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The proof that these elements generate over \mathbf{Z}_q proceeds as above for $p \neq q$, and proceeds as in the proof of Lemma 9 when $p = q$. Details are easily supplied. \square

§6. ON \mathbf{Q} -LINEAR RELATIONS

S. Chowla and P. Chowla have suggested the following conjecture in a private communication to the author. Let a_1, a_2, \dots be a sequence of integers which is periodic, $a_n = a_{n+p}$, for some prime p . Then

$$(11) \quad \sum_1^\infty a_n/n^2 \neq 0$$

except in the special case

$$a_1 = \dots = a_{p-1} = a_p/(1-p^2).$$

If we use the Hurwitz function

$$\zeta_2(k/p) = p^2(k^{-2} + (k+p)^{-2} + \dots),$$

then the inequality (11) can be written as

$$\sum_1^p a_k \zeta_2(k/p) \neq 0;$$

and the exceptional case corresponds to the Kubert relation

$$\zeta_2(1) = p^{-2} \sum_1^p \zeta_2(k/p).$$

Thus the Chowlas' conjecture is true if and only if the real numbers

$$\zeta_2(1/p), \dots, \zeta_2((p-1)/p)$$

are linearly independent over the rational numbers. More generally, for any $m \geq 2$ one might conjecture that the $\varphi(m)$ real numbers $\zeta_2(k/m)$, where k varies over all relatively prime integers between 1 and $m-1$, are \mathbf{Q} -linearly independent. Using Lemma 9, a completely equivalent statement would be the following.

Conjecture: Every \mathbf{Q} -linear relation between the real numbers $\zeta_2(x)$, where x is rational with $0 < x \leq 1$ is a consequence of the Kubert relations $(*_{-1})$.

In fact, since $\zeta_2(x+1) \equiv \zeta_2(x) \pmod{\mathbf{Q}}$ for positive rational x , it might be more natural to sharpen this conjecture by taking the values of ζ_2 modulo \mathbf{Q} . In other words, it is conjectured that the mapping

$$\mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Q}$$

induced by ζ_2 is a "universal" function satisfying $(*_{-1})$. It follows easily from Theorem 3 below that the corresponding conjecture for the even part,

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x,$$

of ζ_2 is indeed true; but the odd part of ζ_2 seems difficult to work with.

One can make analogous and equally plausible conjectures for the Hurwitz functions ζ_3, ζ_4, \dots . In Appendix 2 we will describe analogous conjectures for certain functions closely related to the gamma function.

Bass [2], studying multiplicative relations between cyclotomic units, has proved the following result. Let

$$f_0(x) = \log |1 - e^{2\pi i x}| = \log(2 \sin \pi x)$$

for $0 < x < 1$. Note that $f_0(1-x) = f_0(x)$.

THEOREM OF BASS. *Every \mathbf{Q} -linear relation between the numbers $f_0(x)$ for rational $x \in (0, 1)$ is a consequence of the Kubert relations $(*_1)$, together with evenness.*

A proof will be indicated at the end of this section.

Note that this is the exceptional case in which Lemma 7 does not apply, so that $f_0(0)$ cannot be defined.

Bass' theorem is equivalent, using the results of §5, to the following classical statement. Fixing some integer $m \geq 3$, let $\xi = e^{2\pi i/m}$, and let V_m be the multiplicative group generated by the elements

$$1 - \xi, 1 - \xi^2, \dots, 1 - \xi^{m-1}$$

in the cyclotomic field $\mathbf{Q}[\xi]$. Elements of the intersection $V_m \cap \mathbf{Z}[\xi]^*$ are called *circular units* (or cyclotomic units).

COROLLARY. *This group $V_m \cap \mathbf{Z}[\xi]^*$ of circular units has finite index in the group $\mathbf{Z}[\xi]^*$ consisting of all units of the cyclotomic field.*

Compare Hilbert [8], as well as Sinnott [25].

