Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 29 (1983)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

Autor: Milnor, John

Kapitel: §2. Classical examples

DOI: https://doi.org/10.5169/seals-52983

Here the symbol Λ stands for the function

$$\Lambda(\pi x) = -\int_0^{\pi x} \log|2\sin\theta|\,d\theta = \sum_1^{\infty} \sin(2\pi n x)/2n^2\,,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from (0, 1) to the circle $\mathbf{R}/\mathbf{Z}$. For any *integer* constant s, §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \to U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in $\mathbf{Q}/\mathbf{Z}$ and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to $\mathbf{Q}/\mathbf{Z}$. For example, if $f : (0, 1) \to \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every $\mathbf{Q}$-linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples.* This question is settled in the relatively easy cases where the values of f on $\mathbf{Q}/\mathbf{Z}$ are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

The author is indebted to conversations with S. Chowla, B. H. Gross, Werner Meyer, and W. Sinnott.

§2. Classical examples

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m = c} (t-b)\,,$$

where b varies over all m-th roots of c. Hence, setting $t = 1$, we see that

$$\log|1-c| = \sum_{b^m=c} \log|1-b|\,.$$

If we define

$$f(x) = \log |1 - e^{2\pi i x}| = \log |2 \sin \pi x|,$$

then it follows that

$$f(x) = \sum_{my \equiv x \bmod 1} f(y).$$

Thus f satisfies the Kubert identities $(*_1)$. Note that $f(x)$ is defined and smooth on the open interval (0, 1). Differentiating $(*_1)$, we see that the derivative

$$f'(x) = \pi \cot \pi x$$

satisfies $(*_0)$. Similarly, the second derivative

$$f''(x) = -\pi^2 \csc^2 \pi x$$

satisfies $(*_{-1})$, and so on.

Next let us look at the *Hurwitz zeta function* $\zeta_s(x) = \zeta(s, x)$, which is defined by the series

(1) $$\zeta_s(x) = x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \ldots$$

for $x > 0$. Here s can be any complex number with $Re(s) > 1$.

An easy computation shows that the function $\zeta_{1-s}(x)$ satisfies the Kubert identities $(*_s)$. (Here x is not an element of $\mathbf{R}/\mathbf{Z}$ but rather a positive real number. In fact, it is sometimes useful to let x take complex values also.) It will often be convenient to work with the function

$$\beta_s(x) = -s\zeta_{1-s}(x).$$

We will prove the following.

LEMMA 1. *This product* $\beta_s(x) = -s\zeta_{1-s}(x)$ *extends to a function which is defined and holomorphic in both variables for all complex* s, *and for all* x *in the simply connected region* $\mathbf{C} - (-\infty, 0]$.

Hence $\zeta_{1-s}(x)$ is defined and holomorphic in the same region, except at $s = 0$. Evidently, by analytic continuation, these functions β_s and ζ_{1-s} always satisfy the Kubert identities $(*_s)$.

Proof. Clearly the function x^{s-1} is defined and holomorphic for x in $\mathbf{C} - (-\infty, 0]$ and for all complex s. If $Re(s) < 0$, then it is easy to check that the series

$$\beta_s(x) = -s(x^{s-1} + (x+1)^{s-1} + \ldots)$$

converges to a holomorphic function. Note that

$$(2)\qquad \partial\beta_s(x)/\partial x = s\beta_{s-1}(x)\,.$$

Integrating from x to $x+1$, and then substituting $s+1$ for s, we obtain

$$(3_x)\qquad \int_x^{x+1} \beta_s(\xi)d\xi = x^s$$

whenever $Re(s) < -1$. It follows by analytic continuation that this is true when $Re(s) < 0$ also. In particular,

$$(3_1)\qquad \int_1^2 \beta_s(x)dx = 1\,.$$

Suppose inductively that $\beta_s(x)$ has been defined so as to be holomorphic in both variables for $Re(s) < n$. Then for $Re(s) < n+1$ we can set

$$\beta_s(x) = \int_1^x s\beta_{s-1}(\xi)d\xi + c_s\,,$$

choosing the constant c_s so that (3_1) is satisfied. Evidently this defines a holomorphic function which satisfies (2) and (3_1), and hence coincides with the previously defined function in the common range of definition. It follows by induction that β_s is defined for all s. □

