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Artikel: ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

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Here the symbol Λ stands for the function

$$\Lambda(\pi x) = -\int_0^{\pi x} \log|2\sin\theta|\, d\theta = \sum_1^{\infty} \sin(2\pi n x)/2n^2,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from $(0, 1)$ to the circle $\mathbf{R}/\mathbf{Z}$. For any *integer* constant s, §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \to U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in $\mathbf{Q}/\mathbf{Z}$ and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to $\mathbf{Q}/\mathbf{Z}$. For example, if $f : (0, 1) \to \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every $\mathbf{Q}$-linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples.* This question is settled in the relatively easy cases where the values of f on $\mathbf{Q}/\mathbf{Z}$ are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

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§2. Classical examples

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m = c} (t - b),$$

where b varies over all m-th roots of c. Hence, setting $t = 1$, we see that

$$\log|1 - c| = \sum_{b^m = c} \log|1 - b|.$$

If we define

$$f(x) = \log | 1 - e^{2\pi i x} | = \log | 2 \sin \pi x | ,$$

then it follows that

$$f(x) = \sum_{my \equiv x \bmod 1} f(y) .$$

Thus f satisfies the Kubert identities $(*_1)$. Note that $f(x)$ is defined and smooth on the open interval (0, 1). Differentiating $(*_1)$, we see that the derivative

$$f'(x) = \pi \cot \pi x$$

satisfies $(*_0)$. Similarly, the second derivative

$$f''(x) = -\pi^2 \csc^2 \pi x$$

satisfies $(*_{-1})$, and so on.

Next let us look at the *Hurwitz zeta function* $\zeta_s(x) = \zeta(s, x)$, which is defined by the series

$$\zeta_s(x) = x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \ldots \tag{1}$$

for $x > 0$. Here s can be any complex number with $Re(s) > 1$.

An easy computation shows that the function $\zeta_{1-s}(x)$ satisfies the Kubert identities $(*_s)$. (Here x is not an element of $\mathbf{R}/\mathbf{Z}$ but rather a positive real number. In fact, it is sometimes useful to let x take complex values also.) It will often be convenient to work with the function

$$\beta_s(x) = -s\zeta_{1-s}(x) .$$

We will prove the following.

LEMMA 1. *This product* $\beta_s(x) = -s\zeta_{1-s}(x)$ *extends to a function which is defined and holomorphic in both variables for all complex* s, *and for all* x *in the simply connected region* $\mathbf{C} - (-\infty, 0]$.

Hence $\zeta_{1-s}(x)$ is defined and holomorphic in the same region, except at $s = 0$. Evidently, by analytic continuation, these functions β_s and ζ_{1-s} always satisfy the Kubert identities $(*_s)$.

Proof. Clearly the function x^{s-1} is defined and holomorphic for x in $\mathbf{C} - (-\infty, 0]$ and for all complex s. If $Re(s) < 0$, then it is easy to check that the series

$$\beta_s(x) = -s(x^{s-1} + (x+1)^{s-1} + \ldots)$$

converges to a holomorphic function. Note that

$$(2) \qquad \partial\beta_s(x)/\partial x = s\beta_{s-1}(x)\,.$$

Integrating from x to $x + 1$, and then substituting $s + 1$ for s, we obtain

$$(3_x) \qquad \int_x^{x+1} \beta_s(\xi)d\xi = x^s$$

whenever $Re(s) < -1$. It follows by analytic continuation that this is true when $Re(s) < 0$ also. In particular,

$$(3_1) \qquad \int_1^2 \beta_s(x)dx = 1\,.$$

Suppose inductively that $\beta_s(x)$ has been defined so as to be holomorphic in both variables for $Re(s) < n$. Then for $Re(s) < n + 1$ we can set

$$\beta_s(x) = \int_1^x s\beta_{s-1}(\xi)d\xi + c_s\,,$$

choosing the constant c_s so that (3_1) is satisfied. Evidently this defines a holomorphic function which satisfies (2) and (3_1), and hence coincides with the previously defined function in the common range of definition. It follows by induction that β_s is defined for all s. □

The case where s is a non-negative integer is of particular interest. Using (2) and (3_0) or (3_1) we see inductively that the functions

$$\beta_0(x) = 1\,,$$

$$\beta_1(x) = x - \frac{1}{2},$$

$$\beta_2(x) = x^2 - x + \frac{1}{6}, \ldots$$

are polynomials with rational coefficients. By definition, $\beta_s(x)$ is the s-th *Bernoulli polynomial* for $s = 0, 1, 2, \ldots$. It can be characterized as the unique polynomial satisfying the identity

$$\int_1^n \beta_s(x)dx = 1^s + 2^s + \ldots + (n-1)^s$$

for every n. Note the symmetry condition

$$\beta_s(1-x) = (-1)^s \beta_s(x)\,,$$

which can be proved inductively using (3_0).

For a more explicit computation, define the *Bernoulli numbers*

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, \ldots$$

by the formal power series

$$t/(e^t - 1) = \sum b_k t^k/k!\,.$$

Then

$$\beta_s(x) = \frac{D}{e^D - I}\, x^s = \sum_0^s b_k \binom{s}{k} x^{s-k}\,,$$

where D stands for the differentiation operator d/dx. For example it follows that

$$\beta_s(0) = b_s\,.$$

To prove this formula, simply apply the inverse operator $(e^D - I)/D$ to both sides, noting by Taylor's theorem that

$$\frac{e^D - I}{D}\,\beta_s(x) = \int_x^{x+1} \beta_s(\xi)d\xi = x^s\,.$$

If we substitute $x = 1$, then the Hurwitz zeta function $\zeta_s(x)$ reduces to the Riemann zeta function $\zeta(s)$. Thus our discussion implies the following well known result. *The product*

$$-s\zeta(1-s) = \beta_s(1)$$

can be extended as a function which is holomorphic for all complex s, *and takes rational values for* $s = 0, 1, 2, \ldots$.