Proof. Let $m = q_1 \dots q_n$ be the factorization of m into powers of distinct primes. By Lemmas 8 and 10, Bass' theorem is equivalent to the statement that the additive group generated by the elements

$$f_0(k/m) = \log |1 - \xi^k|$$

has rank $\varphi(m)/2 + n - 1$. Since each generator of V_m is equal to a real number multiplied by a root of unity, this is equivalent to the statement that V_m has rank $\varphi(m)/2 + n - 1$. However it is not difficult to check that V_m splits as the direct sum of the group of circular units and a free abelian group generated by the elements $1 - e^{2\pi i/q_j}$. Hence Bass' theorem is also equivalent to the statement that the group of circular units has rank $\varphi(m)/2 - 1$. According to the Dirichlet unit theorem, this implies that it has finite index in the group of all units of $\mathbf{Z}[\xi]$. \square

The author [21] has conjectured that the function $\mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{R}$ defined by

$$x \mapsto \Lambda(\pi x) = - \int_0^{\pi x} \log | 2 \sin \theta | d\theta$$

is a universal odd function satisfying $(*_2)$. This seems very difficult. However, W. Sinnott has pointed out to the author this the situation for the derivatives of $\log 2 \sin \theta$ is much easier to analyze.

Let $f_t(x)$ be the t -th derivative of $\log | 2 \sin \theta |$, evaluated at $\theta = \pi x$. For example $f_1(x) = \cot(\pi x)$, $f_2(x) = -\csc^2(\pi x)$. Note that $f_1(1-x) = (-1)^t f_t(x)$. The values at $x = 0$ are to be defined as in §4.

THEOREM 3. For each fixed $t = 1, 2, \dots$, the function

$$f_t : \mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{R}$$

is a universal even or odd function satisfying $(*_1-t)$.

That is every \mathbf{Q} -linear relation between the values $f_t(x)$ for x in \mathbf{Q}/\mathbf{Z} follows from $(*_1-t)$, together with evenness or oddness according as t is even or odd.

Fixing some integer $m \geq 3$, let $\xi = e^{2\pi i/m}$. If t is even, the proof will show that the values

$$f_t(1/m), \dots, f_t((m-1)/m)$$

span the real part of the cyclotomic field $\mathbf{Q}[\xi]$. Similarly, if t is odd, the values $if_t(k/m)$ span the totally imaginary subspace of $\mathbf{Q}[\xi]$. In either case, these values span a rational vector space of dimension $\varphi(m)/2$, as required by Lemma 8.

Compare Ewing [7] for an analogous discussion of the values of $\csc(\pi x)$ and its derivatives at rational x .

The proof will depend upon well known properties of Dirichlet L -functions. Fixing some positive integer m , let

$$\chi : (\mathbf{Z}/m\mathbf{Z})^\times \rightarrow \mathbf{C}^\times$$

be an arbitrary Dirichlet character modulo m . We allow the degenerate case $m = 1$ with the understanding that the only character modulo 1 is the constant function $\chi_0(k) = 1$. Recall that such a character is *primitive* (or has conductor generated by m) if it cannot be factored through the projection

$$(\mathbf{Z}/m\mathbf{Z})^\times \rightarrow (\mathbf{Z}/d\mathbf{Z})^\times$$

for any divisor $d < m$. As usual, we set $\chi(k) = 0$ if k is a non-unit modulo m .

The associated L -function is defined by

$$L(s, \chi) = \sum_1^\infty \chi(k)/k^s$$

for $Re(s) > 1$. In terms of the Hurwitz function

$$\zeta_s(k/m)/m^s = k^{-s} + (k+m)^{-s} + \dots$$

we can clearly write this as a finite sum

$$(12) \quad L(s, \chi) = \sum_1^m \chi(k) \zeta_s(k/m)/m^s .$$

It follows that $L(s, \chi)$ extends to a function which is holomorphic in s for all complex s , whenever $\chi \neq \chi_0$. For it is easy to check that the difference $\zeta_s(x) - (s-1)^{-1}$ is holomorphic in s ; and the $(s-1)^{-1}$ terms cancel whenever $\chi \neq \chi_0$.

On the other hand, for the trivial character χ_0 , evidently $L(s, \chi_0)$ is equal to the Riemann zeta function, with a pole at $s = 1$.

Now let us restrict to integer values of s .

LEMMA 13. For primitive $\chi \neq \chi_0$, and for integer values of s , the function $L(s, \chi)$ is zero if and only if $s \leq 0$ and $\chi(-1) = (-1)^s$.

