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For the function $l_0(x) = e^{2\pi i x}/(1-e^{2\pi i x})$, a brief computation shows that

(9)
$$l_0(x) = (-1 + i \cot(\pi x))/2$$
.

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x)$, Note in particular that $l_s(x)$ is either an odd or an even function according as s - 1 is odd or even, for every negative integer s.

For further information about these functions, see Appendix 1.

§3. CONTINUOUS KUBERT FUNCTIONS

Fixing some complex parameter s, let \mathscr{K}_s be the complex vector space consisting of all continuous maps

$$f:(0,1)\to \mathbb{C}$$

which satisfy the Kubert identity

$$(*_{s}) f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m, and every x in (0, 1). We will prove the following.

THEOREM 1. This complex vector space \mathscr{K}_s has dimension 2, spanned by one even element (f(x) = f(1-x)) and one odd element (f(x) = -f(1-x)). Each function f(x) in \mathscr{K}_s is necessarily real analytic.

If f(x) satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if s = 0. Hence an immediate consequence is the following. (Compare Lemma 5.)

COROLLARY. The correspondence $f(x) \mapsto df(x)/dx$ maps the vector space \mathscr{K}_s bijectively onto \mathscr{K}_{s-1} , except when s = 0.

The proof of Theorem 1 will yield explicit bases for \mathscr{K}_s as follows, with notations as in §2. For $s \neq -1, -2, -3, ...,$ the space \mathscr{K}_s is spanned by the two linearly independent functions $l_s(x)$ and $l_s(1-x)$. On the other hand, for $s \neq 0, 1, 2, ...,$ this space is spanned by the linearly independent functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of *s*, we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f:(0, 1) \rightarrow \mathbb{C}$ be a continuous function satisfying $(*_s)$.

LEMMA 3. If Re(s) > 0, then $\int_0^1 |f(x)| dx$ is finite.

Proof. Let C be an upper bound for |f(x)| on the closed interval $\left\lfloor \frac{1}{4}, \frac{3}{4} \right\rfloor$ and let $\alpha = |2^{1-s}| < 2$. Using the identity

$$f(x) = 2^{1-s}f(2x) - f\left(x + \frac{1}{2}\right)$$

we see that

$$|f(x)| \leq (\alpha+1)C$$
 for $\frac{1}{8} \leq x \leq \frac{1}{4}$,

hence

$$|f(x)| \leq (\alpha^2 + \alpha + 1)C$$
 for $\frac{1}{16} \leq x \leq \frac{1}{8}$,

and so on. Therefore $\int_0^{1/2} |f(x)| dx$ is less than the finite sum

$$C\left(\frac{1}{4} + (\alpha+1)/8 + (\alpha^2 + \alpha + 1)/16 + ...\right)$$

Applying the same argument to f(1-x), this completes the proof.

LEMMA 4. (Rohrlich) Let $f:(0, 1) \rightarrow \mathbb{C}$ be a non-constant continuous function satisfying $(*_s)$, and suppose that

 $\int_0^1 |f(x)| \, dx < \infty \; .$

Then Re(s) > 0, and f(x) is equal to some linear combination of $l_s(x)$ and $l_s(1-x)$.

Proof. We will make use of the easily proved fact that a continuous function on (0, 1) with $\int_0^1 |f(x)| dx < \infty$ is uniquely determined by its Fourier coefficients

$$a_n = \int_0^1 f(x) e^{-2\pi i n x} dx .$$

Furthermore, according to the Riemann-Lebesgue Lemma, these coefficients tend to zero as $|n| \rightarrow \infty$.

If f satisfies $(*_s)$, then a straightforward computation shows that

$$a_{nm} = a_n/m^s$$
 for $m = 2, 3, ...$

In particular,

$$a_{\pm m} = a_{\pm 1}/m^s.$$

Furthermore, $a_0 = 0$ except in the special case s = 0.

First suppose that $Re(s) \leq 0$. Then the numbers $1/m^s$ are bounded away from zero. Using the Riemann-Lebesgue Lemma, this implies that f has the Fourier series of a constant function, and hence is constant, contrary to our hypothesis.