Next let us study the *polylogarithm function*, which is defined for any complex numbers s and z with $|z| < 1$ by the convergent power series

$$\mathscr{L}_s(z) = z + z^2/2^s + z^3/3^s + \ldots\,. \tag{4}$$

(Compare [3], [4], [6], [11], [19], [20], [22], [26].)

LEMMA 2. *This extends to a function which is defined, and holomorphic in both variables, for all complex* s *and all* z *in the simply connected region* $\mathbf{C} - [1, \infty)$.

Proof. First note the identity

$$\mathscr{L}_{s-1}(z) = z\partial\mathscr{L}_s(z)/\partial z\,. \tag{5}$$

If $Re(s) > 0$ and $|z| < 1$, then according to Jonquière:

$$\mathscr{L}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z}{e^t - z}\, t^{s-1}\, dt\,. \tag{6}$$

This is proved by substituting $\Sigma z^n e^{-nt}$ for $z/(e^t - z)$, and noting that

$$\int_0^\infty e^{-nt} t^{s-1} dt = \int_0^\infty e^{-u} u^{s-1} du/n^s = \Gamma(s)/n^s\,.$$

Now if $Re(s) > 0$, the right side of (6) clearly defines a function which is holomorphic in both variables for all $z \in \mathbf{C} - [1, \infty)$. The extension to other values of s follows inductively using (5). □

The polylogarithm function satisfies a multiplicative analogue of the Kubert identities. For any positive integer m:

$$\mathscr{L}_s(z) = m^{s-1} \sum_{w^m = z} \mathscr{L}_s(w)\,, \tag{7}$$

to be summed over all m-th roots of z. This is proved by a straightforward power series computation when $|z| < 1$, and by analytic continuation otherwise.

It will be convenient to introduce the abbreviation

$$l_s(x) = \mathscr{L}_s(e^{2\pi i x})$$

for $x \in \mathbf{R}/\mathbf{Z}$, $x \neq 0$, and for all complex s. Evidently $l_s(x)$ satisfies the Kubert identities in their original form $(*_s)$, and also the identity

$$\partial l_s(x)/\partial x = 2\pi i\, l_{s-1}(x)\,. \tag{8}$$

If $Re(s) > 1$, then we can write

$$l_s(x) = \sum \cos(2\pi n x)/n^s + \sum i \sin(2\pi n x)/n^s\,,$$

where the two summands on the right are the even and odd parts of $l_s(x)$. (If s is real, these can be identified with the real and imaginary parts of $l_s(x)$.)

For integer values of the parameter s, the functions $\mathscr{L}_s(z)$ and $l_s(x)$ can be described more explicitly as follows. *Summing the series*

$$\mathscr{L}_0(z) = z + z^2 + z^3 + \ldots$$

and using (5), we see inductively that the functions

$$\mathscr{L}_0(z) = z/(1-z),$$

$$\mathscr{L}_{-1}(z) = z/(1-z)^2,$$

$$\mathscr{L}_{-2}(z) = z(1+z)/(1-z)^3, \ldots$$

are rational, with rational coefficients, holomorphic in z *except for a pole at* $z = 1$. On the other hand, the series $z + z^2/2 + \ldots$ evidently sums to

$$\mathscr{L}_1(z) = -\log(1-z)\,,$$

and the integral

$$\mathscr{L}_2(z) = \int_0^z \mathscr{L}_1(w)dw/w$$

is the classical *dilogarithm function.*

For the function $l_0(x) = e^{2\pi ix}/(1-e^{2\pi ix})$, a brief computation shows that

$$l_0(x) = (-1 + i\cot(\pi x))/2\,. \tag{9}$$

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x), \dots$. *Note in particular that* $l_s(x)$ *is either an odd or an even function according as* $s-1$ *is odd or even, for every negative integer* s.

For further information about these functions, see Appendix 1.

§3. Continuous Kubert functions

Fixing some complex parameter s, let $\mathscr{K}_s$ be the complex vector space consisting of all continuous maps

$$f : (0, 1) \to \mathbf{C}$$

which satisfy the Kubert identity

$$f(x) = m^{s-1}\textstyle\sum_{k=0}^{m-1} f((x+k)/m) \tag{$*_s$}$$

for every positive integer m, and every x in $(0, 1)$. We will prove the following.

THEOREM 1. *This complex vector space* $\mathscr{K}_s$ *has dimension* 2, *spanned by one even element* $(f(x) = f(1-x))$ *and one odd element* $(f(x) = -f(1-x))$. *Each function* $f(x)$ *in* $\mathscr{K}_s$ *is necessarily real analytic.*

If $f(x)$ satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if $s = 0$. Hence an immediate consequence is the following. (Compare Lemma 5.)

COROLLARY. *The correspondence* $f(x) \mapsto df(x)/dx$ *maps the vector space* $\mathscr{K}_s$ *bijectively onto* $\mathscr{K}_{s-1}$, *except when* $s = 0$.

The proof of Theorem 1 will yield explicit bases for $\mathscr{K}_s$ as follows, with notations as in §2. *For* $s \neq -1, -2, -3, \dots,$ *the space* $\mathscr{K}_s$ *is spanned by the two linearly independent functions* $l_s(x)$ *and* $l_s(1-x)$. *On the other hand, for* $s \neq 0, 1, 2, \dots,$ *this space is spanned by the linearly independent functions* $\zeta_{1-s}(x)$ *and* $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of s, we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f : (0, 1) \to \mathbf{C}$ be a continuous function satisfying $(*_s)$.