For $s > 1$, the statement that $L(s, \chi) \neq 1$ is fairly easy to prove, while for $s = 1$ it is a basic result of Dirichlet. See for example [5] or [23]. For $s \leq 0$, this lemma is proved using the functional equation relating $L(s, \chi)$ and $L(1-s, \bar{\chi})$. (Compare [10].) Details of this last argument may be found in Appendix 1. □

In the case of the trivial character χ_0 , this lemma remains true except for anomalous behavior at $s = 0$ (where $\zeta(s)$ is non-zero) and $s = 1$ (where $\zeta(s)$ has a pole).

These Dirichlet L -functions can also be expressed as finite linear combinations of polylogarithms, via Fourier analysis, as follows. Let $\xi = e^{2\pi i/m}$.

LEMMA 14. If $\chi \neq \chi_0$ is primitive modulo m , then

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau ,$$

where

$$\tau = \tau(\chi) = \sum_1^m \chi(k) \xi^k$$

is a complex constant with absolute value \sqrt{m} .

In the case of the trivial character χ_0 , this lemma remains true provided that $l_s(1)$ is interpreted as in §4.

Proof of Lemma 14. Since both sides are holomorphic in s for all complex s , it will suffice to consider the case $Re(s) > 1$. First note that the "Fourier

transform" of the complex valued function χ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is equal to $\tau\bar{\chi}$; that is

$$(13) \quad \sum_{j \bmod m} \chi(j)\xi^{jk} = \tau\bar{\chi}(k).$$

If k is a unit modulo m , this follows from the equation $\chi(j) = \bar{\chi}(k)\chi(jk)$, while if k is a non-unit modulo m then, using the hypothesis that χ is primitive, it is not difficult to check that both sides of this equation are zero. Now dividing both sides by k^s and summing over all positive integers k , we obtain

$$\sum_{j \bmod m} \chi(j)\mathcal{L}_s(\xi^j) = \tau L(s, \bar{\chi}).$$

Since $\mathcal{L}_s(\xi^j) = l_s(j/m)$, this implies the required equation.

To compute $|\tau|$ combine (13) with the complex conjugate equation to obtain

$$\begin{aligned} m\chi(n) &= \sum_j \chi(j) \sum_k \xi^{k(j-n)} = \sum_k \xi^{-kn} \sum_j \chi(j)\xi^{kj} \\ &= \sum_k \xi^{-kn} \tau\bar{\chi}(k) = \tau\bar{\chi}(n); \end{aligned}$$

hence $m = \tau\bar{\tau}$ as asserted. □

Remark. Similar arguments prove that the Fourier transform of the Hurwitz function $\zeta_s(j/m)$ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is a multiple of $l_s(k/m)$. More generally, one can show that any function on $\mathbf{Z}/m\mathbf{Z}$ satisfies $(*_s)$ if and only if its Fourier transform satisfies $(*_{1-s})$.

Proof of Theorem 3. We will work with the polylogarithm function

$$\mathcal{L}_s(\xi^k) = l_s(k/m)$$

where $\xi = e^{2\pi i/m}$. If $s = 1 - t$ is a non-positive integer, recall from §2 that $\mathcal{L}_s(z)$ is a rational function with rational coefficients. Hence $l_s(k/m)$ takes values in the cyclotomic field $\mathbf{Q}[\xi]$.

The Galois group G of $\mathbf{Q}[\xi]$ over \mathbf{Q} can be identified with $(\mathbf{Z}/m\mathbf{Z})^\times$. Evidently the mapping

$$U_s(A_m) \rightarrow \mathbf{Q}[\xi]$$

induced by l_s is G -equivariant, in the sense that the automorphism $u(k/m) \mapsto u(gk/m)$ of $U_s(A_m)$ corresponds to the automorphism $f(\xi) \mapsto f(\xi^g)$ of $\mathbf{Q}[\xi]$ for every g in $G \cong (\mathbf{Z}/m\mathbf{Z})^\times$. Tensoring both sides with the complex numbers, each splits into a direct sum of 1-dimensional eigenspaces under the action of G . Hence, to compute the rank of this map, we need only decide how many eigenspaces are mapped non-trivially.

