Proof of Theorem 1 for $Re(s) < 0$. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \to 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \to 0$. Applying a similar argument to the function $f(1-x)$, we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \to 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \to 0$. □

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, \ldots$, as one can check by repeated differentiation.

§4. Extending from (0, 1) to R/Z

We will prove the following. Let s be a complex constant.

Lemma 7. *If a function* $f : (0, 1) \to \mathbf{C}$ *satisfies the Kubert identities* $(*_s)$ *with* $s \neq 1$, *then it extends uniquely to a function* $\mathbf{R}/\mathbf{Z} \to \mathbf{C}$ *satisfying* $(*_s)$.

Here no mention is made of continuity. If $Re(s) > 1$ and if f happens to be continuous, then we have seen that the extension is also continuous. *However, if* $Re(s) \leqslant 1$ *then the extension cannot be continuous, except in the trivial case of a constant function with* $s = 0$.

Proof. We must choose $f(0)$ so as to satisfy all of the equations

$$f(0) = m^{s-1}\big(f(0) + f(1/m) + \ldots + f((m-1)/n)\big).$$

Setting

$$c_m = f(1/m) + \ldots + f\big((m-1)/m\big),$$

we can write this as

$$(m^{1-s}-1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn}-c_m)$$

hence

$$c_{mn} = m^{1-s}\,c_n + c_m = n^{1-s}\,c_m + c_n$$

and

$$(m^{1-s}-1)c_n = (n^{1-s}-1)c_m.$$

Since $s \neq 1$, these factors $m^{1-s} - 1$ cannot all be zero. It now follows easily that $f(0)$ exists and is unique. □

For the functions $f(x)$ studied in §2, it is interesting to note that $f(0)$ is always an appropriate value of the Riemann zeta function. Thus for the version $f(x) = l_s(x)$ of the polylogarithm function, the appropriate choice is

$$f(0) = \zeta(s) .$$

In fact, if $Re(s) > 1$, then $l_s(x)$ is continuous on **R**/**Z** with $l_s(0) = \zeta(s)$, so the required identity

$$(m^{1-s}-1)\zeta(s) = l_s(1/m) + \ldots + l_s((m-1)/m)$$

holds by continuity as $x \to 0$. It follows by analytic continuation that this formula is true for all $s \neq 1$. (Since the right side is holomorphic for all s, this identity provides an alternative proof that $\zeta(s)$ extends to an holomorphic function for $s \neq 1$.)

Similarly, if $f(x) = \zeta_{1-s}(x)$ for $0 < x < 1$, then by continuity as $x \to 1$ the appropriate choice is

$$f(0) = \zeta(1-s) .$$

Note that Lemma 7 is definitely false in the exceptional case $s = 1$. In the case of the even function

$$f(x) = \log | 2 \sin \pi x | = \log | 1 - e^{2\pi i x} | ,$$

which satisfies $(*_1)$ in the open unit interval, the identity

(10) $$f(1/m) + f(2/m) + \ldots + f((m-1)/m) = \log m \neq 0$$

shows that it is not possible to define $f(0)$ so as to satisfy $(*_1)$ at zero. This identity is proved by substituting $t = 1$ in the equation

$$1 + t + \ldots + t^{m-1} = \prod_{1}^{m-1} (t-\xi^k)$$

where $\xi = e^{2\pi i/m}$, and then taking the logarithm of the absolute value of both sides.

On the other hand, for the Bernoulli polynomial

$$f(x) = x - 1/2 \qquad \text{for} \qquad 0 < x < 1 ,$$

the value $f(0)$ can be defined arbitrarily and $(*_1)$ will always be satisfied.