Next suppose that Re(s) > 1. Then the series $\sum 1/m^s$ converges absolutely. Therefore the Fourier series of f

$$a_{+1} \sum_{m=1}^{\infty} e^{2\pi i m x} / m^{s} + a_{-1} \sum_{m=1}^{\infty} e^{-2\pi i m x} / m^{s}$$

converges uniformly on the circle \mathbf{R}/\mathbf{Z} to the continuous function

 $a_{+1} l_s(x) + a_{-1} l_s(1-x)$.

It follows that f is equal to this expression.

Finally, suppose that $0 < Re(s) \le 1$. If F is any indefinite integral of f, then F is continuous on [0, 1] by Lemma 3. We can integrate by parts to relate the Fourier coefficients of f and F; and it follows easily that F equals a linear combination of $l_{s+1}(x)$ and $l_{s+1}(1-x)$ plus a constant. Differentiating, we obtain the corresponding assertion for f.

Proof of Theorem 1 when Re(s) > 0. Let $f: (0, 1) \to \mathbb{C}$ be a non-zero continuous function satisfying $(*_s)$. Then f is non-constant since $s \neq 0$. Hence f is a linear combination of $l_s(x)$ and $l_s(1-x)$ by Lemmas 3, 4. These two functions are linearly independent since they have independent Fourier expansions.

REMARK. If Re(s) > 1, then this proof shows also that f extends to a continuous function on the circle \mathbf{R}/\mathbf{Z} . Whenever Re(s) > 0, it shows that $\int_0^1 f(x)dx = 0$.

We can extend this proof to all values of s except -1, -2, ... by using the following lemma. Let $f : (0, 1) \rightarrow \mathbb{C}$ be a continuous function satisfying $(*_s)$, and let

$$F(x) = \int f(x) dx$$

be any indefinite integral of f.

LEMMA 5. If $s \neq -1$, then there is one and only one constant c so that the function F(x) + c satisfies $(*_{s+1})$.

Proof. Integrating $(*_s)$, we have

$$F(x) = m^{s} \sum_{k=0}^{m-1} F((x+k)/m) + c_{m}$$

for some constants c_m . Comparing the formulas for different values of m, we see easily that

hence

$$(m^{s+1}-1)c_l = (l^{s+1}-1)c_m$$
.

 $c_{lm} = m^{s+1} c_l + c_m = l^{s+1} c_m + c_l$

These numbers $m^{s+1} - 1$ cannot all be zero, since $s \neq -1$. Therefore there exists one and only one c with

$$c_m = (m^{s+1} - 1)c$$

for every *m*. It is now easy to check that F + c has the required property, and that *c* is unique.

Remark. This lemma definitely fails for s = -1. In fact Gauss' formula

$$\Gamma(x) = \frac{m^{x-1/2}}{(2\pi)^{(m-1)/2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{x+k}{m}\right)$$

implies that the logarithmic derivative $F(x) = \Gamma'(x)/\Gamma(x)$ satisfies

$$F(x) = m^{-1} \sum_{k=0}^{m-1} F\left(\frac{x+k}{m}\right) + \log m$$
.

Differentiating, we see that F'(x) satisfies the Kubert identities $(*_{-1})$. (In fact $F'(x) = \zeta_2(x)$.) But there is no constant c so that F + c satisfies $(*_0)$. See Appendix 2 for details.

Proof of Theorem 1 for $s \neq -1, -2, ...$ Given any continuous $f:(0, 1) \rightarrow \mathbf{C}$ satisfying $(*_s)$ we can integrate *n* times, using Lemma 5, to obtain a continuous function *F* satisfying $(*_{s+n})$ with Re(s+n) > 1. Then

$$F(x) = al_{s+n}(x) + bl_{s+n}(1-x)$$

by Lemmas 3, 4, as above. Differentiating *n* times, and using (8), we see that f(x) equals a linear combination of $l_s(x)$ and $l_s(1-x)$. These last two functions are linearly independent; for otherwise applying Lemma 5 *n* times we would obtain a contradiction.

