

<b>Zeitschrift:</b>	L'Enseignement Mathématique
<b>Herausgeber:</b>	Commission Internationale de l'Enseignement Mathématique
<b>Band:</b>	29 (1983)
<b>Heft:</b>	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
<b>Artikel:</b>	ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES
<b>Autor:</b>	Milnor, John
<b>Kapitel:</b>	§5. Universal Kubert functions
<b>DOI:</b>	<a href="https://doi.org/10.5169/seals-52983">https://doi.org/10.5169/seals-52983</a>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 11.02.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## §5. UNIVERSAL KUBERT FUNCTIONS

The results in this section are either due to Kubert, or are minor variations on results of Kubert.

Let  $A \subset \mathbf{Q}/\mathbf{Z}$  be a subgroup, and let  $s$  be a fixed integer. A function

$$f : A \rightarrow V$$

to a rational vector space will be called a *Kubert function* if it satisfies

$$(*)_s' \quad f(ma) = m^{s-1} \sum_0^{m-1} f(a + k/m)$$

for every integer  $m$  such that  $1/m$  belongs to  $A$ . It will be convenient to say that  $f$  is *universal* if every  $\mathbf{Q}$ -linear relation between the values  $f(a)$  follows from these Kubert relations.

Let  $U_s(A)$  be the additive group with one generator  $u(a)$  for each element of  $A$ , and with defining relations  $(*)_s'$ . Then evidently  $f$  is universal if and only if the induced mapping

$$u(a) \mapsto f(a)$$

from  $U_s(A) \otimes \mathbf{Q}$  to  $V$  is injective.

We are primarily interested in the case where  $A$  is the entire group  $\mathbf{Q}/\mathbf{Z}$ . However, it is very useful to consider finite subgroups of  $\mathbf{Q}/\mathbf{Z}$ , and requires no extra work to consider arbitrary subgroups.

Note that every automorphism of  $A$  gives rise to an automorphism of  $U_s(A)$ . We will use the notation  $\text{Hom}(A, A)^\circ$  for the automorphism group of  $A$ , identifying it with the group of invertible elements in the ring  $\text{Hom}(A, A)$  consisting of all homomorphisms from  $A$  to itself.

**THEOREM 2.** *The complex vector space  $U_s(A) \otimes \mathbf{C}$  splits, under the action of the automorphism group of  $A$ , into a direct sum of 1-dimensional eigenspaces, with just one eigenspace corresponding to each continuous character*

$$\chi : \text{Hom}(A, A)^\circ \rightarrow \mathbf{C}^\circ.$$

Furthermore, any inclusion  $A \subset A' \subset \mathbf{Q}/\mathbf{Z}$  gives rise to an embedding  $U_s(A) \otimes \mathbf{C} \subset U_s(A') \otimes \mathbf{C}$ .

Proofs will be given at the end of this section.

If  $A = A_m$  is the cyclic group of order  $m$ , note that  $\text{Hom}(A, A)$  can be identified with the ring  $\mathbf{Z}/m\mathbf{Z}$ , and  $\text{Hom}(A, A)^\circ$  is an abelian group of order  $\phi(m)$ . In general,  $\text{Hom}(A, A)^\circ$  is to be topologized as the inverse limit of these groups

$$\text{Hom}(A_m, A_m)^\bullet = (\mathbf{Z}/m\mathbf{Z})^\bullet$$

as  $A_m$  varies over all finite subgroups of  $A$ . Similarly, the character group of  $\text{Hom}(A, A)^\bullet$  is the direct limit of the corresponding Dirichlet character groups  $\text{Hom}((\mathbf{Z}/m\mathbf{Z})^\bullet, \mathbf{C})$ .

One interesting consequence of Theorem 2 is the following statement, which is reminiscent of Galois theory.

**COROLLARY.** *If  $A \subset A' \subset \mathbf{Q}/\mathbf{Z}$ , then  $U_s(A) \otimes \mathbf{Q}$  can be identified with the subspace of  $U_s(A') \otimes \mathbf{Q}$  which is fixed by all automorphisms of  $A'$  over  $A$ .*

A proof is easily supplied.  $\square$

Here is another consequence.