The case where s is a non-negative integer is of particular interest. Using (2) and (3_0) or (3_1) we see inductively that the functions

$$\beta_0(x) = 1\,,$$

$$\beta_1(x) = x - \frac{1}{2},$$

$$\beta_2(x) = x^2 - x + \frac{1}{6}, \ldots$$

are polynomials with rational coefficients. By definition, $\beta_s(x)$ is the s-th *Bernoulli polynomial* for $s = 0, 1, 2, \ldots$. It can be characterized as the unique polynomial satisfying the identity

$$\int_1^n \beta_s(x)dx = 1^s + 2^s + \ldots + (n-1)^s$$

for every n. Note the symmetry condition

$$\beta_s(1-x) = (-1)^s\,\beta_s(x)\,,$$

which can be proved inductively using (3_0).

For a more explicit computation, define the *Bernoulli numbers*

$$b_0 = 1,\ b_1 = -\frac{1}{2},\ b_2 = \frac{1}{6},\ b_3 = 0,\ b_4 = -\frac{1}{30}, \ldots$$

by the formal power series

$$t/(e^t-1) = \sum b_k t^k/k!\,.$$

Then

$$\beta_s(x) = \frac{D}{e^D - I}\, x^s = \sum_0^s b_k \binom{s}{k} x^{s-k}\,,$$

where D stands for the differentiation operator d/dx. For example it follows that

$$\beta_s(0) = b_s\,.$$

To prove this formula, simply apply the inverse operator $(e^D - I)/D$ to both sides, noting by Taylor's theorem that

$$\frac{e^D - I}{D}\,\beta_s(x) = \int_x^{x+1} \beta_s(\xi)d\xi = x^s\,.$$

If we substitute $x = 1$, then the Hurwitz zeta function $\zeta_s(x)$ reduces to the Riemann zeta function $\zeta(s)$. Thus our discussion implies the following well known result. *The product*

$$-s\zeta(1-s) = \beta_s(1)$$

can be extended as a function which is holomorphic for all complex s, *and takes rational values for* $s = 0, 1, 2, \ldots$.

Next let us study the *polylogarithm function*, which is defined for any complex numbers s and z with $|z| < 1$ by the convergent power series

$$\mathscr{L}_s(z) = z + z^2/2^s + z^3/3^s + \ldots\,. \tag{4}$$

(Compare [3], [4], [6], [11], [19], [20], [22], [26].)

LEMMA 2. *This extends to a function which is defined, and holomorphic in both variables, for all complex* s *and all* z *in the simply connected region* $\mathbf{C} - [1, \infty)$.

Proof. First note the identity

$$\mathscr{L}_{s-1}(z) = z\partial\mathscr{L}_s(z)/\partial z\,. \tag{5}$$

If $Re(s) > 0$ and $|z| < 1$, then according to Jonquière:

$$\mathscr{L}_s(z) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{z}{e^t - z}\, t^{s-1}\, dt\,. \tag{6}$$

This is proved by substituting $\Sigma z^n e^{-nt}$ for $z/(e^t - z)$, and noting that

$$\int_0^\infty e^{-nt}t^{s-1}dt = \int_0^\infty e^{-u}u^{s-1}du/n^s = \Gamma(s)/n^s\,.$$

Now if $Re(s) > 0$, the right side of (6) clearly defines a function which is holomorphic in both variables for all $z \in \mathbf{C} - [1, \infty)$. The extension to other values of s follows inductively using (5). □

The polylogarithm function satisfies a multiplicative analogue of the Kubert identities. For any positive integer m:

$$(7) \qquad \mathscr{L}_s(z) = m^{s-1} \textstyle\sum_{w^m = z} \mathscr{L}_s(w),$$

to be summed over all m-th roots of z. This is proved by a straightforward power series computation when $|z| < 1$, and by analytic continuation otherwise.