The proof for negative integer values of s will require a precise description of the behavior of f(x) as $x \to 0$.

LEMMA 6. If $f:(0, 1) \to \mathbb{C}$ is continuous and satisfies $(*_s)$ with Re(s) < 1, then there exists a constant A so that $f(x) - Ax^{s-1}$ tends to a finite limit as $x \to 0$.

Proof. We will first show that the function $g(x) = f(x)/x^{s-1}$ tends to a limit A as $x \to 0$. Let $c_m = f(1/m) + f(2/m) + \dots + f((m-1)/m)$. Then

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$$f(x) = m^{s-1} (f(x/m) + f((x+1)/m) + \dots + f((x+m-1)/m))$$

= $m^{s-1} (f(x/m) + c_m + o(1))$

as $x \to 0$. Hence

 $g(x) = g(x/m) + O(x^{1-s}),$

and it follows easily that the sequence of functions g(x), g(x/m), $g(x/m^2)$, ... converges uniformly to a limit $A_m(x)$. Evidently this limit function is defined and continuous for all x > 0, and satisfies

$$A_m(x) = A_m(x/m) \, .$$

Further, for any m, n > 1 we have

$$g(x) = A_m(x) + o(1) = A_n(x) + o(1)$$

as $x \to 0$. Therefore

$$A_m(x) = A_n(x) + o(1) = A_n(x/n) + o(1) = A_m(x/n) + o(1).$$

Substituting x/m^k for x and letting $k \to \infty$, we see that

$$A_m(x) = A_m(x/n) \, .$$

But clearly any continuous function on the positive reals which satisfies all of these periodicity conditions must be constant. Therefore $A = A_m(x)$ is independent of m and x.

Now take m = 2, and define f(0) by the equation $f(0) = 2^{s-1}(f(0) + f(1/2))$. (Compare §4.) Subtracting this from $f(x) = 2^{s-1}(f(x/2) + f((x+1)/2))$ and dividing by x^{s-1} we obtain

$$\frac{f(x) - f(0)}{x^{s-1}} = \frac{f(x/2) - f(0)}{(x/2)^{s-1}} + o(x^{1-s})$$

as $x \to 0$. Taking the corresponding statements for x/2, x/4, ..., it follows that

$$\frac{f(x) - f(0)}{x^{s-1}} = A + o(x^{1-s}),$$

or in other words

 $f(x) = Ax^{s-1} + f(0) + o(1)$

as $x \to 0$.

To illustrate this lemma, note that the Hurwitz zeta function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \dots$$

is equal to the sum of x^{s-1} and a function $\zeta_{1-s}(x+1)$ which is continuous as $x \to 0$.

 \square

Proof of Theorem 1 for Re(s) < 0. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \to 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \to 0$. Applying a similar argument to the function f(1-x), we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \to 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \to 0$.

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, ..., as$ one can check by repeated differentiation.

§4. Extending from (0, 1) to \mathbf{R}/\mathbf{Z}

We will prove the following. Let s be a complex constant.

LEMMA 7. If a function $f:(0, 1) \to \mathbb{C}$ satisfies the Kubert identities $(*_s)$ with $s \neq 1$, then it extends uniquely to a function $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$ satisfying $(*_s)$.

Here no mention is made of continuity. If Re(s) > 1 and if f happens to be continuous, then we have seen that the extension is also continuous. However, if $Re(s) \leq 1$ then the extension cannot be continuous, except in the trivial case of a constant function with s = 0.

Proof. We must choose f(0) so as to satisfy all of the equations

$$f(0) = m^{s-1} (f(0) + f(1/m) + \dots + f((m-1)/n)).$$

Setting

$$c_m = f(1/m) + \dots + f((m-1)/m),$$

we can write this as

$$(m^{1-s}-1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn}-c_m)$$

hence

$$c_{mn} = m^{1-s} c_n + c_m = n^{1-s} c_m + c_m$$

and

$$(m^{1-s}-1)c_n = (n^{1-s}-1)c_m$$