

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 29 (1983)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES  
**Autor:** Milnor, John  
**Kapitel:** §1. Introduction  
**DOI:** <https://doi.org/10.5169/seals-52983>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

by John MILNOR

Cat

### §1. INTRODUCTION

D. Kubert [12] has studied functions  $f(x)$ , where  $x$  varies over  $\mathbf{Q}/\mathbf{Z}$  or  $\mathbf{R}/\mathbf{Z}$ , which satisfy the identity

$$(*) \quad f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer  $m$ . (See also Lang [16-18], as well as Kubert and Lang [13-15].) Here  $s$  is some fixed parameter. Note that  $(x+k)/m$  varies precisely over all solutions  $y$  to the equation  $my = x$  in the group  $\mathbf{Q}/\mathbf{Z}$  or  $\mathbf{R}/\mathbf{Z}$ . However, the equation is set up so that it also makes sense for  $x$  in the interval  $(0, 1)$  or  $(0, \infty)$ . Evidently it would suffice to assume the equation  $(*)_s$  for prime values of  $m$ .

Classical examples of such functions are provided by the uniformly convergent Fourier series  $l_s(x) = \sum_{n=1}^{\infty} e^{2\pi i n x} / n^s$  for  $x \in \mathbf{R}/\mathbf{Z}$  and  $\text{Re}(s) > 1$ , the Hurwitz function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \dots$$

for  $0 < x$  and  $\text{Re}(s) < 0$ , and by the Bernoulli polynomial  $\beta_s(x)$  of degree  $s$  for  $s = 0, 1, 2, 3, \dots$ . See §2.

For each complex constant  $s$ , it is shown in §3 that there are exactly two linearly independent functions, defined and continuous on the open interval  $(0, 1)$ , which satisfy these Kubert identities  $(*)_s$ . The two generators may be chosen so that one is even and one is odd under the involution  $f(x) \mapsto f(1-x)$ . They are then uniquely determined up to a multiplicative constant. Here is a table of examples, for small integer values of  $s$ .

	-2	-1	0	1	2
even	$\zeta_3(x) + \zeta_3(1-x)$	$\csc^2 \pi x$	$\beta_0(x) = 1$	$\log(2 \sin \pi x)$	$\beta_2(x) = x^2 - x + \frac{1}{6}$
odd	$\cos \pi x / \sin^3 \pi x$	$\zeta_2(x) - \zeta_2(1-x)$	$\cot \pi x$	$\beta_1(x) = x - \frac{1}{2}$	$\Lambda(\pi x)$

Here the symbol  $\Lambda$  stands for the function

$$\Lambda(\pi x) = - \int_0^{\pi x} \log | 2 \sin \theta | d\theta = \sum_1^{\infty} \sin(2\pi n x)/2n^2,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from  $(0, 1)$  to the circle  $\mathbf{R}/\mathbf{Z}$ . For any integer constant  $s$ , §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \rightarrow U_s$$

satisfying the identities  $(*_s)$ . Here  $U_s$  is the abelian group with one generator  $u(x)$  for each  $x$  in  $\mathbf{Q}/\mathbf{Z}$  and with defining relations  $(*_s)$ .

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to  $\mathbf{Q}/\mathbf{Z}$ . For example, if  $f : (0, 1) \rightarrow \mathbf{R}$  is the essentially unique even [or odd] continuous function satisfying  $(*_s)$ , where  $s$  is an integer, does every  $\mathbf{Q}$ -linear relation between the values of  $f$  at rational arguments follow from  $(*_s)$  together with evenness [or oddness]? The Bernoulli polynomials  $\beta_s(x)$  provide obvious counterexamples; but *it is conjectured that these are the only counterexamples*. This question is settled in the relatively easy cases where the values of  $f$  on  $\mathbf{Q}/\mathbf{Z}$  are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing  $\Gamma(x)$  and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

The author is indebted to conversations with S. Chowla, B. H. Gross, Werner Meyer, and W. Sinnott.

## §2. CLASSICAL EXAMPLES

This section describes several well known functions. Since the identities  $(*_s)$  are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant  $c$  the polynomial  $t^m - c$  factors as

$$t^m - c = \prod_{b^m=c} (t-b),$$

where  $b$  varies over all  $m$ -th roots of  $c$ . Hence, setting  $t = 1$ , we see that

$$\log | 1 - c | = \sum_{b^m=c} \log | 1 - b |.$$