formula*

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = m^{x-1/2} \prod_0^{m-1} \frac{\Gamma((x+k)/m)}{\sqrt{2\pi}} .$$

As an example, taking $x = 1$ and $m = 2$, we obtain another proof that $\Gamma(1/2) = \sqrt{\pi}$.

Note that each γ_{s+1} is essentially just an indefinite integral of γ_s, up to a constant factor and a polynomial summand. More precisely, differentiating the equation

$$\zeta_t'(x) = -t\zeta_{t+1}(x)$$

with respect to t and setting $s = -t$, we find that

$$(26) \qquad \gamma_{s+1}'(x) = \partial\gamma_{s+1}(x)/\partial x = s\gamma_s(x) + \beta_s(x)/s .$$

The function $\exp(\gamma_s(x))$ can be thought of as a kind of higher order gamma function, satisfying

$$\exp(\gamma_s(n+1) - \gamma_s(1)) = 1^{1^{s-1}} 2^{2^{s-1}} \dots n^{n^{s-1}} .$$

(Compare Shintani [24].)

As a final remark, let us apply these methods to derive the Stirling asymptotic series for $\gamma_1(x)$ as $x \to \infty$. Using (26), together with (3) and (20), we have

$$\int_x^{x+1} \gamma_1(u)du = x \log x - x .$$

As in the discussion of Bernoulli polynomials in §2, the left side of this equation can be expanded as a Taylor series

$$\frac{e^D - I}{D} \gamma_1(x) = \sum_0^{\infty} D^n\gamma_1(x)/(n+1)! ,$$

which converges whenever $\gamma_1(x)$ is analytic throughout a unit disk centered at x, or in other words whenever $x > 1$. Here D stands for d/dx. Recall from §2 that the inverse operator is given formally by

$$\frac{D}{e^D - I} = \sum_0^\infty b_n D^n/n! \,.$$

Hence, applying this inverse operator to both sides of the equation

$$\frac{e^D - I}{D}\gamma_1(x) = x \log x - x \,,$$

we might hope that

$$\gamma_1(x) \stackrel{?}{=} \frac{D}{e^D - I}(x \log x - x) = \sum_0^\infty b_n D^n(x \log x - x)/n! \,.$$

Unfortunately, this series does not converge. However, if we truncate, setting

$$s_N(x) = \sum_0^N b_n D^n(x \log x - x)/n!$$

for some integer $N \geqslant 1$, then we will prove that

$$\gamma_1(x) = s_N(x) + O(x^{-N})$$

as $x \to \infty$. This is the required asymptotic series. More explicitly, we can write it as

$$\gamma_1(x) = (x \log x - x) - \frac{1}{2}\log x + \sum_2^N \frac{b_n x^{1-n}}{n(n-1)} + O(x^{-N}) \,. \tag{27}$$

(For a more precise description of the error term, see [1, p. 31]. Using (19) this yields the corresponding asymptotic formula for $\Gamma(x)$.)

To prove this formula, substitute the identity

$$x \log x - x = \sum_0^\infty \frac{D^m}{(m+1)!}\gamma_1(x)$$

in the definition of $s_N(x)$ to obtain a double series

$$s_N(x) = \sum_{n=0}^N \sum_{m=0}^\infty \frac{b_n D^n}{n!}\frac{D^m}{(m+1)!}\gamma_1(x) \,,$$

which converges absolutely whenever $x > 1$. If we collect terms involving the same total power of D, then evidently all the terms involving $D^1, D^2, \ldots, D^N$ must cancel. Since

$$D^n\gamma_1(x) = \pm(n-1)!\zeta_n(x)$$

for $n \geqslant 2$, it follows that the resulting series has the form

$$s_N(x) = \gamma_1(x) + \sum_{N+1}^{\infty} a_n \zeta_n(x)$$

for suitable constants a_n. Setting

$$E(x) = \sum_{N+1}^{\infty} a_n x^{-n},$$

we can write the error term as

$$s_N(x) - \gamma_1(x) = E(x) + E(x+1) + \dots .$$

Note that all of these series converge absolutely for $x > 1$. Evidently

$$E(x) = O(x^{-N-1})$$

as $x \to \infty$, for any fixed N, so

$$s_N(x) - \gamma_1(x) = O(x^{-N})$$

as required. □

This argument yields similar asymptotic series for related functions such as $\zeta_s(x)$, $\gamma_s(x)$, and $\gamma_s'(x)$. Such estimates work also for complex values of x, as long as x stays well away from the negative real axis.

Appendix 3

Volume and the Dehn invariant in hyperbolic 3-space

We will describe some constructions in hyperbolic space involving the dilogarithm function $\mathscr{L}_2(z)$ and its Kubert identity (7). Further details may be found in the paper "Scissors Congruences, II" by J. L. Dupont and C.-H. Sah (*J. Pure Appl. Algebra* 25 (1982), 159-195).

Using the upper half-space model for hyperbolic 3-space, consider a totally asymptotic 3-simplex Δ. In other words, we assume that the vertices a, b, c, d of Δ all lie on the 2-sphere of points at infinity, which we identify with the extended complex plane $\mathbf{C} \cup \infty$. Then Δ is determined up to orientation preserving isometry by the cross ratio

$$z = (a, b; c, d) = (c-a)(d-b)/(c-b)(d-a) .$$

[The semicolon is inserted in our cross ratio symbol as a remainder of its symmetry properties, which are similar to those of the four index symbol R_{hijk} in Riemannian geometry.] In particular, the volume of Δ can be expressed as a function of the cross ratio z.