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§5. UNIVERSAL KUBERT FUNCTIONS

The results in this section are either due to Kubert, or are minor variations on results of Kubert.

Let $A \subset \mathbf{Q}/\mathbf{Z}$ be a subgroup, and let s be a fixed integer. A function

$$f : A \rightarrow V$$

to a rational vector space will be called a *Kubert function* if it satisfies

$$(*'_s) \quad f(ma) = m^{s-1} \sum_0^{m-1} f(a + k/m)$$

for every integer m such that $1/m$ belongs to A . It will be convenient to say that f is *universal* if every \mathbf{Q} -linear relation between the values $f(a)$ follows from these Kubert relations.

Let $U_s(A)$ be the additive group with one generator $u(a)$ for each element of A , and with defining relations $(*_s)$. Then evidently f is universal if and only if the induced mapping

$$u(a) \mapsto f(a)$$

from $U_s(A) \otimes \mathbf{Q}$ to V is injective.

We are primarily interested in the case where A is the entire group \mathbf{Q}/\mathbf{Z} . However, it is very useful to consider finite subgroups of \mathbf{Q}/\mathbf{Z} , and requires no extra work to consider arbitrary subgroups.

Note that every automorphism of A gives rise to an automorphism of $U_s(A)$. We will use the notation $\text{Hom}(A, A)^\bullet$ for the automorphism group of A , identifying it with the group of invertible elements in the ring $\text{Hom}(A, A)$ consisting of all homomorphisms from A to itself.

THEOREM 2. *The complex vector space $U_s(A) \otimes \mathbf{C}$ splits, under the action of the automorphism group of A , into a direct sum of 1-dimensional eigenspaces, with just one eigenspace corresponding to each continuous character*

$$\chi : \text{Hom}(A, A)^\bullet \rightarrow \mathbf{C}^\bullet.$$

Furthermore, any inclusion $A \subset A' \subset \mathbf{Q}/\mathbf{Z}$ gives rise to an embedding $U_s(A) \otimes \mathbf{C} \subset U_s(A') \otimes \mathbf{C}$.

Proofs will be given at the end of this section.

If $A = A_m$ is the cyclic group of order m , note that $\text{Hom}(A, A)$ can be identified with the ring $\mathbf{Z}/m\mathbf{Z}$, and $\text{Hom}(A, A)^\bullet$ is an abelian group of order $\phi(m)$. In general, $\text{Hom}(A, A)^\bullet$ is to be topologized as the inverse limit of these groups

$$\text{Hom}(A_m, A_m)^{\cdot} = (\mathbb{Z}/m\mathbb{Z})^{\cdot}$$

as A_m varies over all finite subgroups of A . Similarly, the character group of $\text{Hom}(A, A)^{\cdot}$ is the direct limit of the corresponding Dirichlet character groups $\text{Hom}((\mathbb{Z}/m\mathbb{Z})^{\cdot}, \mathbb{C}^{\cdot})$.

One interesting consequence of Theorem 2 is the following statement, which is reminiscent of Galois theory.

COROLLARY. *If $A \subset A' \subset \mathbb{Q}/\mathbb{Z}$, then $U_s(A) \otimes \mathbb{Q}$ can be identified with the subspace of $U_s(A') \otimes \mathbb{Q}$ which is fixed by all automorphisms of A' over A .*

A proof is easily supplied. \square

Here is another consequence.

LEMMA 8. *If $A = A_m$ is cyclic of order m , then the rational vector space $U_s(A_m) \otimes \mathbb{Q}$ has dimension $\phi(m)$. For $m > 2$ this splits as the direct sum of even and odd parts with respect to the involution*

$$u(a) \mapsto u(-a),$$

where each of these summands has dimension $\phi(m)/2$.

Proof. This follows immediately from the corresponding statement for $U_s(A) \otimes \mathbb{C}$. The two summands have equal dimension since there are as many even characters ($\chi(-1) = 1$) as odd characters ($\chi(-1) = -1$) modulo m . \square

If $s \neq 1$, then Lemma 8 could also be derived from the following more explicit statement.

LEMMA 9. *If $s \neq 1$, and if $A = A_m$ is cyclic of order m , then $U_s(A) \otimes \mathbb{Q}$ has a basis consisting of the $\phi(m)$ elements $u(k/m)$ with k relatively prime modulo m .*

However, this statement definitely fails for $s = 1$.