For each character $\chi \pmod m$, let $\chi' : (\mathbf{Z}/d\mathbf{Z})^* \rightarrow \mathbf{C}^*$ be the associated primitive character, where $d \mid m$ generates the conductor of χ . Evidently the sum

$$\sum_{k \pmod d} l_s(k/d) \otimes \chi'(k)$$

belongs to the $\bar{\chi}$ -eigenspace under the action of G on $\mathbf{Q}[\xi] \otimes \mathbf{C}$. By Lemmas 13 and 14, its image $\sum \chi'(k)l_s(k/d)$ in \mathbf{C} is zero if and only if $\chi(-1) = (-1)^s$; except for the single anomalous case when $s = 0$ and $\chi = \chi_0$. Thus the rank of this mapping

$$U_s(A_m) \rightarrow \mathbf{Q}[\xi]$$

is at least $\varphi(m)/2$ for $s < 0$, and at least $1 + \varphi(m)/2$ when $s = 0$.

It follows that the image $l_s(A_m)$ spans the real part of the cyclotomic field $\mathbf{Q}[\xi]$ when $s = 1 - t < 0$ is odd, and the totally imaginary part of $\mathbf{Q}[\xi]$ when s is even. Here l_s is related to the real valued functions f_t of Theorem 3 by the identity

$$l_{1-t}(x) + f_t(x)/(2i)^t = 0$$

for $t \geq 2$; which follows from (8) and (9). Similarly, for $t = 1$, the image of the function

$$if_1(k/m) = 2l_0(k/m) + 1$$

spans the totally imaginary subspace of $\mathbf{Q}[\xi]$.

Since the dimension $\varphi(m)/2$ of this image is the maximum allowed by Lemma 8, this completes the proof of Theorem 3. □

Proof of Bass' Theorem. Recall that V_m is the multiplicative group in $\mathbf{Q}[\xi]$ spanned by the $1 - \xi^k$. Evidently the Galois group G of $\mathbf{Q}[\xi]$ operates on V_m . Since each generator is the product of a real number and a root of unity, G operates also on the additive group $\log |V_m|$, generated by the images

$$f_0(k/m) = \log |1 - \xi^k|.$$

Note that $f_0(x)$ is precisely the even part $-(l_1(x) + l_1(-x))/2$ of the function $-l_1(x) = \log(1 - e^{2\pi ix})$.

As in the proof of Theorem 3, we can consider the map

$$U_1(A_m - 0) \rightarrow \log |V_m|$$

induced by f_0 , and split both sides into eigenspaces under the action of $G \cong (\mathbf{Z}/m\mathbf{Z})^*$. For each even character $\chi \neq \chi_0$, with conductor generated by $d \mid m$, the corresponding L -function

$$\sum_{k \pmod d} \chi'(k)f_0(k/d) = - \sum_{k \pmod d} \chi'(k)l_1(k/d) = -\tau L(1, \bar{\chi}')$$

is non-zero according to Dirichlet. Thus we obtain a contribution of $-1 + \varphi(m)/2$ to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing m . Lemmas 8 and 10 of §5 now complete the proof. \square

APPENDIX 1

RELATIONS BETWEEN POLYLOGARITHM AND HURWITZ FUNCTION

For every complex number s , it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function $l_s(x)$ and of the function $\zeta_{1-s}(x)$ or $\beta_s(x) = -s\zeta_{1-s}(x)$. This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of s , the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1 + i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0.$$

Integrating, we see that

$$\begin{aligned} l_1(x) - l_1(1-x) + 2\pi i \beta_1(x)/1! &= 0 \\ l_2(x) + l_2(1-x) + (2\pi i)^2 \beta_2(x)/2! &= 0 \end{aligned}$$

and so on, for $0 < x < 1$. For even values of the subscript, specializing to $x = 0$ as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k} b_{2k}/(2k)! = 0.$$

In particular, it follows that $\zeta(0) = -\frac{1}{2}$, and that the numbers $b_2, -b_4, b_6, -b_8, \dots$ are strictly positive. On the other hand, differentiating the formula for $l_0(x)$, we obtain

$$l_{-1}(x) = -\operatorname{csc}^2(\pi x)/4.$$

This is an even function satisfying $(*_{-1})$, so it must be some multiple of $\zeta_2(x) + \zeta_2(1-x)$. Comparing asymptotic behavior as $x \rightarrow 0$, we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1!.$$