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Artikel: ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

Autor: Milnor, John

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For the function $l_0(x) = e^{2\pi ix}/(1-e^{2\pi ix})$, a brief computation shows that

$$l_0(x) = (-1 + i\cot(\pi x))/2\,. \tag{9}$$

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x), \ldots$. *Note in particular that* $l_s(x)$ *is either an odd or an even function according as* $s-1$ *is odd or even, for every negative integer* s.

For further information about these functions, see Appendix 1.

§3. Continuous Kubert functions

Fixing some complex parameter s, let $\mathscr{K}_s$ be the complex vector space consisting of all continuous maps

$$f : (0, 1) \to \mathbf{C}$$

which satisfy the Kubert identity

$$f(x) = m^{s-1} \textstyle\sum_{k=0}^{m-1} f((x+k)/m) \tag{$*_s$}$$

for every positive integer m, and every x in $(0, 1)$. We will prove the following.

Theorem 1. *This complex vector space* $\mathscr{K}_s$ *has dimension* 2, *spanned by one even element* $(f(x) = f(1-x))$ *and one odd element* $(f(x) = -f(1-x))$. *Each function* $f(x)$ *in* $\mathscr{K}_s$ *is necessarily real analytic.*

If $f(x)$ satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if $s = 0$. Hence an immediate consequence is the following. (Compare Lemma 5.)

Corollary. *The correspondence* $f(x) \mapsto df(x)/dx$ *maps the vector space* $\mathscr{K}_s$ *bijectively onto* $\mathscr{K}_{s-1}$, *except when* $s = 0$.

The proof of Theorem 1 will yield explicit bases for $\mathscr{K}_s$ as follows, with notations as in §2. *For* $s \neq -1, -2, -3, \ldots,$ *the space* $\mathscr{K}_s$ *is spanned by the two linearly independent functions* $l_s(x)$ *and* $l_s(1-x)$. *On the other hand, for* $s \neq 0, 1, 2, \ldots,$ *this space is spanned by the linearly independent functions* $\zeta_{1-s}(x)$ *and* $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of s, we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f : (0, 1) \to \mathbf{C}$ be a continuous function satisfying $(*_s)$.

LEMMA 3. *If* $Re(s) > 0$, *then* $\int_0^1 |f(x)|\, dx$ *is finite.*

Proof. Let C be an upper bound for $|f(x)|$ on the closed interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ and let $\alpha = |2^{1-s}| < 2$. Using the identity

$$f(x) = 2^{1-s} f(2x) - f\left(x + \frac{1}{2}\right)$$

we see that

$$|f(x)| \leqslant (\alpha+1)C \qquad \text{for} \qquad \frac{1}{8} \leqslant x \leqslant \frac{1}{4},$$

hence

$$|f(x)| \leqslant (\alpha^2+\alpha+1)C \qquad \text{for} \qquad \frac{1}{16} \leqslant x \leqslant \frac{1}{8},$$

and so on. Therefore $\int_0^{1/2} |f(x)|\, dx$ is less than the finite sum

$$C\left(\frac{1}{4} + (\alpha+1)/8 + (\alpha^2+\alpha+1)/16 + \dots\right).$$

Applying the same argument to $f(1-x)$, this completes the proof. □

LEMMA 4. (Rohrlich) *Let* $f : (0, 1) \to \mathbf{C}$ *be a non-constant continuous function satisfying* $(*_s)$, *and suppose that*

$$\int_0^1 |f(x)|\, dx < \infty .$$

Then $Re(s) > 0$, *and* $f(x)$ *is equal to some linear combination of* $l_s(x)$ *and* $l_s(1-x)$.

Proof. We will make use of the easily proved fact that a continuous function on $(0, 1)$ with $\int_0^1 |f(x)|\, dx < \infty$ is uniquely determined by its Fourier coefficients

$$a_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx .$$

Furthermore, according to the Riemann-Lebesgue Lemma, these coefficients tend to zero as $|n| \to \infty$.

If f satisfies $(*_s)$, then a straightforward computation shows that

$$a_{nm} = a_n/m^s \qquad \text{for} \qquad m = 2, 3, \dots .$$

In particular,

$$a_{\pm m} = a_{\pm 1}/m^s .$$

Furthermore, $a_0 = 0$ except in the special case $s = 0$.