**LEMMA 8.** *If  $A = A_m$  is cyclic of order  $m$ , then the rational vector space  $U_s(A_m) \otimes \mathbf{Q}$  has dimension  $\phi(m)$ . For  $m > 2$  this splits as the direct sum of even and odd parts with respect to the involution*

$$u(a) \mapsto u(-a),$$

where each of these summands has dimension  $\phi(m)/2$ .

*Proof.* This follows immediately from the corresponding statement for  $U_s(A) \otimes \mathbf{C}$ . The two summands have equal dimension since there are as many even characters ( $\chi(-1) = 1$ ) as odd characters ( $\chi(-1) = -1$ ) modulo  $m$ .  $\square$

If  $s \neq 1$ , then Lemma 8 could also be derived from the following more explicit statement.

**LEMMA 9.** *If  $s \neq 1$ , and if  $A = A_m$  is cyclic of order  $m$ , then  $U_s(A) \otimes \mathbf{Q}$  has a basis consisting of the  $\phi(m)$  elements  $u(k/m)$  with  $k$  relatively prime modulo  $m$ .*

However, this statement definitely fails for  $s = 1$ .

Another complication when  $s = 1$  is that Lemma 7 fails, so that we must also consider “punctured” Kubert functions, which are not defined at zero.

**Definition.** Let  $U_s(A - 0)$  be the universal group with one generator  $u(a)$  for each  $a \neq 0$  in  $A$ , and with defining relations

$$u(ma) = m^{s-1} \sum_0^{m-1} u(a + k/m)$$

for all  $m$  and  $a$  with  $ma \neq 0$  and  $1/m \in A$ .

If  $s \neq 1$ , then the proof of Lemma 7 can be used to show that the kernel and cokernel of the natural maps

$$U_s(A_m - 0) \rightarrow U_s(A_m)$$

are finite groups of order prime to  $m$ . Taking the direct limit over  $m$ , it follows that

$$U_s(\mathbf{Q}/\mathbf{Z} - 0) \cong U_s(\mathbf{Q}/\mathbf{Z}).$$

However, for  $s = 1$  the situation is different.

LEMMA 10. *The kernel of the natural homomorphism*

$$U_1(A - 0) \rightarrow U_1(A)$$

*is a free abelian group freely generated by the elements*

$$u(1/p) + u(2/p) + \dots + u((p-1)/p),$$

*as  $p$  ranges over all primes with  $1/p \in A$ . The cokernel of this homomorphism is free cyclic, generated by  $u(0)$ .*

A proof is easily supplied, using formula (10) of §4 to prove that there are no relations between these generators.  $\square$

The precise structure of  $U_s(A)$  can be given as follows.

LEMMA 11. *If  $s \leq 1$ , or if  $A$  is finite, then the group  $U_s(A)$  is free abelian. In any case,  $U_s(A)$  is torsion free, and any inclusion  $A \subset A'$  gives rise to an embedding of  $U_s(A)$  into  $U_s(A')$ .*

If  $s \geq 2$ , it is interesting to note that  $U_s(\mathbf{Q}/\mathbf{Z})$  is actually a vector space over the rational numbers. For this lemma asserts that it is torsion free, and the relations  $(*_s)$  clearly imply that it is divisible.

The proof of Theorem 2 will be based on the following. Let  $s$  be any complex number and let  $\chi : \text{Hom}(A, A)^\circ \rightarrow \mathbf{C}^\circ$  be a continuous character.

LEMMA 12. *There is one and, up to a constant multiple, only one function*

$$f = f_\chi : A \rightarrow \mathbf{C}$$

*satisfying  $(*_s)$  and satisfying  $f(ua) = \chi(u)f(a)$  for every  $u$  in  $\text{Hom}(A, A)^\circ$  and every  $a$  in  $A$ .*

*Proof.* To fix our ideas, let us consider only the case  $A = \mathbf{Q}/\mathbf{Z}$ , so that  $\text{Hom}(A, A) = \varprojlim \mathbf{Z}/m\mathbf{Z}$  is the profinite completion  $\hat{\mathbf{Z}}$  of the integers. The general case is completely analogous.

