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ON POLYLOGARITHMS, HURWITZ ZETA FUNCTIONS, AND THE KUBERT IDENTITIES

by John MILNOR

§1. INTRODUCTION

D. Kubert [12] has studied functions $f(x)$, where x varies over $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$, which satisfy the identity

$$(*_s) \qquad f(x) = m^{s-1} \sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m. (See also Lang [16-18], as well as Kubert and Lang [13-15].) Here s is some fixed parameter. Note that $(x+k)/m$ varies precisely over all solutions y to the equation $my = x$ in the group $\mathbf{Q}/\mathbf{Z}$ or $\mathbf{R}/\mathbf{Z}$. However, the equation is set up so that it also makes sense for x in the interval $(0, 1)$ or $(0, \infty)$. Evidently it would suffice to assume the equation $(*_s)$ for prime values of m.

Classical examples of such functions are provided by the uniformly convergent Fourier series $l_s(x) = \sum_{n=1}^{\infty} e^{2\pi i n x}/n^s$ for $x \in \mathbf{R}/\mathbf{Z}$ and $Re(s) > 1$, the Hurwitz function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \ldots$$

for $0 < x$ and $Re(s) < 0$, and by the Bernoulli polynomial $\beta_s(x)$ of degree s for $s = 0, 1, 2, 3, \ldots,$. See §2.

For each complex constant s, it is shown in §3 that there are exactly two linearly independent functions, defined and continuous on the open interval $(0, 1)$, which satisfy these Kubert identities $(*_s)$. The two generators may be chosen so that one is even and one is odd under the involution $f(x) \mapsto f(1-x)$. They are then uniquely determined up to a multiplicative constant. Here is a table of examples, for small integer values of s.

s	-2	-1	0	1	2
even	$\zeta_3(x) + \zeta_3(1-x)$	$csc^2 \pi x$	$\beta_0(x) = 1$	$\log(2 \sin \pi x)$	$\beta_2(x) = x^2 - x + \frac{1}{6}$
odd	$\cos \pi x / \sin^3 \pi x$	$\zeta_2(x) - \zeta_2(1-x)$	$\cot \pi x$	$\beta_1(x) = x - \frac{1}{2}$	$\Lambda(\pi x)$

Here the symbol Λ stands for the function

$$\Lambda(\pi x) = - \int_0^{\pi x} \log |2 \sin \theta| \, d\theta = \sum_1^\infty \sin(2\pi n x)/2n^2 ,$$

which is closely related to Lobachevsky's computations of volume in hyperbolic 3-space. Compare Appendix 3.

Section 4 extends such functions from $(0, 1)$ to the circle $\mathbf{R}/\mathbf{Z}$. For any *integer* constant s, §5 computes the universal function

$$u : \mathbf{Q}/\mathbf{Z} \to U_s$$

satisfying the identities $(*_s)$. Here U_s is the abelian group with one generator $u(x)$ for each x in $\mathbf{Q}/\mathbf{Z}$ and with defining relations $(*_s)$.

Section 6 attempts to study the extent to which the continuous Kubert functions of §3 are actually universal, when restricted to $\mathbf{Q}/\mathbf{Z}$. For example, if $f : (0, 1) \to \mathbf{R}$ is the essentially unique even [or odd] continuous function satisfying $(*_s)$, where s is an integer, does every $\mathbf{Q}$-linear relation between the values of f at rational arguments follow from $(*_s)$ together with evenness [or oddness]? The Bernoulli polynomials $\beta_s(x)$ provide obvious counterexamples; but *it is conjectured that these are the only counterexamples.* This question is settled in the relatively easy cases where the values of f on $\mathbf{Q}/\mathbf{Z}$ are known to be algebraic numbers, or logarithms of algebraic numbers.

There are three appendices, one describing a functional equation relating polylogarithms and Hurwitz functions, one describing $\Gamma(x)$ and related functions, and one describing the use of dilogarithms to compute volume in Lobachevsky space.

The author is indebted to conversations with S. Chowla, B. H. Gross, Werner Meyer, and W. Sinnott.