First suppose that $Re(s) \leqslant 0$. Then the numbers $1/m^s$ are bounded away from zero. Using the Riemann-Lebesgue Lemma, this implies that f has the Fourier series of a constant function, and hence is constant, contrary to our hypothesis.

Next suppose that $Re(s) > 1$. Then the series $\Sigma\ 1/m^s$ converges absolutely. Therefore the Fourier series of f

$$a_{+1} \sum_{m=1}^{\infty} e^{2\pi imx}/m^s + a_{-1} \sum_{m=1}^{\infty} e^{-2\pi imx}/m^s$$

converges uniformly on the circle $\mathbf{R}/\mathbf{Z}$ to the continuous function

$$a_{+1}\, l_s(x) + a_{-1}\, l_s(1-x)\,.$$

It follows that f is equal to this expression.

Finally, suppose that $0 < Re(s) \leqslant 1$. If F is any indefinite integral of f, then F is continuous on $[0, 1]$ by Lemma 3. We can integrate by parts to relate the Fourier coefficients of f and F; and it follows easily that F equals a linear combination of $l_{s+1}(x)$ and $l_{s+1}(1-x)$ plus a constant. Differentiating, we obtain the corresponding assertion for f. □

Proof of Theorem 1 when $Re(s) > 0$. Let $f : (0, 1) \to \mathbf{C}$ be a non-zero continuous function satisfying $(*_s)$. Then f is non-constant since $s \neq 0$. Hence f is a linear combination of $l_s(x)$ and $l_s(1-x)$ by Lemmas 3, 4. These two functions are linearly independent since they have independent Fourier expansions. □

REMARK. *If* $Re(s) > 1$, *then this proof shows also that* f *extends to a continuous function on the circle* $\mathbf{R}/\mathbf{Z}$. *Whenever* $Re(s) > 0$, *it shows that* $\int_0^1 f(x)dx = 0$.

We can extend this proof to all values of s except $-1, -2, \ldots$ by using the following lemma. Let $f : (0, 1) \to \mathbf{C}$ be a continuous function satisfying $(*_s)$, and let

$$F(x) = \int f(x)dx$$

be any indefinite integral of f.

LEMMA 5. *If* $s \neq -1$, *then there is one and only one constant* c *so that the function* $F(x) + c$ *satisfies* $(*_{s+1})$.

Proof. Integrating $(*_s)$, we have

$$F(x) = m^s \sum_{k=0}^{m-1} F((x+k)/m) + c_m$$

for some constants c_m. Comparing the formulas for different values of m, we see easily that

$$c_{lm} = m^{s+1} c_l + c_m = l^{s+1} c_m + c_l ,$$

hence

$$(m^{s+1} - 1)c_l = (l^{s+1} - 1)c_m .$$

These numbers $m^{s+1} - 1$ cannot all be zero, since $s \neq -1$. Therefore there exists one and only one c with

$$c_m = (m^{s+1} - 1)c$$

for every m. It is now easy to check that $F + c$ has the required property, and that c is unique. □

Remark. This lemma definitely fails for $s = -1$. In fact Gauss' formula

$$\Gamma(x) = \frac{m^{x-1/2}}{(2\pi)^{(m-1)/2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{x+k}{m}\right)$$

implies that the logarithmic derivative $F(x) = \Gamma'(x)/\Gamma(x)$ satisfies

$$F(x) = m^{-1} \sum_{k=0}^{m-1} F\left(\frac{x+k}{m}\right) + \log m .$$

Differentiating, we see that $F'(x)$ satisfies the Kubert identities $(*_{-1})$. (In fact $F'(x) = \zeta_2(x)$.) But there is no constant c so that $F + c$ satisfies $(*_0)$. See Appendix 2 for details.

Proof of Theorem 1 for $s \neq -1, -2, \ldots$. Given any continuous $f : (0, 1) \to \mathbf{C}$ satisfying $(*_s)$ we can integrate n times, using Lemma 5, to obtain a continuous function F satisfying $(*_{s+n})$ with $Re(s+n) > 1$. Then

$$F(x) = a l_{s+n}(x) + b l_{s+n}(1-x)$$

by Lemmas 3, 4, as above. Differentiating n times, and using (8), we see that $f(x)$ equals a linear combination of $l_s(x)$ and $l_s(1-x)$. These last two functions are linearly independent; for otherwise applying Lemma 5 n times we would obtain a contradiction. □

The proof for negative integer values of s will require a precise description of the behavior of $f(x)$ as $x \to 0$.

