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The proof that these elements generate over $\mathbf{Z}_q$ proceeds as above for $p \neq q$, and proceeds as in the proof of Lemma 9 when $p = q$. Details are easily supplied. □

§6. On Q-linear relations

S. Chowla and P. Chowla have suggested the following conjecture in a private communication to the author. Let $a_1, a_2, \ldots$ be a sequence of integers which is periodic, $a_n = a_{n+p}$, for some prime p. Then

$$\sum_1^\infty a_n/n^2 \neq 0 \tag{11}$$

except in the special case

$$a_1 = \ldots = a_{p-1} = a_p/(1-p^2)\,.$$

If we use the Hurwitz function

$$\zeta_2(k/p) = p^2(k^{-2} + (k+p)^{-2} + \ldots)\,,$$

then the inequality (11) can be written as

$$\sum_1^p a_k\, \zeta_2(k/p) \neq 0\,;$$

and the exceptional case corresponds to the Kubert relation

$$\zeta_2(1) = p^{-2} \sum_1^p \zeta_2(k/p)\,.$$

Thus the Chowlas' conjecture is true if and only if the real numbers

$$\zeta_2(1/p), \ldots, \zeta_2((p-1)/p)$$

are linearly independent over the rational numbers. More generally, for any $m \geqslant 2$ one might conjecture that the $\varphi(m)$ real numbers $\zeta_2(k/m)$, where k varies over all relatively prime integers between 1 and $m-1$, are $\mathbf{Q}$-linearly independent. Using Lemma 9, a completely equivalent statement would be the following.

Conjecture: Every $\mathbf{Q}$-linear relation between the real numbers $\zeta_2(x)$, where x is rational with $0 < x \leqslant 1$ is a consequence of the Kubert relations $(*_{-1})$.

In fact, since $\zeta_2(x+1) \equiv \zeta_2(x) \bmod \mathbf{Q}$ for positive rational x, it might be more natural to sharpen this conjecture by taking the values of ζ_2 modulo $\mathbf{Q}$. *In other words, it is conjectured that the mapping*

$$\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/\mathbf{Q}$$

induced by ζ_2 *is a "universal" function satisfying* $(*_{-1})$. It follows easily from Theorem 3 below that the corresponding conjecture for the even part,

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x\,,$$

of ζ_2 is indeed true; but the odd part of ζ_2 seems difficult to work with.

One can make analogous and equally plausible conjectures for the Hurwitz functions $\zeta_3, \zeta_4, \ldots$. In Appendix 2 we will describe analogous conjectures for certain functions closely related to the gamma function.

Bass [2], studying multiplicative relations between cyclotomic units, has proved the following result. Let

$$f_0(x) = \log |1 - e^{2\pi i x}| = \log(2 \sin \pi x)$$

for $0 < x < 1$. Note that $f_0(1-x) = f_0(x)$.

THEOREM OF BASS. *Every* **Q**-*linear relation between the numbers* $f_0(x)$ *for rational* $x \in (0, 1)$ *is a consequence of the Kubert relations* $(*_1)$, *together with evenness.*

A proof will be indicated at the end of this section.

Note that this is the exceptional case in which Lemma 7 does not apply, so that $f_0(0)$ cannot be defined.

Bass' theorem is equivalent, using the results of §5, to the following classical statement. Fixing some integer $m \geqslant 3$, let $\xi = e^{2\pi i/m}$, and let V_m be the multiplicative group generated by the elements

$$1 - \xi, 1 - \xi^2, \ldots, 1 - \xi^{m-1}$$

in the cyclotomic field $\mathbf{Q}[\xi]$. Elements of the intersection $V_m \cap \mathbf{Z}[\xi]^{\bullet}$ are called *circular units* (or cyclotomic units).

COROLLARY. *This group* $V_m \cap \mathbf{Z}[\xi]^{\bullet}$ *of circular units has finite index in the group* $\mathbf{Z}[\xi]^{\bullet}$ *consisting of all units of the cyclotomic field.*

Compare Hilbert [8], as well as Sinnott [25].