Another complication when $s = 1$ is that Lemma 7 fails, so that we must also consider "punctured" Kubert functions, which are not defined at zero.

Definition. Let $U_s(A - 0)$ be the universal group with one generator $u(a)$ for each $a \neq 0$ in A , and with defining relations

$$u(ma) = m^{s-1} \sum_{k=0}^{m-1} u(a + k/m)$$

for all m and a with $ma \neq 0$ and $1/m \in A$.

If $s \neq 1$, then the proof of Lemma 7 can be used to show that the kernel and cokernel of the natural maps

$$U_s(A_m - 0) \rightarrow U_s(A_m)$$

are finite groups of order prime to m . Taking the direct limit over m , it follows that

$$U_s(\mathbf{Q}/\mathbf{Z} - 0) \cong U_s(\mathbf{Q}/\mathbf{Z}).$$

However, for $s = 1$ the situation is different.

LEMMA 10. *The kernel of the natural homomorphism*

$$U_1(A - 0) \rightarrow U_1(A)$$

is a free abelian group freely generated by the elements

$$u(1/p) + u(2/p) + \dots + u((p-1)/p),$$

as p ranges over all primes with $1/p \in A$. The cokernel of this homomorphism is free cyclic, generated by $u(0)$.

A proof is easily supplied, using formula (10) of §4 to prove that there are no relations between these generators. \square

The precise structure of $U_s(A)$ can be given as follows.

LEMMA 11. *If $s \leq 1$, or if A is finite, then the group $U_s(A)$ is free abelian. In any case, $U_s(A)$ is torsion free, and any inclusion $A \subset A'$ gives rise to an embedding of $U_s(A)$ into $U_s(A')$.*

If $s \geq 2$, it is interesting to note that $U_s(\mathbf{Q}/\mathbf{Z})$ is actually a vector space over the rational numbers. For this lemma asserts that it is torsion free, and the relations $(*)_s$ clearly imply that it is divisible.

The proof of Theorem 2 will be based on the following. Let s be any complex number and let $\chi : \text{Hom}(A, A)^* \rightarrow \mathbf{C}^*$ be a continuous character.

LEMMA 12. *There is one and, up to a constant multiple, only one function*

$$f = f_\chi : A \rightarrow \mathbf{C}$$

*satisfying $(*_s)$ and satisfying $f(ua) = \chi(u)f(a)$ for every u in $\text{Hom}(A, A)^*$ and every a in A .*

Proof. To fix our ideas, let us consider only the case $A = \mathbf{Q}/\mathbf{Z}$, so that $\text{Hom}(A, A) = \varprojlim \mathbf{Z}/m\mathbf{Z}$ is the profinite completion $\hat{\mathbf{Z}}$ of the integers. The general case is completely analogous.

Since χ is continuous, there exists an integer $m \neq 0$ so that χ is identically equal to 1 on the congruence class $1 + m\hat{\mathbf{Z}}$ intersected with $\hat{\mathbf{Z}}$. The collection of

all m with this property forms an ideal \mathcal{F} called the *conductor* of χ . Evidently χ is equal to the composition

$$\hat{\mathbf{Z}}^* \rightarrow (\mathbf{Z}/\mathcal{F})^* \rightarrow \mathbf{C}^*$$

for some Dirichlet character modulo \mathcal{F} , and \mathcal{F} is the unique largest ideal with this property. We will use the same symbol χ for this character on $(\mathbf{Z}/\mathcal{F})^*$. If k is any integer relatively prime to \mathcal{F} , it follows that $\chi(k)$ is a well defined root of unity.

Any fraction in \mathbf{Q}/\mathbf{Z} with denominator n can be written as u/n for some unit u in $\hat{\mathbf{Z}}^*$. In view of the identity

$$f(u/n) = \chi(u)f(1/n),$$

we need only compute the values $f(1/n)$ in order to determine f completely.

Note that the unit u in this equation is well defined modulo $n\hat{\mathbf{Z}}$. If n belongs to the ideal \mathcal{F} , then it follows that the root of unity $\chi(u)$ is uniquely determined. However, if $n \notin \mathcal{F}$, then we can choose $u \equiv 1 \pmod{n}$ with $\chi(u) \neq 1$. This proves that $f(1/n) = 0$ whenever n is not in the ideal \mathcal{F} .