LEMMA 6. *If* $f : (0, 1) \to \mathbf{C}$ *is continuous and satisfies* $(*_s)$ *with* $Re(s) < 1$, *then there exists a constant* A *so that* $f(x) - Ax^{s-1}$ *tends to a finite limit as* $x \to 0$.

Proof. We will first show that the function $g(x) = f(x)/x^{s-1}$ tends to a limit A as $x \to 0$. Let $c_m = f(1/m) + f(2/m) + \ldots + f((m-1)/m)$. Then

$$f(x) = m^{s-1}\big(f(x/m) + f((x+1)/m) + \dots + f((x+m-1)/m)\big)$$
$$= m^{s-1}\big(f(x/m) + c_m + o(1)\big)$$

as $x \to 0$. Hence

$$g(x) = g(x/m) + O(x^{1-s}),$$

and it follows easily that the sequence of functions $g(x)$, $g(x/m)$, $g(x/m^2)$, ... converges uniformly to a limit $A_m(x)$. Evidently this limit function is defined and continuous for all $x > 0$, and satisfies

$$A_m(x) = A_m(x/m).$$

Further, for any $m, n > 1$ we have

$$g(x) = A_m(x) + o(1) = A_n(x) + o(1)$$

as $x \to 0$. Therefore

$$A_m(x) = A_n(x) + o(1) = A_n(x/n) + o(1) = A_m(x/n) + o(1).$$

Substituting x/m^k for x and letting $k \to \infty$, we see that

$$A_m(x) = A_m(x/n).$$

But clearly any continuous function on the positive reals which satisfies all of these periodicity conditions must be constant. Therefore $A = A_m(x)$ is independent of m and x.

Now take $m = 2$, and define $f(0)$ by the equation $f(0) = 2^{s-1}\big(f(0) + f(1/2)\big)$. (Compare §4.) Subtracting this from $f(x) = 2^{s-1}\big(f(x/2) + f((x+1)/2)\big)$ and dividing by x^{s-1} we obtain

$$\frac{f(x) - f(0)}{x^{s-1}} = \frac{f(x/2) - f(0)}{(x/2)^{s-1}} + o(x^{1-s})$$

as $x \to 0$. Taking the corresponding statements for $x/2$, $x/4$, ..., it follows that

$$\frac{f(x) - f(0)}{x^{s-1}} = A + o(x^{1-s}),$$

or in other words

$$f(x) = Ax^{s-1} + f(0) + o(1)$$

as $x \to 0$. □

To illustrate this lemma, note that the Hurwitz zeta function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \dots$$

is equal to the sum of x^{s-1} and a function $\zeta_{1-s}(x+1)$ which is continuous as $x \to 0$.

Proof of Theorem 1 for $Re(s) < 0$. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \to 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \to 0$. Applying a similar argument to the function $f(1-x)$, we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \to 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \to 0$. □

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, \ldots$, as one can check by repeated differentiation.

§4. Extending from (0, 1) to R/Z

We will prove the following. Let s be a complex constant.

Lemma 7. *If a function* $f : (0, 1) \to \mathbf{C}$ *satisfies the Kubert identities* $(*_s)$ *with* $s \neq 1$, *then it extends uniquely to a function* $\mathbf{R}/\mathbf{Z} \to \mathbf{C}$ *satisfying* $(*_s)$.

Here no mention is made of continuity. If $Re(s) > 1$ and if f happens to be continuous, then we have seen that the extension is also continuous. *However, if* $Re(s) \leqslant 1$ *then the extension cannot be continuous, except in the trivial case of a constant function with* $s = 0$.

Proof. We must choose $f(0)$ so as to satisfy all of the equations

$$f(0) = m^{s-1}(f(0) + f(1/m) + \ldots + f((m-1)/n).$$

Setting

$$c_m = f(1/m) + \ldots + f((m-1)/m),$$

we can write this as

$$(m^{1-s}-1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn}-c_m)$$

hence

$$c_{mn} = m^{1-s}\, c_n + c_m = n^{1-s}\, c_m + c_n$$

and

$$(m^{1-s}-1)c_n = (n^{1-s}-1)c_m.$$