Since  $\chi$  is continuous, there exists an integer  $m \neq 0$  so that  $\chi$  is identically equal to 1 on the congruence class  $1 + m\hat{\mathbf{Z}}$  intersected with  $\hat{\mathbf{Z}}$ . The collection of

all  $m$  with this property forms an ideal  $\mathcal{F}$  called the *conductor* of  $\chi$ . Evidently  $\chi$  is equal to the composition

$$\hat{\mathbf{Z}}^* \rightarrow (\mathbf{Z}/\mathcal{F})^* \rightarrow \mathbf{C}^*$$

for some Dirichlet character modulo  $\mathcal{F}$ , and  $\mathcal{F}$  is the unique largest ideal with this property. We will use the same symbol  $\chi$  for this character on  $(\mathbf{Z}/\mathcal{F})^*$ . If  $k$  is any integer relatively prime to  $\mathcal{F}$ , it follows that  $\chi(k)$  is a well defined root of unity.

Any fraction in  $\mathbf{Q}/\mathbf{Z}$  with denominator  $n$  can be written as  $u/n$  for some unit  $u$  in  $\hat{\mathbf{Z}}^*$ . In view of the identity

$$f(u/n) = \chi(u)f(1/n),$$

we need only compute the values  $f(1/n)$  in order to determine  $f$  completely.

Note that the unit  $u$  in this equation is well defined modulo  $n\hat{\mathbf{Z}}$ . If  $n$  belongs to the ideal  $\mathcal{F}$ , then it follows that the root of unity  $\chi(u)$  is uniquely determined. However, if  $n \notin \mathcal{F}$ , then we can choose  $u \equiv 1 \pmod{n}$  with  $\chi(u) \neq 1$ . This proves that  $f(1/n) = 0$  whenever  $n$  is not in the ideal  $\mathcal{F}$ .

The proof will show that  $f$  is some constant multiple of the expression

$$f(1/n) = n^{-s} \prod_{p|n} (p - p^s \bar{\chi}(p))/(p-1) \quad \text{for } n > 0, n \in \mathcal{F}.$$

Here  $\bar{\chi}(p)$  is a well defined root of unity if the prime  $p$  is a unit modulo  $\mathcal{F}$ , and is to be set equal to zero otherwise.

First consider the Kubert identity

$$(\star) \quad p^{1-s} f\left(\frac{1}{n}\right) = \sum_0^{p-1} f\left(\frac{1+kn}{pn}\right)$$

for  $n \in \mathcal{F}$ .

*Case 1.* If  $p \mid n$ , then each  $1 + kn$  is a unit modulo  $pn$ , with  $\chi(1+kn) = 1$ . Hence this equation reduces simply to

$$p^{-s} f\left(\frac{1}{n}\right) = f\left(\frac{1}{pn}\right).$$

*Case 2.* If  $n$  is not a multiple of  $p$ , then there is exactly one  $k_0$  between 1 and  $p-1$  so that  $1 + k_0 n$  is some multiple, say  $lp$ , of  $p$ . Then

$$f\left(\frac{1+k_0n}{np}\right) = f\left(\frac{l}{n}\right) = \chi(l)f\left(\frac{1}{n}\right),$$

where  $\chi(l) = \bar{\chi}(p)$  since  $lp \equiv 1 \pmod{\mathcal{F}}$ . Thus the Kubert identity takes the form

$$(p^{1-s} - \bar{\chi}(p))f\left(\frac{1}{n}\right) = (p-1)f\left(\frac{1}{pn}\right).$$

Evidently this completes the proof that  $f$  is uniquely defined up to multiplication by a constant.