Proof. Let $m = q_1 \ldots q_n$ be the factorization of m into powers of distinct primes. By Lemmas 8 and 10, Bass' theorem is equivalent to the statement that the additive group generated by the elements

$$f_0(k/m) = \log |1 - \xi^k|$$

has rank $\varphi(m)/2 + n - 1$. Since each generator of V_m is equal to a real number multiplied by a root of unity, this is equivalent to the statement that V_m has rank $\varphi(m)/2 + n - 1$. However it is not difficult to check that V_m splits as the direct sum of the group of circular units and a free abelian group generated by the elements $1 - e^{2\pi i/q_j}$. Hence Bass' theorem is also equivalent to the statement that the group of circular units has rank $\varphi(m)/2 - 1$. According to the Dirichlet unit theorem, this implies that it has finite index in the group of all units of $\mathbf{Z}[\xi]$. □

The author [21] has conjectured that the function $\mathbf{Q}/\mathbf{Z} \to \mathbf{R}$ defined by

$$x \mapsto \Lambda(\pi x) = -\int_0^{\pi x} \log |2 \sin \theta| \, d\theta$$

is a universal odd function satisfying $(*_2)$. This seems very difficult. However, W. Sinnott has pointed out to the author this the situation for the derivatives of log 2 sin θ is much easier to analyze.

Let $f_t(x)$ be the t-th derivative of $\log |2 \sin \theta|$, evaluated at $\theta = \pi x$. For example $f_1(x) = \cot(\pi x)$, $f_2(x) = -csc^2(\pi x)$. Note that $f_1(1-x) = (-1)^t f_t(x)$. The values at $x = 0$ are to be defined as in §4.

THEOREM 3. *For each fixed* $t = 1, 2, \ldots,$ *the function*

$$f_t : \mathbf{Q}/\mathbf{Z} \to \mathbf{R}$$

is a universal even or odd function satisfying $(*_{1-t})$.

That is every **Q**-linear relation between the values $f_t(x)$ for x in $\mathbf{Q}/\mathbf{Z}$ follows from $(*_{1-t})$, together with evenness or oddnes according as t is even or odd.

Fixing some integer $m \geqslant 3$, let $\xi = e^{2\pi i/m}$. If t is even, the proof will show that the values

$$f_t(1/m), \ldots, f_t((m-1)/m)$$

span the real part of the cyclotomic field $\mathbf{Q}[\xi]$. Similarly, if t is odd, the values $if_t(k/m)$ span the totally imaginary subspace of $\mathbf{Q}[\xi]$. In either case, these values span a rational vector space of dimension $\varphi(m)/2$, as required by Lemma 8.

Compare Ewing [7] for an analogous discussion of the values of $csc(\pi x)$ and its derivatives at rational x.

The proof will depend upon well known properties of Dirichlet L-functions. Fixing some positive integer m, let

$$\chi : (\mathbf{Z}/m\mathbf{Z})^{\bullet} \to \mathbf{C}^{\bullet}$$

be an arbitrary Dirichlet character modulo m. We allow the degenerate case $m = 1$ with the understanding that the only character modulo 1 is the constant function $\chi_0(k) = 1$. Recall that such a character is *primitive* (or has conductor generated by m) if it cannot be factored through the projection

$$(\mathbf{Z}/m\mathbf{Z})^{\bullet} \to (\mathbf{Z}/d\mathbf{Z})^{\bullet}$$

for any divisor $d < m$. As usual, we set $\chi(k) = 0$ if k is a non-unit modulo m.

The associated L-function is defined by

$$L(s, \chi) = \sum_{1}^{\infty} \chi(k)/k^s$$

for $Re(s) > 1$. In terms of the Hurwitz function

$$\zeta_s(k/m)/m^s = k^{-s} + (k+m)^{-s} + \dots$$

we can clearly write this as a finite sum

$$L(s, \chi) = \sum_1^m \chi(k)\, \zeta_s(k/m)/m^s\,. \tag{12}$$

It follows that $L(s, \chi)$ *extends to a function which is holomorphic in* s *for all complex* s, *whenever* $\chi \neq \chi_0$. For it is easy to check that the difference $\zeta_s(x) - (s-1)^{-1}$ is holomorphic in s; and the $(s-1)^{-1}$ terms cancel whenever $\chi \neq \chi_0$.

On the other hand, for the trivial character χ_0, evidently $L(s, \chi_0)$ is equal to the Riemann zeta function, with a pole at $s = 1$.

Now let us restrict to integer values of s.

LEMMA 13. *For primitive* $\chi \neq \chi_0$, *and for integer values of* s, *the function* $L(s, \chi)$ *is zero if and only if* $s \leqslant 0$ *and* $\chi(-1) = (-1)^s$.

