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Here the symbol  $\Lambda$  stands for the function

$$\Lambda(\pi x) = - \int_0^{\pi x} \log |2 \sin \theta| d\theta = \sum_1^{\infty} \sin(2\pi n x)/2n^2,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from  $(0, 1)$  to the circle  $\mathbf{R}/\mathbf{Z}$ . For any integer constant  $s$ , §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \rightarrow U_s$$

satisfying the identities  $(*_s)$ . Here  $U_s$  is the abelian group with one generator  $u(x)$  for each  $x$  in  $\mathbf{Q}/\mathbf{Z}$  and with defining relations  $(*_s)$ .

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to  $\mathbf{Q}/\mathbf{Z}$ . For example, if  $f : (0, 1) \rightarrow \mathbf{R}$  is the essentially unique even [or odd] continuous function satisfying  $(*_s)$ , where  $s$  is an integer, does every  $\mathbf{Q}$ -linear relation between the values of  $f$  at rational arguments follow from  $(*_s)$  together with evenness [or oddness]? The Bernoulli polynomials  $\beta_s(x)$  provide obvious counterexamples; but *it is conjectured that these are the only counterexamples*. This question is settled in the relatively easy cases where the values of  $f$  on  $\mathbf{Q}/\mathbf{Z}$  are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing  $\Gamma(x)$  and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

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## §2. CLASSICAL EXAMPLES

This section describes several well known functions. Since the identities  $(*_s)$  are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant  $c$  the polynomial  $t^m - c$  factors as

$$t^m - c = \prod_{b^m=c} (t-b),$$

where  $b$  varies over all  $m$ -th roots of  $c$ . Hence, setting  $t = 1$ , we see that

$$\log |1 - c| = \sum_{b^m=c} \log |1 - b|.$$

If we define

$$f(x) = \log |1 - e^{2\pi i x}| = \log |2 \sin \pi x|,$$

then it follows that

$$f(x) = \sum_{my \equiv x \pmod{1}} f(y).$$

Thus  $f$  satisfies the Kubert identities  $(*_1)$ . Note that  $f(x)$  is defined and smooth on the open interval  $(0, 1)$ . Differentiating  $(*_1)$ , we see that the derivative

$$f'(x) = \pi \cot \pi x$$

satisfies  $(*_0)$ . Similarly, the second derivative

$$f''(x) = -\pi^2 \csc^2 \pi x$$

satisfies  $(*_{-1})$ , and so on.

Next let us look at the *Hurwitz zeta function*  $\zeta_s(x) = \zeta(s, x)$ , which is defined by the series

$$(1) \quad \zeta_s(x) = x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \dots$$

for  $x > 0$ . Here  $s$  can be any complex number with  $\operatorname{Re}(s) > 1$ .

An easy computation shows that the function  $\zeta_{1-s}(x)$  satisfies the Kubert identities  $(*_s)$ . (Here  $x$  is not an element of  $\mathbf{R}/\mathbf{Z}$  but rather a positive real number. In fact, it is sometimes useful to let  $x$  take complex values also.) It will often be convenient to work with the function

$$\beta_s(x) = -s\zeta_{1-s}(x).$$

We will prove the following.

**LEMMA 1.** *This product  $\beta_s(x) = -s\zeta_{1-s}(x)$  extends to a function which is defined and holomorphic in both variables for all complex  $s$ , and for all  $x$  in the simply connected region  $\mathbf{C} - (-\infty, 0]$ .*

Hence  $\zeta_{1-s}(x)$  is defined and holomorphic in the same region, except at  $s = 0$ . Evidently, by analytic continuation, these functions  $\beta_s$  and  $\zeta_{1-s}$  always satisfy the Kubert identities  $(*_s)$ .

*Proof.* Clearly the function  $x^{s-1}$  is defined and holomorphic for  $x$  in  $\mathbf{C} - (-\infty, 0]$  and for all complex  $s$ . If  $\operatorname{Re}(s) < 0$ , then it is easy to check that the series

$$\beta_s(x) = -s(x^{s-1} + (x+1)^{s-1} + \dots)$$

converges to a holomorphic function. Note that

$$(2) \quad \partial \beta_s(x) / \partial x = s \beta_{s-1}(x).$$

Integrating from  $x$  to  $x + 1$ , and then substituting  $s + 1$  for  $s$ , we obtain

