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ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

by John MILNOR

§1. INTRODUCTION

D. Kubert [12] has studied functions $f(x)$, where x varies over $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$, which satisfy the identity

$$(*_s) \qquad f(x) = m^{s-1} \textstyle\sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m. (See also Lang [16-18], as well as Kubert and Lang [13-15].) Here s is some fixed parameter. Note that $(x+k)/m$ varies precisely over all solutions y to the equation $my = x$ in the group $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$. However, the equation is set up so that it also makes sense for x in the interval $(0, 1)$ or $(0, \infty)$. Evidently it would suffice to assume the equation $(*_s)$ for prime values of m.

Classical examples of such functions are provided by the uniformly convergent Fourier series $l_s(x) = \sum_{n=1}^{\infty} e^{2\pi inx}/n^s$ for $x \in \mathbf{R}/\mathbf{Z}$ and $Re(s) > 1$, the Hurwitz function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \ldots$$

for $0 < x$ and $Re(s) < 0$, and by the Bernoulli polynomial $\beta_s(x)$ of degree s for $s = 0, 1, 2, 3, \ldots,$. See §2.

For each complex constant s, it is shown in §3 that there are exactly two linearly independent functions, defined and continuous on the open interval $(0, 1)$, which satisfy these Kubert identities $(*_s)$. The two generators may be chosen so that one is even and one is odd under the involution $f(x) \mapsto f(1-x)$. They are then uniquely determined up to a multiplicative constant. Here is a table of examples, for small integer values of s.

=	-2	-1	0	1	2
en	$\zeta_3(x) + \zeta_3(1-x)$	$csc^2 \pi x$	$\beta_0(x) = 1$	$\log(2 \sin \pi x)$	$\beta_2(x) = x^2 - x + \frac{1}{6}$
dd	$\cos \pi x / \sin^3 \pi x$	$\zeta_2(x) - \zeta_2(1-x)$	$\cot \pi x$	$\beta_1(x) = x - \frac{1}{2}$	$\Lambda(\pi x)$

Here the symbol Λ stands for the function

$$\Lambda(\pi x) = -\int_0^{\pi x} \log|2\sin\theta|\,d\theta = \sum_1^\infty \sin(2\pi n x)/2n^2\,,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from (0, 1) to the circle $\mathbf{R}/\mathbf{Z}$. For any *integer* constant s, §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \to U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in $\mathbf{Q}/\mathbf{Z}$ and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to $\mathbf{Q}/\mathbf{Z}$. For example, if $f : (0, 1) \to \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every $\mathbf{Q}$-linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples.* This question is settled in the relatively easy cases where the values of f on $\mathbf{Q}/\mathbf{Z}$ are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

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§2. Classical examples

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m = c} (t-b)\,,$$

where b varies over all m-th roots of c. Hence, setting $t = 1$, we see that

$$\log|1-c| = \sum_{b^m=c} \log|1-b|\,.$$