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Here the symbol Λ stands for the function

$$\Lambda(\pi x) = - \int_0^{\pi x} \log |2 \sin \theta| d\theta = \sum_1^{\infty} \sin(2\pi n x)/2n^2,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from $(0, 1)$ to the circle \mathbf{R}/\mathbf{Z} . For any integer constant s , §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \rightarrow U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in \mathbf{Q}/\mathbf{Z} and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to \mathbf{Q}/\mathbf{Z} . For example, if $f : (0, 1) \rightarrow \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every \mathbf{Q} -linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples*. This question is settled in the relatively easy cases where the values of f on \mathbf{Q}/\mathbf{Z} are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

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§2. CLASSICAL EXAMPLES

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m=c} (t-b),$$

where b varies over all m -th roots of c . Hence, setting $t = 1$, we see that

$$\log |1 - c| = \sum_{b^m=c} \log |1 - b|.$$

If we define

$$f(x) = \log |1 - e^{2\pi i x}| = \log |2 \sin \pi x|,$$

then it follows that

$$f(x) = \sum_{my \equiv x \pmod{1}} f(y).$$

Thus f satisfies the Kubert identities $(*_1)$. Note that $f(x)$ is defined and smooth on the open interval $(0, 1)$. Differentiating $(*_1)$, we see that the derivative

$$f'(x) = \pi \cot \pi x$$

satisfies $(*_0)$. Similarly, the second derivative

$$f''(x) = -\pi^2 \csc^2 \pi x$$

satisfies $(*_{-1})$, and so on.

Next let us look at the *Hurwitz zeta function* $\zeta_s(x) = \zeta(s, x)$, which is defined by the series

$$(1) \quad \zeta_s(x) = x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \dots$$

for $x > 0$. Here s can be any complex number with $\operatorname{Re}(s) > 1$.

An easy computation shows that the function $\zeta_{1-s}(x)$ satisfies the Kubert identities $(*_s)$. (Here x is not an element of \mathbf{R}/\mathbf{Z} but rather a positive real number. In fact, it is sometimes useful to let x take complex values also.) It will often be convenient to work with the function

$$\beta_s(x) = -s\zeta_{1-s}(x).$$

We will prove the following.

LEMMA 1. *This product $\beta_s(x) = -s\zeta_{1-s}(x)$ extends to a function which is defined and holomorphic in both variables for all complex s , and for all x in the simply connected region $\mathbf{C} - (-\infty, 0]$.*

Hence $\zeta_{1-s}(x)$ is defined and holomorphic in the same region, except at $s = 0$. Evidently, by analytic continuation, these functions β_s and ζ_{1-s} always satisfy the Kubert identities $(*_s)$.

Proof. Clearly the function x^{s-1} is defined and holomorphic for x in $\mathbf{C} - (-\infty, 0]$ and for all complex s . If $\operatorname{Re}(s) < 0$, then it is easy to check that the series

$$\beta_s(x) = -s(x^{s-1} + (x+1)^{s-1} + \dots)$$

converges to a holomorphic function. Note that

$$(2) \quad \partial \beta_s(x) / \partial x = s \beta_{s-1}(x).$$

Integrating from x to $x + 1$, and then substituting $s + 1$ for s , we obtain

$$(3_x) \quad \int_x^{x+1} \beta_s(\xi) d\xi = x^s$$

whenever $\operatorname{Re}(s) < -1$. It follows by analytic continuation that this is true when $\operatorname{Re}(s) < 0$ also. In particular,

$$(3_1) \quad \int_1^2 \beta_s(x) dx = 1.$$

Suppose inductively that $\beta_s(x)$ has been defined so as to be holomorphic in both variables for $\operatorname{Re}(s) < n$. Then for $\operatorname{Re}(s) < n + 1$ we can set

$$\beta_s(x) = \int_1^x s \beta_{s-1}(\xi) d\xi + c_s,$$

choosing the constant c_s so that (3_1) is satisfied. Evidently this defines a holomorphic function which satisfies (2) and (3_1) , and hence coincides with the previously defined function in the common range of definition. It follows by induction that β_s is defined for all s . \square

The case where s is a non-negative integer is of particular interest. Using (2) and (3_0) or (3_1) we see inductively that the functions

$$\beta_0(x) = 1,$$

$$\beta_1(x) = x - \frac{1}{2},$$

$$\beta_2(x) = x^2 - x + \frac{1}{6}, \dots$$

are polynomials with rational coefficients. By definition, $\beta_s(x)$ is the s -th *Bernoulli polynomial* for $s = 0, 1, 2, \dots$. It can be characterized as the unique polynomial satisfying the identity

$$\int_1^n \beta_s(x) dx = 1^s + 2^s + \dots + (n-1)^s$$

for every n . Note the symmetry condition

$$\beta_s(1-x) = (-1)^s \beta_s(x),$$

which can be proved inductively using (3_0) .

For a more explicit computation, define the *Bernoulli numbers*

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, \dots$$

by the formal power series

$$t/(e^t - 1) = \sum b_k t^k/k!.$$

Then

$$\beta_s(x) = \frac{D}{e^D - I} x^s = \sum_0^s b_k \binom{s}{k} x^{s-k},$$

where D stands for the differentiation operator d/dx . For example it follows that

$$\beta_s(0) = b_s.$$

To prove this formula, simply apply the inverse operator $(e^D - I)/D$ to both sides, noting by Taylor's theorem that

$$\frac{e^D - I}{D} \beta_s(x) = \int_x^{x+1} \beta_s(\xi) d\xi = x^s.$$

If we substitute $x = 1$, then the Hurwitz zeta function $\zeta_s(x)$ reduces to the Riemann zeta function $\zeta(s)$. Thus our discussion implies the following well known result. *The product*

$$-s\zeta(1-s) = \beta_s(1)$$

can be extended as a function which is holomorphic for all complex s , and takes rational values for $s = 0, 1, 2, \dots$.