$$(3_x) \quad \int_x^{x+1} \beta_s(\xi) d\xi = x^s$$

whenever  $\operatorname{Re}(s) < -1$ . It follows by analytic continuation that this is true when  $\operatorname{Re}(s) < 0$  also. In particular,

$$(3_1) \quad \int_1^2 \beta_s(x) dx = 1.$$

Suppose inductively that  $\beta_s(x)$  has been defined so as to be holomorphic in both variables for  $\operatorname{Re}(s) < n$ . Then for  $\operatorname{Re}(s) < n + 1$  we can set

$$\beta_s(x) = \int_1^x s \beta_{s-1}(\xi) d\xi + c_s,$$

choosing the constant  $c_s$  so that  $(3_1)$  is satisfied. Evidently this defines a holomorphic function which satisfies (2) and  $(3_1)$ , and hence coincides with the previously defined function in the common range of definition. It follows by induction that  $\beta_s$  is defined for all  $s$ .  $\square$

The case where  $s$  is a non-negative integer is of particular interest. Using (2) and  $(3_0)$  or  $(3_1)$  we see inductively that the functions

$$\beta_0(x) = 1,$$

$$\beta_1(x) = x - \frac{1}{2},$$

$$\beta_2(x) = x^2 - x + \frac{1}{6}, \dots$$

are polynomials with rational coefficients. By definition,  $\beta_s(x)$  is the  $s$ -th *Bernoulli polynomial* for  $s = 0, 1, 2, \dots$ . It can be characterized as the unique polynomial satisfying the identity

$$\int_1^n \beta_s(x) dx = 1^s + 2^s + \dots + (n-1)^s$$

for every  $n$ . Note the symmetry condition

$$\beta_s(1-x) = (-1)^s \beta_s(x),$$

which can be proved inductively using  $(3_0)$ .

For a more explicit computation, define the *Bernoulli numbers*

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, \dots$$

by the formal power series

$$t/(e^t - 1) = \sum b_k t^k/k!.$$

Then

$$\beta_s(x) = \frac{D}{e^D - I} x^s = \sum_0^s b_k \binom{s}{k} x^{s-k},$$

where  $D$  stands for the differentiation operator  $d/dx$ . For example it follows that

$$\beta_s(0) = b_s.$$

To prove this formula, simply apply the inverse operator  $(e^D - I)/D$  to both sides, noting by Taylor's theorem that

$$\frac{e^D - I}{D} \beta_s(x) = \int_x^{x+1} \beta_s(\xi) d\xi = x^s.$$

If we substitute  $x = 1$ , then the Hurwitz zeta function  $\zeta_s(x)$  reduces to the Riemann zeta function  $\zeta(s)$ . Thus our discussion implies the following well known result. *The product*

$$-s\zeta(1-s) = \beta_s(1)$$

can be extended as a function which is holomorphic for all complex  $s$ , and takes rational values for  $s = 0, 1, 2, \dots$ .

Next let us study the *polylogarithm function*, which is defined for any complex numbers  $s$  and  $z$  with  $|z| < 1$  by the convergent power series

$$(4) \quad \mathcal{L}_s(z) = z + z^2/2^s + z^3/3^s + \dots$$

(Compare [3], [4], [6], [11], [19], [20], [22], [26].)

LEMMA 2. *This extends to a function which is defined, and holomorphic in both variables, for all complex  $s$  and all  $z$  in the simply connected region  $\mathbb{C} - [1, \infty)$ .*

*Proof.* First note the identity

$$(5) \quad \mathcal{L}_{s-1}(z) = z \partial \mathcal{L}_s(z) / \partial z.$$

If  $\operatorname{Re}(s) > 0$  and  $|z| < 1$ , then according to Jonquière:

$$(6) \quad \mathcal{L}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z}{e^t - z} t^{s-1} dt.$$

This is proved by substituting  $\Sigma z^n e^{-nt}$  for  $z/(e^t - z)$ , and noting that

$$\int_0^\infty e^{-nt} t^{s-1} dt = \int_0^\infty e^{-u} u^{s-1} du / n^s = \Gamma(s) / n^s.$$

Now if  $\operatorname{Re}(s) > 0$ , the right side of (6) clearly defines a function which is holomorphic in both variables for all  $z \in \mathbf{C} - [1, \infty)$ . The extension to other values of  $s$  follows inductively using (5).  $\square$

The polylogarithm function satisfies a multiplicative analogue of the Kubert identities. For any positive integer  $m$ :

$$(7) \quad \mathcal{L}_s(z) = m^{s-1} \sum_{w^m=z} \mathcal{L}_s(w),$$

to be summed over all  $m$ -th roots of  $z$ . This is proved by a straightforward power series computation when  $|z| < 1$ , and by analytic continuation otherwise.