§2. Classical examples

This section describes several well known functions. Since the identities $(*_s)$ are not immediately perspicuous, let me start with some examples where they are clearly satisfied. For any complex constant c the polynomial $t^m - c$ factors as

$$t^m - c = \prod_{b^m = c} (t-b) ,$$

where b varies over all m-th roots of c. Hence, setting $t = 1$, we see that

$$\log |1 - c| = \sum_{b^m = c} \log |1 - b| .$$

If we define

$$f(x) = \log|1 - e^{2\pi i x}| = \log|2 \sin \pi x|,$$

then it follows that

$$f(x) = \sum_{my \equiv x \bmod 1} f(y).$$

Thus f satisfies the Kubert identities $(*_1)$. Note that $f(x)$ is defined and smooth on the open interval (0, 1). Differentiating $(*_1)$, we see that the derivative

$$f'(x) = \pi \cot \pi x$$

satisfies $(*_0)$. Similarly, the second derivative

$$f''(x) = -\pi^2 \csc^2 \pi x$$

satisfies $(*_{-1})$, and so on.

Next let us look at the *Hurwitz zeta function* $\zeta_s(x) = \zeta(s, x)$, which is defined by the series

$$(1) \qquad \zeta_s(x) = x^{-s} + (x+1)^{-s} + (x+2)^{-s} + \ldots$$

for $x > 0$. Here s can be any complex number with $Re(s) > 1$.

An easy computation shows that the function $\zeta_{1-s}(x)$ satisfies the Kubert identities $(*_s)$. (Here x is not an element of $\mathbf{R}/\mathbf{Z}$ but rather a positive real number. In fact, it is sometimes useful to let x take complex values also.) It will often be convenient to work with the function

$$\beta_s(x) = -s\zeta_{1-s}(x).$$

We will prove the following.

LEMMA 1. *This product* $\beta_s(x) = -s\zeta_{1-s}(x)$ *extends to a function which is defined and holomorphic in both variables for all complex* s, *and for all* x *in the simply connected region* $\mathbf{C} - (-\infty, 0]$.

Hence $\zeta_{1-s}(x)$ is defined and holomorphic in the same region, except at $s = 0$. Evidently, by analytic continuation, these functions β_s and ζ_{1-s} always satisfy the Kubert identities $(*_s)$.

Proof. Clearly the function x^{s-1} is defined and holomorphic for x in $\mathbf{C} - (-\infty, 0]$ and for all complex s. If $Re(s) < 0$, then it is easy to check that the series

$$\beta_s(x) = -s(x^{s-1} + (x+1)^{s-1} + \ldots)$$

converges to a holomorphic function. Note that

$$(2) \qquad \partial\beta_s(x)/\partial x = s\beta_{s-1}(x)\,.$$

Integrating from x to $x + 1$, and then substituting $s + 1$ for s, we obtain

$$(3_x) \qquad \int_x^{x+1} \beta_s(\xi)d\xi = x^s$$

whenever $Re(s) < -1$. It follows by analytic continuation that this is true when $Re(s) < 0$ also. In particular,

$$(3_1) \qquad \int_1^2 \beta_s(x)dx = 1\,.$$

Suppose inductively that $\beta_s(x)$ has been defined so as to be holomorphic in both variables for $Re(s) < n$. Then for $Re(s) < n + 1$ we can set

$$\beta_s(x) = \int_1^x s\beta_{s-1}(\xi)d\xi + c_s\,,$$

choosing the constant c_s so that (3_1) is satisfied. Evidently this defines a holomorphic function which satisfies (2) and (3_1), and hence coincides with the previously defined function in the common range of definition. It follows by induction that β_s is defined for all s. □

The case where s is a non-negative integer is of particular interest. Using (2) and (3_0) or (3_1) we see inductively that the functions

$$\beta_0(x) = 1\,,$$

$$\beta_1(x) = x - \frac{1}{2}\,,$$

$$\beta_2(x) = x^2 - x + \frac{1}{6}\,, \ldots$$

are polynomials with rational coefficients. By definition, $\beta_s(x)$ is the s-th *Bernoulli polynomial* for $s = 0, 1, 2, \ldots$. It can be characterized as the unique polynomial satisfying the identity

$$\int_1^n \beta_s(x)dx = 1^s + 2^s + \ldots + (n-1)^s$$

for every n. Note the symmetry condition

$$\beta_s(1-x) = (-1)^s\,\beta_s(x)\,,$$

which can be proved inductively using (3_0).