The proof will show that f is some constant multiple of the expression

$$f(1/n) = n^{-s} \prod_{p|n} (p - p^s \bar{\chi}(p)) / (p-1) \quad \text{for } n > 0, n \in \mathcal{F}.$$

Here $\bar{\chi}(p)$ is a well defined root of unity if the prime p is a unit modulo \mathcal{F} , and is to be set equal to zero otherwise.

First consider the Kubert identity

$$(\star) \quad p^{1-s} f\left(\frac{1}{n}\right) = \sum_0^{p-1} f\left(\frac{1+kn}{pn}\right)$$

for $n \in \mathcal{F}$.

Case 1. If $p | n$, then each $1 + kn$ is a unit modulo pn , with $\chi(1+kn) = 1$. Hence this equation reduces simply to

$$p^{-s} f\left(\frac{1}{n}\right) = f\left(\frac{1}{pn}\right).$$

Case 2. If n is not a multiple of p , then there is exactly one k_0 between 1 and $p-1$ so that $1 + k_0 n$ is some multiple, say lp , of p . Then

$$f\left(\frac{1+k_0 n}{np}\right) = f\left(\frac{l}{n}\right) = \chi(l)f\left(\frac{1}{n}\right),$$

where $\chi(l) = \bar{\chi}(p)$ since $lp \equiv 1 \pmod{\mathcal{F}}$. Thus the Kubert identity takes the form

$$(p^{1-s} - \bar{\chi}(p))f\left(\frac{1}{n}\right) = (p-1)f\left(\frac{1}{pn}\right).$$

Evidently this completes the proof that f is uniquely defined up to multiplication by a constant.

To prove that the function f defined in this way satisfies all of the Kubert identities, we must also consider the case where n does *not* belong to the ideal \mathcal{F} , so that $f(1/n) = 0$. If pn does belong to \mathcal{F} , then the units $1 + kn$ modulo pn , in the argument above, range precisely over the kernel of the homomorphism

$$(\mathbf{Z}/pn\mathbf{Z})^* \rightarrow (\mathbf{Z}/n\mathbf{Z})^*.$$

Since χ is non-trivial on this kernel, by the definition of \mathcal{F} , it follows that

$$\sum \chi(1 + kn) = 0,$$

taking the sum over all k between 0 and $p - 1$ with $1 + kn$ prime to p . Thus both sides of the required equation (★) are zero. Since every other Kubert identity follows from one of these by applying an automorphism to \mathbf{Q}/\mathbf{Z} , this completes the proof. \square

Proof of Theorem 2. If $A = A_m$ is a finite group of order m , then $U_s(A) \otimes \mathbf{C}$ is finite dimensional, so it certainly splits under the action of the commutative group $\text{Hom}(A, A)^*$ into a direct sum of 1-dimensional spaces. According to Lemma 12, there is exactly one of these spaces for each character $\chi \bmod m$, so the conclusion follows.

The general case now follows by passing to a direct limit over finite subgroups of A . (For any integer n , note that there are only finitely many characters χ whose conductor contains n , hence only finitely many χ with $f_\chi(1/n) \neq 0$.) This completes the proof. \square

Proof of Lemma 9. It will be convenient to consider the various vector spaces $U_s(A_m) \otimes \mathbf{Q}$ as subspaces of $U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$. This is permissible by the Corollary above (or by Lemma 11)).

Let W_m be the rational vector space spanned by all elements

$$u(a) \in U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$$

such that a has denominator precisely m , and hence generates the cyclic group A_m . We will show that $W_m \subset W_{pm}$. Assuming this for the moment, it follows inductively that

$$W_m = U_s(A_m) \otimes \mathbf{Q}.$$

Hence the $\phi(m)$ generators of W_m must be linearly independent, as was to be proved.

Suppose then that a generates A_m . If $p \mid m$, then the Kubert identity

$$u(a) = p^{s-1} \sum_0^{p-1} u((a+k)/p),$$

where each $(a+k)/p$ has denominator precisely pm , proves that $u(a) \equiv 0 \pmod{W_{pm}}$. On the other hand, if p is prime to m , then the relation

$$u(pa) - p^{s-1} u(a) = p^{s-1} \sum_1^{p-1} u(a + k/p)$$

proves that

$$u(pa) \equiv p^{s-1} u(a) \pmod{W_{pm}}.$$

Choosing $r \geq 1$ so that $p^r \equiv 1 \pmod{m}$, it follows that

$$u(a) = u(p^r a) \equiv p^{r(s-1)} u(a) \pmod{W_{pm}}.$$

Since $s \neq 1$, this proves that $u(a) \equiv 0 \pmod{W_{pm}}$, as required. \square

Proof of Lemma 11. For any $a \in \mathbf{Q}/\mathbf{Z}$ let a_p be the p -primary component of a . Thus $a = \sum a_p$, where the denominator of a_p is a power of p . Represent each a_p as a rational in the interval $0 \leq a_p < 1$.