For $s > 1$, the statement that $L(s, \chi) \neq 1$ is fairly easy to prove, while for $s = 1$ it is a basic result of Dirichlet. See for example [5] or [23]. For $s \leqslant 0$, this lemma is proved using the functional equation relating $L(s, \chi)$ and $L(1-s, \bar{\chi})$. (Compare [10].) Details of this last argument may be found in Appendix 1. □

In the case of the trivial character χ_0, this lemma remains true except for anomalous behavior at $s = 0$ (where $\zeta(s)$ is non-zero) and $s = 1$ (where $\zeta(s)$ has a pole).

These Dirichlet L-functions can also be expressed as finite linear combinations of polylogarithms, via Fourier analysis, as follows. Let $\xi = e^{2\pi i/m}$.

LEMMA 14. *If* $\chi \neq \chi_0$ *is primitive modulo* m, *then*

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau\,,$$

where

$$\tau = \tau(\chi) = \sum_1^m \chi(k)\xi^k$$

is a complex constant with absolute value $\sqrt{m}$.

In the case of the trivial character χ_0, this lemma remains true provided that $l_s(1)$ is interpreted as in §4.

Proof of Lemma 14. Since both sides are holomorphic in s for all complex s, it will suffice to consider the case $Re(s) > 1$. First note that the "Fourier transform" of the complex valued function χ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is equal to $\tau\bar{\chi}$; that is

$$\sum_{j \bmod m} \chi(j)\xi^{jk} = \tau\bar{\chi}(k)\,. \tag{13}$$

If k is a unit modulo m, this follows from the equation $\chi(j) = \bar{\chi}(k)\chi(jk)$, while if k is a non-unit modulo m then, using the hypothesis that χ is primitive, it is not difficult to check that both sides of this equation are zero. Now dividing both sides by k^s and summing over all positive integers k, we obtain

$$\sum_{j \bmod m} \chi(j)\mathscr{L}_s(\xi^j) = \tau L(s, \bar{\chi})\,.$$

Since $\mathscr{L}_s(\xi^j) = l_s(j/m)$, this implies the required equation.

To compute $|\tau|$ combine (13) with the complex conjugate equation to obtain

$$\begin{aligned} m\chi(n) &= \sum_j \chi(j) \sum_k \xi^{k(j-n)} = \sum_k \xi^{-kn} \sum_j \chi(j)\xi^{kj} \\ &= \sum_k \xi^{-kn}\,\tau\bar{\chi}(k) = \tau\bar{\tau}\chi(n)\,; \end{aligned}$$

hence $m = \tau\bar{\tau}$ as asserted. □

Remark. Similar arguments prove that the Fourier transform of the Hurwitz function $\zeta_s(j/m)$ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is a multiple of $l_s(k/m)$. More generally, one can show that any function on $\mathbf{Z}/m\mathbf{Z}$ satisfies $(*_s)$ if and only if its Fourier transform satisfies $(*_{1-s})$.

Proof of Theorem 3. We will work with the polylogarithm function

$$\mathscr{L}_s(\xi^k) = l_s(k/m)$$

where $\xi = e^{2\pi i/m}$. If $s = 1 - t$ is a non-positive integer, recall from §2 that $\mathscr{L}_s(z)$ is a rational function with rational coefficients. Hence $l_s(k/m)$ takes values in the cyclotomic field $\mathbf{Q}[\xi]$.

The Galois group G of $\mathbf{Q}[\xi]$ over $\mathbf{Q}$ can be identified with $(\mathbf{Z}/m\mathbf{Z})^{\bullet}$. Evidently the mapping

$$U_s(A_m) \to \mathbf{Q}[\xi]$$

induced by l_s is G-equivariant, in the sense that the automorphism $u(k/m) \mapsto u(gk/m)$ of $U_s(A_m)$ corresponds to the automorphism $f(\xi) \mapsto f(\xi^g)$ of $\mathbf{Q}[\xi]$ for every g in $G \cong (\mathbf{Z}/m\mathbf{Z})^{\bullet}$. Tensoring both sides with the complex numbers, each splits into a direct sum of 1-dimensional eigenspaces under the action of G. Hence, to compute the rank of this map, we need only decide how many eigenspaces are mapped non-trivially.