For a more explicit computation, define the *Bernoulli numbers*

$$b_0 = 1,\, b_1 = -\frac{1}{2},\, b_2 = \frac{1}{6},\, b_3 = 0,\, b_4 = -\frac{1}{30}, \ldots$$

by the formal power series

$$t/(e^t-1) = \sum b_k t^k/k!\,.$$

Then

$$\beta_s(x) = \frac{D}{e^D - I}\,x^s = \sum_0^s b_k \binom{s}{k} x^{s-k}\,,$$

where D stands for the differentiation operator d/dx. For example it follows that

$$\beta_s(0) = b_s\,.$$

To prove this formula, simply apply the inverse operator $(e^D - I)/D$ to both sides, noting by Taylor's theorem that

$$\frac{e^D - I}{D}\,\beta_s(x) = \int_x^{x+1} \beta_s(\xi)d\xi = x^s\,.$$

If we substitute $x = 1$, then the Hurwitz zeta function $\zeta_s(x)$ reduces to the Riemann zeta function $\zeta(s)$. Thus our discussion implies the following well known result. *The product*

$$-s\zeta(1-s) = \beta_s(1)$$

can be extended as a function which is holomorphic for all complex s, *and takes rational values for* $s = 0, 1, 2, \ldots$.

Next let us study the *polylogarithm function*, which is defined for any complex numbers s and z with $|z| < 1$ by the convergent power series

(4) $$\mathcal{L}_s(z) = z + z^2/2^s + z^3/3^s + \ldots\,.$$

(Compare [3], [4], [6], [11], [19], [20], [22], [26].)

LEMMA 2. *This extends to a function which is defined, and holomorphic in both variables, for all complex* s *and all* z *in the simply connected region* $\mathbf{C} - [1, \infty)$.

Proof. First note the identity

(5) $$\mathcal{L}_{s-1}(z) = z\partial\mathcal{L}_s(z)/\partial z\,.$$

If $Re(s) > 0$ and $|z| < 1$, then according to Jonquière:

(6) $$\mathcal{L}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z}{e^t - z}\, t^{s-1}\, dt\,.$$

This is proved by substituting $\Sigma z^n e^{-nt}$ for $z/(e^t - z)$, and noting that

$$\int_0^\infty e^{-nt} t^{s-1} dt = \int_0^\infty e^{-u} u^{s-1} du/n^s = \Gamma(s)/n^s\,.$$

Now if $Re(s) > 0$, the right side of (6) clearly defines a function which is holomorphic in both variables for all $z \in \mathbf{C} - [1, \infty)$. The extension to other values of s follows inductively using (5). □

The polylogarithm function satisfies a multiplicative analogue of the Kubert identities. For any positive integer m:

(7) $$\mathscr{L}_s(z) = m^{s-1} \sum_{w^m = z} \mathscr{L}_s(w),$$

to be summed over all m-th roots of z. This is proved by a straightforward power series computation when $|z| < 1$, and by analytic continuation otherwise.

It will be convenient to introduce the abbreviation

$$l_s(x) = \mathscr{L}_s(e^{2\pi i x})$$

for $x \in \mathbf{R}/\mathbf{Z}$, $x \neq 0$, and for all complex s. Evidently $l_s(x)$ satisfies the Kubert identities in their original form $(*_s)$, and also the identity

(8) $$\partial l_s(x)/\partial x = 2\pi i\, l_{s-1}(x).$$

If $Re(s) > 1$, then we can write

$$l_s(x) = \sum \cos(2\pi n x)/n^s + \sum i \sin(2\pi n x)/n^s,$$

where the two summands on the right are the even and odd parts of $l_s(x)$. (If s is real, these can be identified with the real and imaginary parts of $l_s(x)$.)