Definition. We will say that a is *reduced* if $0 \leq a_p < 1 - p^{-1}$ for every prime p .

Then for $s \leq 1$ we will prove explicitly that $U_s(A)$ is a free abelian group, with one free generator $u(a)$ for each reduced element a of A . Evidently it suffices to check that $U_s(A)$ is generated by these elements. For a simple counting argument shows that the number of reduced elements in any finite subgroup $A_m = m^{-1}\mathbf{Z}/\mathbf{Z}$ is equal to the rank

$$\varphi(m) = m \prod_{p|m} (1 - p^{-1})$$

of $U_s(A_m)$.

Suppose that a is not reduced, say $1 - p^{-1} \leq a_p < 1$ for some prime p . Then the identity

$$p^{1-s} u(pa) = u(a) + u(a - 1/p) + \dots + u(a - (p-1)/p)$$

shows that $u(a)$ is a linear combination of $u(pa)$, where pa has strictly smaller denominator than a , and elements $a - k/p$ which are reduced at the prime p and have q -primary component unchanged for $q \neq p$. A straightforward induction now completes the proof in the case $s \leq 1$.

If $s \geq 2$, this argument shows only that the reduced generators form a basis for the rational vector space $U_s(A) \otimes \mathbf{Q}$. To prove that $U_s(A_m)$ is free abelian, we will show that the tensor product $U_s(A_m) \otimes \mathbf{Z}_q$ is generated by $\varphi(m)$ elements for any prime q . This will show that there cannot be any torsion.

As free generators, we will choose all elements $u(a)$ where $a = \sum a_p$ is "reduced" at all primes p other than q . However, we require that the q -primary component a_q should have denominator equal to the highest power of q dividing m .

The proof that these elements generate over \mathbf{Z}_q proceeds as above for $p \neq q$, and proceeds as in the proof of Lemma 9 when $p = q$. Details are easily supplied. \square

§6. ON \mathbf{Q} -LINEAR RELATIONS

S. Chowla and P. Chowla have suggested the following conjecture in a private communication to the author. Let a_1, a_2, \dots be a sequence of integers which is periodic, $a_n = a_{n+p}$, for some prime p . Then

$$(11) \quad \sum_1^\infty a_n/n^2 \neq 0$$

except in the special case

$$a_1 = \dots = a_{p-1} = a_p/(1-p^2).$$

If we use the Hurwitz function

$$\zeta_2(k/p) = p^2(k^{-2} + (k+p)^{-2} + \dots),$$

then the inequality (11) can be written as

$$\sum_1^p a_k \zeta_2(k/p) \neq 0;$$

and the exceptional case corresponds to the Kubert relation

$$\zeta_2(1) = p^{-2} \sum_1^p \zeta_2(k/p).$$

Thus the Chowlas' conjecture is true if and only if the real numbers

$$\zeta_2(1/p), \dots, \zeta_2((p-1)/p)$$

are linearly independent over the rational numbers. More generally, for any $m \geq 2$ one might conjecture that the $\phi(m)$ real numbers $\zeta_2(k/m)$, where k varies over all relatively prime integers between 1 and $m-1$, are \mathbf{Q} -linearly independent. Using Lemma 9, a completely equivalent statement would be the following.

Conjecture: Every \mathbf{Q} -linear relation between the real numbers $\zeta_2(x)$, where x is rational with $0 < x \leq 1$ is a consequence of the Kubert relations $(*_1)$.

In fact, since $\zeta_2(x+1) \equiv \zeta_2(x) \pmod{\mathbf{Q}}$ for positive rational x , it might be more natural to sharpen this conjecture by taking the values of ζ_2 modulo \mathbf{Q} . In other words, it is conjectured that the mapping

$$\mathbf{Q}/\mathbf{Z} \rightarrow \mathbf{R}/\mathbf{Q}$$

induced by ζ_2 is a "universal" function satisfying $(*_1)$. It follows easily from Theorem 3 below that the corresponding conjecture for the even part,

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x,$$

of ζ_2 is indeed true; but the odd part of ζ_2 seems difficult to work with.