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Proof of Theorem 1 for $\operatorname{Re}(s) < 0$. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \rightarrow 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \rightarrow 0$. Applying a similar argument to the function $f(1-x)$, we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \rightarrow 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \rightarrow 0$. \square

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, \dots$, as one can check by repeated differentiation.

§4. EXTENDING FROM $(0, 1)$ TO \mathbf{R}/\mathbf{Z}

We will prove the following. Let s be a complex constant.

LEMMA 7. *If a function $f : (0, 1) \rightarrow \mathbf{C}$ satisfies the Kubert identities $(*_s)$ with $s \neq 1$, then it extends uniquely to a function $\mathbf{R}/\mathbf{Z} \rightarrow \mathbf{C}$ satisfying $(*_s)$.*

Here no mention is made of continuity. If $\operatorname{Re}(s) > 1$ and if f happens to be continuous, then we have seen that the extension is also continuous. However, if $\operatorname{Re}(s) \leq 1$ then the extension cannot be continuous, except in the trivial case of a constant function with $s = 0$.

Proof. We must choose $f(0)$ so as to satisfy all of the equations

$$f(0) = m^{s-1}(f(0) + f(1/m) + \dots + f((m-1)/n)).$$

Setting

$$c_m = f(1/m) + \dots + f((m-1)/m),$$

we can write this as

$$(m^{1-s} - 1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn} - c_m)$$

hence

$$c_{mn} = m^{1-s}c_n + c_m = n^{1-s}c_m + c_n$$

and

$$(m^{1-s} - 1)c_n = (n^{1-s} - 1)c_m.$$

Since $s \neq 1$, these factors $m^{1-s} - 1$ cannot all be zero. It now follows easily that $f(0)$ exists and is unique. \square

For the functions $f(x)$ studied in §2, it is interesting to note that $f(0)$ is always an appropriate value of the Riemann zeta function. Thus for the version $f(x) = l_s(x)$ of the polylogarithm function, the appropriate choice is

$$f(0) = \zeta(s).$$

In fact, if $\operatorname{Re}(s) > 1$, then $l_s(x)$ is continuous on \mathbf{R}/\mathbf{Z} with $l_s(0) = \zeta(s)$, so the required identity

$$(m^{1-s} - 1)\zeta(s) = l_s(1/m) + \dots + l_s((m-1)/m)$$

holds by continuity as $x \rightarrow 0$. It follows by analytic continuation that this formula is true for all $s \neq 1$. (Since the right side is holomorphic for all s , this identity provides an alternative proof that $\zeta(s)$ extends to an holomorphic function for $s \neq 1$.)

Similarly, if $f(x) = \zeta_{1-s}(x)$ for $0 < x < 1$, then by continuity as $x \rightarrow 1$ the appropriate choice is

$$f(0) = \zeta(1-s).$$

Note that Lemma 7 is definitely false in the exceptional case $s = 1$. In the case of the even function

$$f(x) = \log |2 \sin \pi x| = \log |1 - e^{2\pi i x}|,$$

which satisfies $(*_1)$ in the open unit interval, the identity

$$(10) \quad f(1/m) + f(2/m) + \dots + f((m-1)/m) = \log m \neq 0$$

shows that it is not possible to define $f(0)$ so as to satisfy $(*_1)$ at zero. This identity is proved by substituting $t = 1$ in the equation

$$1 + t + \dots + t^{m-1} = \prod_1^{m-1} (t - \xi^k)$$

where $\xi = e^{2\pi i/m}$, and then taking the logarithm of the absolute value of both sides.

On the other hand, for the Bernoulli polynomial

$$f(x) = x - 1/2 \quad \text{for} \quad 0 < x < 1,$$

the value $f(0)$ can be defined arbitrarily and $(*_1)$ will always be satisfied.