Next let us study the *polylogarithm function*, which is defined for any complex numbers s and z with $|z| < 1$ by the convergent power series

$$(4) \quad \mathcal{L}_s(z) = z + z^2/2^s + z^3/3^s + \dots$$

(Compare [3], [4], [6], [11], [19], [20], [22], [26].)

LEMMA 2. *This extends to a function which is defined, and holomorphic in both variables, for all complex s and all z in the simply connected region $\mathbb{C} - [1, \infty)$.*

Proof. First note the identity

$$(5) \quad \mathcal{L}_{s-1}(z) = z \partial \mathcal{L}_s(z) / \partial z.$$

If $\operatorname{Re}(s) > 0$ and $|z| < 1$, then according to Jonquière:

$$(6) \quad \mathcal{L}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z}{e^t - z} t^{s-1} dt.$$

This is proved by substituting $\sum z^n e^{-nt}$ for $z/(e^t - z)$, and noting that

$$\int_0^\infty e^{-nt} t^{s-1} dt = \int_0^\infty e^{-u} u^{s-1} du / n^s = \Gamma(s) / n^s.$$

Now if $\operatorname{Re}(s) > 0$, the right side of (6) clearly defines a function which is holomorphic in both variables for all $z \in \mathbf{C} - [1, \infty)$. The extension to other values of s follows inductively using (5). \square

The polylogarithm function satisfies a multiplicative analogue of the Kubert identities. For any positive integer m :

$$(7) \quad \mathcal{L}_s(z) = m^{s-1} \sum_{w^m=z} \mathcal{L}_s(w),$$

to be summed over all m -th roots of z . This is proved by a straightforward power series computation when $|z| < 1$, and by analytic continuation otherwise.

It will be convenient to introduce the abbreviation

$$l_s(x) = \mathcal{L}_s(e^{2\pi i x})$$

for $x \in \mathbf{R}/\mathbf{Z}$, $x \neq 0$, and for all complex s . Evidently $l_s(x)$ satisfies the Kubert identities in their original form $(*)_s$, and also the identity

$$(8) \quad \partial l_s(x)/\partial x = 2\pi i l_{s-1}(x).$$

If $\operatorname{Re}(s) > 1$, then we can write

$$l_s(x) = \sum \cos(2\pi n x)/n^s + \sum i \sin(2\pi n x)/n^s,$$

where the two summands on the right are the even and odd parts of $l_s(x)$. (If s is real, these can be identified with the real and imaginary parts of $l_s(x)$.)

For integer values of the parameter s , the functions $\mathcal{L}_s(z)$ and $l_s(x)$ can be described more explicitly as follows. *Summing the series*

$$\mathcal{L}_0(z) = z + z^2 + z^3 + \dots$$

and using (5), we see inductively that the functions

$$\mathcal{L}_0(z) = z/(1-z),$$

$$\mathcal{L}_{-1}(z) = z/(1-z)^2,$$

$$\mathcal{L}_{-2}(z) = z(1+z)/(1-z)^3, \dots$$

are rational, with rational coefficients, holomorphic in z except for a pole at $z = 1$. On the other hand, the series $z + z^2/2 + \dots$ evidently sums to

$$\mathcal{L}_1(z) = -\log(1-z),$$

and the integral

$$\mathcal{L}_2(z) = \int_0^z \mathcal{L}_1(w) dw/w$$

is the classical *dilogarithm function*.

For the function $l_0(x) = e^{2\pi ix}/(1 - e^{2\pi ix})$, a brief computation shows that

$$(9) \quad l_0(x) = (-1 + i \cot(\pi x))/2.$$

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x)$, Note in particular that $l_s(x)$ is either an odd or an even function according as $s - 1$ is odd or even, for every negative integer s .

For further information about these functions, see Appendix 1.

§3. CONTINUOUS KUBERT FUNCTIONS

Fixing some complex parameter s , let \mathcal{K}_s be the complex vector space consisting of all continuous maps

$$f : (0, 1) \rightarrow \mathbb{C}$$

which satisfy the Kubert identity

$$(*_s) \quad f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m , and every x in $(0, 1)$. We will prove the following.

THEOREM 1. *This complex vector space \mathcal{K}_s has dimension 2, spanned by one even element ($f(x) = f(1-x)$) and one odd element ($f(x) = -f(1-x)$). Each function $f(x)$ in \mathcal{K}_s is necessarily real analytic.*

If $f(x)$ satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if $s = 0$. Hence an immediate consequence is the following. (Compare Lemma 5.)

COROLLARY. *The correspondence $f(x) \mapsto df(x)/dx$ maps the vector space \mathcal{K}_s bijectively onto \mathcal{K}_{s-1} , except when $s = 0$.*

The proof of Theorem 1 will yield explicit bases for \mathcal{K}_s as follows, with notations as in §2. For $s \neq -1, -2, -3, \dots$, the space \mathcal{K}_s is spanned by the two linearly independent functions $l_s(x)$ and $l_s(1-x)$. On the other hand, for $s \neq 0, 1, 2, \dots$, this space is spanned by the linearly independent functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of s , we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f : (0, 1) \rightarrow \mathbb{C}$ be a continuous function satisfying $(*_s)$.