To prove that the function  $f$  defined in this way satisfies all of the Kubert identities, we must also consider the case where  $n$  does *not* belong to the ideal  $\mathcal{F}$ , so that  $f(1/n) = 0$ . If  $pn$  does belong to  $\mathcal{F}$ , then the units  $1 + kn$  modulo  $pn$ , in the argument above, range precisely over the kernel of the homomorphism

$$(\mathbf{Z}/pn\mathbf{Z})^\circ \rightarrow (\mathbf{Z}/n\mathbf{Z})^\circ.$$

Since  $\chi$  is non-trivial on this kernel, by the definition of  $\mathcal{F}$ , it follows that

$$\sum \chi(1 + kn) = 0,$$

taking the sum over all  $k$  between 0 and  $p - 1$  with  $1 + kn$  prime to  $p$ . Thus both sides of the required equation ( $\star$ ) are zero. Since every other Kubert identity follows from one of these by applying an automorphism to  $\mathbf{Q}/\mathbf{Z}$ , this completes the proof.  $\square$

*Proof of Theorem 2.* If  $A = A_m$  is a finite group of order  $m$ , then  $U_s(A) \otimes \mathbf{C}$  is finite dimensional, so it certainly splits under the action of the commutative group  $\text{Hom}(A, A)^\circ$  into a direct sum of 1-dimensional spaces. According to Lemma 12, there is exactly one of these spaces for each character  $\chi \bmod m$ , so the conclusion follows.

The general case now follows by passing to a direct limit over finite subgroups of  $A$ . (For any integer  $n$ , note that there are only finitely many characters  $\chi$  whose conductor contains  $n$ , hence only finitely many  $\chi$  with  $f_\chi(1/n) \neq 0$ .) This completes the proof.  $\square$

*Proof of Lemma 9.* It will be convenient to consider the various vector spaces  $U_s(A_m) \otimes \mathbf{Q}$  as subspaces of  $U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$ . This is permissible by the Corollary above (or by Lemma 11)).

Let  $W_m$  be the rational vector space spanned by all elements

$$u(a) \in U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$$

such that  $a$  has denominator precisely  $m$ , and hence generates the cyclic group  $A_m$ . We will show that  $W_m \subset W_{pm}$ . Assuming this for the moment, it follows inductively that

$$W_m = U_s(A_m) \otimes \mathbf{Q}.$$

Hence the  $\varphi(m)$  generators of  $W_m$  must be linearly independent, as was to be proved.

Suppose then that  $a$  generates  $A_m$ . If  $p \mid m$ , then the Kubert identity

$$u(a) = p^{s-1} \sum_0^{p-1} u((a+k)/p),$$

where each  $(a+k)/p$  has denominator precisely  $pm$ , proves that  $u(a) \equiv 0 \pmod{W_{pm}}$ . On the other hand, if  $p$  is prime to  $m$ , then the relation

$$u(pa) - p^{s-1} u(a) = p^{s-1} \sum_1^{p-1} u(a+k/p)$$

proves that

$$u(pa) \equiv p^{s-1} u(a) \pmod{W_{pm}}.$$

Choosing  $r \geq 1$  so that  $p^r \equiv 1 \pmod{m}$ , it follows that

$$u(a) = u(p^r a) \equiv p^{r(s-1)} u(a) \pmod{W_{pm}}.$$

Since  $s \neq 1$ , this proves that  $u(a) \equiv 0 \pmod{W_{pm}}$ , as required.  $\square$

*Proof of Lemma 11.* For any  $a \in \mathbf{Q}/\mathbf{Z}$  let  $a_p$  be the  $p$ -primary component of  $a$ . Thus  $a = \sum a_p$ , where the denominator of  $a_p$  is a power of  $p$ . Represent each  $a_p$  as a rational in the interval  $0 \leq a_p < 1$ .

*Definition.* We will say that  $a$  is *reduced* if  $0 \leq a_p < 1 - p^{-1}$  for every prime  $p$ .

Then for  $s \leq 1$  we will prove explicitly that  $U_s(A)$  is a free abelian group, with one free generator  $u(a)$  for each reduced element  $a$  of  $A$ . Evidently it suffices to check that  $U_s(A)$  is generated by these elements. For a simple counting argument shows that the number of reduced elements in any finite subgroup  $A_m = m^{-1}\mathbf{Z}/\mathbf{Z}$  is equal to the rank

$$\varphi(m) = m \prod_{p|m} (1 - p^{-1})$$

of  $U_s(A_m)$ .