For each character χ mod m, let $\chi' : (\mathbf{Z}/d\mathbf{Z})^{\cdot} \to \mathbf{C}^{\cdot}$ be the associated primitive character, where $d \mid m$ generates the conductor of χ. Evidently the sum

$$\sum_{k \bmod d} l_s(k/d) \otimes \chi'(k)$$

belongs to the $\bar{\chi}$-eigenspace under the action of G on $\mathbf{Q}[\xi] \otimes \mathbf{C}$. By Lemmas 13 and 14, its image $\sum \chi'(k) l_s(k/d)$ in $\mathbf{C}$ is zero if and only if $\chi(-1) = (-1)^s$; except for the single anomalous case when $s = 0$ and $\chi = \chi_0$. Thus the rank of this mapping

$$U_s(A_m) \to \mathbf{Q}[\xi]$$

is at least $\varphi(m)/2$ for $s < 0$, and at least $1 + \varphi(m)/2$ when $s = 0$.

It follows that the image $l_s(A_m)$ *spans the real part of the cyclotomic field* $\mathbf{Q}[\xi]$ *when* $s = 1 - t < 0$ *is odd, and the totally imaginary part of* $\mathbf{Q}[\xi]$ *when* s *is even.* Here l_s is related to the real valued functions f_t of Theorem 3 by the identity

$$l_{1-t}(x) + f_t(x)/(2i)^t = 0$$

for $t \geqslant 2$; which follows from (8) and (9). Similarly, for $t = 1$, the image of the function

$$if_1(k/m) = 2l_0(k/m) + 1$$

spans the totally imaginary subspace of $\mathbf{Q}[\xi]$.

Since the dimension $\varphi(m)/2$ of this image is the maximum allowed by Lemma 8, this completes the proof of Theorem 3. □

Proof of Bass' Theorem. Recall that V_m is the multiplicative group in $\mathbf{Q}[\xi]$ spanned by the $1 - \xi^k$. Evidently the Galois group G of $\mathbf{Q}[\xi]$ operates on V_m. Since each generator is the product of a real number and a root of unity, G operates also on the additive group $\log | V_m |$, generated by the images

$$f_0(k/m) = \log | 1 - \xi^k | .$$

Note that $f_0(x)$ is precisely the even part $-\big(l_1(x) + l_1(-x)\big)/2$ of the function $-l_1(x) = \log(1 - e^{2\pi i x})$.

As in the proof of Theorem 3, we can consider the map

$$U_1(A_m - 0) \to \log | V_m |$$

induced by f_0, and split both sides into eigenspaces under the action of $G \cong (\mathbf{Z}/m\mathbf{Z})^{\cdot}$. For each even character $\chi \neq \chi_0$, with conductor generated by $d \mid m$, the corresponding L-function

$$\sum_{k \bmod d} \chi'(k) f_0(k/d) = - \sum_{k \bmod d} \chi'(k) l_1(k/d) = -\tau L(1, \bar{\chi}')$$

is non-zero according to Dirichlet. Thus we obtain a contribution of $-1 + \varphi(m)/2$ to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing m. Lemmas 8 and 10 of §5 now complete the proof. □

Appendix 1

Relations between polylogarithm and Hurwitz function

For every complex number s, it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function $l_s(x)$ and of the function $\zeta_{1-s}(x)$ or $\beta_s(x) = -s\zeta_{1-s}(x)$. This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of s, the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1+i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0\,.$$

Integrating, we see that

$$l_1(x) - l_1(1-x) + 2\pi i\, \beta_1(x)/1! = 0$$

$$l_2(x) + l_2(1-x) + (2\pi i)^2\beta_2(x)/2! = 0$$

and so on, for $0 < x < 1$. For even values of the subscript, specializing to $x = 0$ as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k}b_{2k}/(2k)! = 0\,.$$

In particular, it follows that $\zeta(0) = -\frac{1}{2}$, and that the numbers b_2, $-b_4$, b_6, $-b_8$, ... are strictly positive. On the other hand, differentiating the formula for $l_0(x)$, we obtain

$$l_{-1}(x) = -csc^2(\pi x)/4\,.$$

This is an even function satisfying $(*_{-1})$, so it must be some multiple of $\zeta_2(x) + \zeta_2(1-x)$. Comparing asymptotic behavior as $x \to 0$, we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1!\,.$$