For integer values of the parameter s, the functions $\mathscr{L}_s(z)$ and $l_s(x)$ can be described more explicitly as follows. *Summing the series*

$$\mathscr{L}_0(z) = z + z^2 + z^3 + \ldots$$

and using (5), we see inductively that the functions

$$\mathscr{L}_0(z) = z/(1-z),$$

$$\mathscr{L}_{-1}(z) = z/(1-z)^2,$$

$$\mathscr{L}_{-2}(z) = z(1+z)/(1-z)^3, \ldots$$

are rational, with rational coefficients, holomorphic in z *except for a pole at* $z = 1$. On the other hand, the series $z + z^2/2 + \ldots$ evidently sums to

$$\mathscr{L}_1(z) = -\log(1-z),$$

and the integral

$$\mathscr{L}_2(z) = \int_0^z \mathscr{L}_1(w)dw/w$$

is the classical *dilogarithm function.*

For the function $l_0(x) = e^{2\pi i x}/(1-e^{2\pi i x})$, a brief computation shows that

(9) $$l_0(x) = (-1 + i\cot(\pi x))/2\,.$$

Differentiating this expression, we obtain corresponding formulas for $l_{-1}(x)$, $l_{-2}(x), \ldots$. *Note in particular that* $l_s(x)$ *is either an odd or an even function according as* $s-1$ *is odd or even, for every negative integer* s.

For further information about these functions, see Appendix 1.

§3. Continuous Kubert functions

Fixing some complex parameter s, let $\mathscr{K}_s$ be the complex vector space consisting of all continuous maps

$$f:(0,1)\to \mathbf{C}$$

which satisfy the Kubert identity

$(*_s)$ $$f(x) = m^{s-1}\textstyle\sum_{k=0}^{m-1} f((x+k)/m)$$

for every positive integer m, and every x in $(0,1)$. We will prove the following.

Theorem 1. *This complex vector space* $\mathscr{K}_s$ *has dimension* 2, *spanned by one even element* $(f(x) = f(1-x))$ *and one odd element* $(f(x) = -f(1-x))$. *Each function* $f(x)$ *in* $\mathscr{K}_s$ *is necessarily real analytic.*

If $f(x)$ satisfies $(*_s)$, then evidently the derivative of f satisfies $(*_{s-1})$. Note that a non-zero constant function satisfies $(*_s)$ if and only if $s = 0$. Hence an immediate consequence is the following. (Compare Lemma 5.)

Corollary. *The correspondence* $f(x)\mapsto df(x)/dx$ *maps the vector space* $\mathscr{K}_s$ *bijectively onto* $\mathscr{K}_{s-1}$, *except when* $s = 0$.

The proof of Theorem 1 will yield explicit bases for $\mathscr{K}_s$ as follows, with notations as in §2. *For* $s \neq -1, -2, -3, \ldots$, *the space* $\mathscr{K}_s$ *is spanned by the two linearly independent functions* $l_s(x)$ *and* $l_s(1-x)$. *On the other hand, for* $s \neq 0, 1, 2, \ldots$, *this space is spanned by the linearly independent functions* $\zeta_{1-s}(x)$ *and* $\zeta_{1-s}(1-x)$.

Thus, for every non-integer value of s, we obtain two alternative bases for the same vector space. See Appendix 1 for a precise description of the linear relations between Hurwitz zeta function and polylogarithm which are implied by this statement.

The proof of Theorem 1 will be based on several preliminary statements. Let $f:(0,1)\to\mathbf{C}$ be a continuous function satisfying $(*_s)$.