Suppose that  $a$  is not reduced, say  $1 - p^{-1} \leq a_p < 1$  for some prime  $p$ . Then the identity

$$p^{1-s} u(pa) = u(a) + u(a - 1/p) + \dots + u(a - (p-1)/p)$$

shows that  $u(a)$  is a linear combination of  $u(pa)$ , where  $pa$  has strictly smaller denominator than  $a$ , and elements  $a - k/p$  which are reduced at the prime  $p$  and have  $q$ -primary component unchanged for  $q \neq p$ . A straightforward induction now completes the proof in the case  $s \leq 1$ .

If  $s \geq 2$ , this argument shows only that the reduced generators form a basis for the rational vector space  $U_s(A) \otimes \mathbf{Q}$ . To prove that  $U_s(A_m)$  is free abelian, we will show that the tensor product  $U_s(A_m) \otimes \mathbf{Z}_q$  is generated by  $\varphi(m)$  elements for any prime  $q$ . This will show that there cannot be any torsion.

As free generators, we will choose all elements  $u(a)$  where  $a = \sum a_p$  is “reduced” at all primes  $p$  other than  $q$ . However, we require that the  $q$ -primary component  $a_q$  should have denominator equal to the highest power of  $q$  dividing  $m$ .

The proof that these elements generate over  $\mathbf{Z}_q$  proceeds as above for  $p \neq q$ , and proceeds as in the proof of Lemma 9 when  $p = q$ . Details are easily supplied.  $\square$

## §6. ON $\mathbf{Q}$ -LINEAR RELATIONS

S. Chowla and P. Chowla have suggested the following conjecture in a private communication to the author. Let  $a_1, a_2, \dots$  be a sequence of integers which is periodic,  $a_n = a_{n+p}$ , for some prime  $p$ . Then

$$(11) \quad \sum_1^\infty a_n/n^2 \neq 0$$

except in the special case

$$a_1 = \dots = a_{p-1} = a_p/(1-p^2).$$

If we use the Hurwitz function

$$\zeta_2(k/p) = p^2(k^{-2} + (k+p)^{-2} + \dots),$$

then the inequality (11) can be written as

$$\sum_1^p a_k \zeta_2(k/p) \neq 0;$$

and the exceptional case corresponds to the Kubert relation

$$\zeta_2(1) = p^{-2} \sum_1^p \zeta_2(k/p).$$

*Thus the Chowlas' conjecture is true if and only if the real numbers*

$$\zeta_2(1/p), \dots, \zeta_2((p-1)/p)$$

*are linearly independent over the rational numbers.* More generally, for any  $m \geq 2$  one might conjecture that the  $\varphi(m)$  real numbers  $\zeta_2(k/m)$ , where  $k$  varies over all relatively prime integers between 1 and  $m-1$ , are  $\mathbf{Q}$ -linearly independent. Using Lemma 9, a completely equivalent statement would be the following.

*Conjecture:* Every  $\mathbf{Q}$ -linear relation between the real numbers  $\zeta_2(x)$ , where  $x$  is rational with  $0 < x \leq 1$  is a consequence of the Kubert relations  $(*_1)$ .

In fact, since  $\zeta_2(x+1) \equiv \zeta_2(x) \pmod{\mathbf{Q}}$  for positive rational  $x$ , it might be more natural to sharpen this conjecture by taking the values of  $\zeta_2$  modulo  $\mathbf{Q}$ . *In other words, it is conjectured that the mapping*

$$\mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Q}$$

*induced by  $\zeta_2$  is a “universal” function satisfying  $(*_1)$ .* It follows easily from Theorem 3 below that the corresponding conjecture for the even part,

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x,$$

of  $\zeta_2$  is indeed true; but the odd part of  $\zeta_2$  seems difficult to work with.