It will be convenient to introduce the abbreviation

$$l_s(x) = \mathcal{L}_s(e^{2\pi i x})$$

for  $x \in \mathbf{R}/\mathbf{Z}$ ,  $x \neq 0$ , and for all complex  $s$ . Evidently  $l_s(x)$  satisfies the Kubert identities in their original form  $(*)_s$ , and also the identity

$$(8) \quad \partial l_s(x)/\partial x = 2\pi i l_{s-1}(x).$$

If  $\operatorname{Re}(s) > 1$ , then we can write

$$l_s(x) = \sum \cos(2\pi n x)/n^s + \sum i \sin(2\pi n x)/n^s,$$

where the two summands on the right are the even and odd parts of  $l_s(x)$ . (If  $s$  is real, these can be identified with the real and imaginary parts of  $l_s(x)$ .)

For integer values of the parameter  $s$ , the functions  $\mathcal{L}_s(z)$  and  $l_s(x)$  can be described more explicitly as follows. *Summing the series*

$$\mathcal{L}_0(z) = z + z^2 + z^3 + \dots$$

and using (5), we see inductively that the functions

$$\mathcal{L}_0(z) = z/(1-z),$$

$$\mathcal{L}_{-1}(z) = z/(1-z)^2,$$

$$\mathcal{L}_{-2}(z) = z(1+z)/(1-z)^3, \dots$$

are rational, with rational coefficients, holomorphic in  $z$  except for a pole at  $z = 1$ . On the other hand, the series  $z + z^2/2 + \dots$  evidently sums to

$$\mathcal{L}_1(z) = -\log(1-z),$$

and the integral

$$\mathcal{L}_2(z) = \int_0^z \mathcal{L}_1(w) dw/w$$

is the classical *dilogarithm function*.

For the function  $l_0(x) = e^{2\pi ix}/(1 - e^{2\pi ix})$ , a brief computation shows that

$$(9) \quad l_0(x) = (-1 + i \cot(\pi x))/2.$$

Differentiating this expression, we obtain corresponding formulas for  $l_{-1}(x)$ ,  $l_{-2}(x)$ , ... . Note in particular that  $l_s(x)$  is either an odd or an even function according as  $s - 1$  is odd or even, for every negative integer  $s$ .

For further information about these functions, see Appendix 1.

### §3. CONTINUOUS KUBERT FUNCTIONS

Fixing some complex parameter  $s$ , let  $\mathcal{K}_s$  be the complex vector space consisting of all continuous maps

$$f : (0, 1) \rightarrow \mathbb{C}$$

which satisfy the Kubert identity

$$(*_s) \quad f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer  $m$ , and every  $x$  in  $(0, 1)$ . We will prove the following.

**THEOREM 1.** *This complex vector space  $\mathcal{K}_s$  has dimension 2, spanned by one even element ( $f(x) = f(1-x)$ ) and one odd element ( $f(x) = -f(1-x)$ ). Each function  $f(x)$  in  $\mathcal{K}_s$  is necessarily real analytic.*

If  $f(x)$  satisfies  $(*_s)$ , then evidently the derivative of  $f$  satisfies  $(*_{s-1})$ . Note that a non-zero constant function satisfies  $(*_s)$  if and only if  $s = 0$ . Hence an immediate consequence is the following. (Compare Lemma 5.)

**COROLLARY.** *The correspondence  $f(x) \mapsto df(x)/dx$  maps the vector space  $\mathcal{K}_s$  bijectively onto  $\mathcal{K}_{s-1}$ , except when  $s = 0$ .*

The proof of Theorem 1 will yield explicit bases for  $\mathcal{K}_s$  as follows, with notations as in §2. For  $s \neq -1, -2, -3, \dots$ , the space  $\mathcal{K}_s$  is spanned by the two linearly independent functions  $l_s(x)$  and  $l_s(1-x)$ . On the other hand, for  $s \neq 0, 1, 2, \dots$ , this space is spanned by the linearly independent functions  $\zeta_{1-s}(x)$  and  $\zeta_{1-s}(1-x)$ .

Thus, for every non-integer value of  $s$ , we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let  $f : (0, 1) \rightarrow \mathbb{C}$  be a continuous function satisfying  $(*_s)$ .