It will be convenient to introduce the abbreviation

$$l_s(x) = \mathscr{L}_s(e^{2\pi i x})$$

for $x \in \mathbf{R}/\mathbf{Z}$, $x \neq 0$, and for all complex s. Evidently $l_s(x)$ satisfies the Kubert identities in their original form $(*_s)$, and also the identity

$$(8) \qquad \partial l_s(x)/\partial x = 2\pi i \, l_{s-1}(x).$$

If $Re(s) > 1$, then we can write

$$l_s(x) = \textstyle\sum \cos(2\pi n x)/n^s + \sum i \sin(2\pi n x)/n^s,$$

where the two summands on the right are the even and odd parts of $l_s(x)$. (If s is real, these can be identified with the real and imaginary parts of $l_s(x)$.)

For integer values of the parameter s, the functions $\mathscr{L}_s(z)$ and $l_s(x)$ can be described more explicitly as follows. *Summing the series*

$$\mathscr{L}_0(z) = z + z^2 + z^3 + \ldots$$

and using (5), we see inductively that the functions

$$\mathscr{L}_0(z) = z/(1-z),$$

$$\mathscr{L}_{-1}(z) = z/(1-z)^2,$$

$$\mathscr{L}_{-2}(z) = z(1+z)/(1-z)^3, \ldots$$

are rational, with rational coefficients, holomorphic in z except for a pole at $z = 1$. On the other hand, the series $z + z^2/2 + \ldots$ evidently sums to

$$\mathscr{L}_1(z) = -\log(1-z),$$

and the integral

$$\mathscr{L}_2(z) = \textstyle\int_0^z \mathscr{L}_1(w)dw/w$$

is the classical *dilogarithm function.*

For the function $l_0(x) = e^{2\pi ix}/(1-e^{2\pi ix})$, a brief computation shows that

$$l_0(x) = (-1 + i\cot(\pi x))/2\,. \tag{9}$$

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x), \ldots$. *Note in particular that* $l_s(x)$ *is either an odd or an even function according as* $s-1$ *is odd or even, for every negative integer* s.

For further information about these functions, see Appendix 1.

§3. Continuous Kubert functions

Fixing some complex parameter s, let $\mathscr{K}_s$ be the complex vector space consisting of all continuous maps

$$f : (0, 1) \to \mathbf{C}$$

which satisfy the Kubert identity

$$f(x) = m^{s-1} \textstyle\sum_{k=0}^{m-1} f\big((x+k)/m\big) \tag{$*_s$}$$

for every positive integer m, and every x in $(0, 1)$. We will prove the following.

Theorem 1. *This complex vector space* $\mathscr{K}_s$ *has dimension* 2, *spanned by one even element* $(f(x) = f(1-x))$ *and one odd element* $(f(x) = -f(1-x))$. *Each function* $f(x)$ *in* $\mathscr{K}_s$ *is necessarily real analytic.*

If $f(x)$ satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if $s = 0$. Hence an immediate consequence is the following. (Compare Lemma 5.)

Corollary. *The correspondence* $f(x) \mapsto df(x)/dx$ *maps the vector space* $\mathscr{K}_s$ *bijectively onto* $\mathscr{K}_{s-1}$, *except when* $s = 0$.

The proof of Theorem 1 will yield explicit bases for $\mathscr{K}_s$ as follows, with notations as in §2. *For* $s \neq -1, -2, -3, \ldots$, *the space* $\mathscr{K}_s$ *is spanned by the two linearly independent functions* $l_s(x)$ *and* $l_s(1-x)$. *On the other hand, for* $s \neq 0, 1, 2, \ldots$, *this space is spanned by the linearly independent functions* $\zeta_{1-s}(x)$ *and* $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of s, we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f : (0, 1) \to \mathbf{C}$ be a continuous function satisfying $(*_s)$.