

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	29 (1983)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES
Autor:	Milnor, John
Kapitel:	§4. EXTENDING FROM (0, 1) TO R/Z
DOI:	https://doi.org/10.5169/seals-52983

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Proof of Theorem 1 for $\operatorname{Re}(s) < 0$. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \rightarrow 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \rightarrow 0$. Applying a similar argument to the function $f(1-x)$, we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \rightarrow 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \rightarrow 0$. \square

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, \dots$, as one can check by repeated differentiation.

§4. EXTENDING FROM $(0, 1)$ TO \mathbf{R}/\mathbf{Z}

We will prove the following. Let s be a complex constant.

LEMMA 7. *If a function $f : (0, 1) \rightarrow \mathbf{C}$ satisfies the Kubert identities $(*_s)$ with $s \neq 1$, then it extends uniquely to a function $\mathbf{R}/\mathbf{Z} \rightarrow \mathbf{C}$ satisfying $(*_s)$.*

Here no mention is made of continuity. If $\operatorname{Re}(s) > 1$ and if f happens to be continuous, then we have seen that the extension is also continuous. However, if $\operatorname{Re}(s) \leq 1$ then the extension cannot be continuous, except in the trivial case of a constant function with $s = 0$.

Proof. We must choose $f(0)$ so as to satisfy all of the equations

$$f(0) = m^{s-1}(f(0) + f(1/m) + \dots + f((m-1)/m)).$$

Setting

$$c_m = f(1/m) + \dots + f((m-1)/m),$$

we can write this as

$$(m^{1-s} - 1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn} - c_m)$$

hence

$$c_{mn} = m^{1-s}c_n + c_m = n^{1-s}c_m + c_n$$

and

$$(m^{1-s} - 1)c_n = (n^{1-s} - 1)c_m.$$

Since $s \neq 1$, these factors $m^{1-s} - 1$ cannot all be zero. It now follows easily that $f(0)$ exists and is unique. \square

For the functions $f(x)$ studied in §2, it is interesting to note that $f(0)$ is always an appropriate value of the Riemann zeta function. Thus for the version $f(x) = l_s(x)$ of the polylogarithm function, the appropriate choice is

$$f(0) = \zeta(s).$$

In fact, if $Re(s) > 1$, then $l_s(x)$ is continuous on \mathbf{R}/\mathbf{Z} with $l_s(0) = \zeta(s)$, so the required identity

$$(m^{1-s} - 1)\zeta(s) = l_s(1/m) + \dots + l_s((m-1)/m)$$

holds by continuity as $x \rightarrow 0$. It follows by analytic continuation that this formula is true for all $s \neq 1$. (Since the right side is holomorphic for all s , this identity provides an alternative proof that $\zeta(s)$ extends to an holomorphic function for $s \neq 1$.)

Similarly, if $f(x) = \zeta_{1-s}(x)$ for $0 < x < 1$, then by continuity as $x \rightarrow 1$ the appropriate choice is

$$f(0) = \zeta(1-s).$$

Note that Lemma 7 is definitely false in the exceptional case $s = 1$. In the case of the even function

$$f(x) = \log |2 \sin \pi x| = \log |1 - e^{2\pi i x}|,$$

which satisfies $(*_1)$ in the open unit interval, the identity

$$(10) \quad f(1/m) + f(2/m) + \dots + f((m-1)/m) = \log m \neq 0$$

shows that it is not possible to define $f(0)$ so as to satisfy $(*_1)$ at zero. This identity is proved by substituting $t = 1$ in the equation

$$1 + t + \dots + t^{m-1} = \prod_1^{m-1} (t - \xi^k)$$

where $\xi = e^{2\pi i/m}$, and then taking the logarithm of the absolute value of both sides.

On the other hand, for the Bernoulli polynomial

$$f(x) = x - 1/2 \quad \text{for} \quad 0 < x < 1,$$

the value $f(0)$ can be defined arbitrarily and $(*_1)$ will always be satisfied.