LEMMA 3. *If* $Re(s) > 0$, *then* $\int_0^1 | f(x) | \, dx$ *is finite.*

Proof. Let C be an upper bound for $| f(x) |$ on the closed interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ and let $\alpha = | 2^{1-s} | < 2$. Using the identity

$$f(x) = 2^{1-s} f(2x) - f\left(x + \frac{1}{2}\right)$$

we see that

$$|f(x)| \leqslant (\alpha+1)C \qquad \text{for} \qquad \frac{1}{8} \leqslant x \leqslant \frac{1}{4},$$

hence

$$| f(x) | \leqslant (\alpha^2+\alpha+1)C \qquad \text{for} \qquad \frac{1}{16} \leqslant x \leqslant \frac{1}{8},$$

and so on. Therefore $\int_0^{1/2} | f(x) | \, dx$ is less than the finite sum

$$C\left(\frac{1}{4} + (\alpha+1)/8 + (\alpha^2+\alpha+1)/16 + \ldots\right).$$

Applying the same argument to $f(1-x)$, this completes the proof. □

LEMMA 4. (Rohrlich) *Let* $f : (0, 1) \to \mathbf{C}$ *be a non-constant continuous function satisfying* $(*_s)$, *and suppose that*

$$\int_0^1 | f(x) | \, dx < \infty .$$

Then $Re(s) > 0$, *and* $f(x)$ *is equal to some linear combination of* $l_s(x)$ *and* $l_s(1-x)$.

Proof. We will make use of the easily proved fact that a continuous function on $(0, 1)$ with $\int_0^1 | f(x) | \, dx < \infty$ is uniquely determined by its Fourier coefficients

$$a_n = \int_0^1 f(x) e^{-2\pi i n x} \, dx .$$

Furthermore, according to the Riemann-Lebesgue Lemma, these coefficients tend to zero as $| n | \to \infty$.

If f satisfies $(*_s)$, then a straightforward computation shows that

$$a_{nm} = a_n / m^s \qquad \text{for} \qquad m = 2, 3, \ldots .$$

In particular,

$$a_{\pm m} = a_{\pm 1} / m^s .$$

Furthermore, $a_0 = 0$ except in the special case $s = 0$.

First suppose that $Re(s) \leqslant 0$. Then the numbers $1/m^s$ are bounded away from zero. Using the Riemann-Lebesgue Lemma, this implies that f has the Fourier series of a constant function, and hence is constant, contrary to our hypothesis.

Next suppose that $Re(s) > 1$. Then the series $\Sigma\ 1/m^s$ converges absolutely. Therefore the Fourier series of f

$$a_{+1} \sum_{m=1}^{\infty} e^{2\pi imx}/m^s + a_{-1} \sum_{m=1}^{\infty} e^{-2\pi imx}/m^s$$

converges uniformly on the circle $\mathbf{R}/\mathbf{Z}$ to the continuous function

$$a_{+1}\, l_s(x) + a_{-1}\, l_s(1-x)\,.$$

It follows that f is equal to this expression.

Finally, suppose that $0 < Re(s) \leqslant 1$. If F is any indefinite integral of f, then F is continuous on $[0, 1]$ by Lemma 3. We can integrate by parts to relate the Fourier coefficients of f and F; and it follows easily that F equals a linear combination of $l_{s+1}(x)$ and $l_{s+1}(1-x)$ plus a constant. Differentiating, we obtain the corresponding assertion for f. □

Proof of Theorem 1 when $Re(s) > 0$. Let $f : (0, 1) \to \mathbf{C}$ be a non-zero continuous function satisfying $(*_s)$. Then f is non-constant since $s \neq 0$. Hence f is a linear combination of $l_s(x)$ and $l_s(1-x)$ by Lemmas 3, 4. These two functions are linearly independent since they have independent Fourier expansions. □

REMARK. *If* $Re(s) > 1$, *then this proof shows also that* f *extends to a continuous function on the circle* $\mathbf{R}/\mathbf{Z}$. *Whenever* $Re(s) > 0$, *it shows that* $\int_0^1 f(x)dx = 0$.

We can extend this proof to all values of s except $-1, -2, \ldots$ by using the following lemma. Let $f : (0, 1) \to \mathbf{C}$ be a continuous function satisfying $(*_s)$, and let

$$F(x) = \int f(x)dx$$

be any indefinite integral of f.

LEMMA 5. *If* $s \neq -1$, *then there is one and only one constant* c *so that the function* $F(x) + c$ *satisfies* $(*_{s+1})$.

Proof. Integrating $(*_s)$, we have

$$F(x) = m^s \sum_{k=0}^{m-1} F((x+k)/m) + c_m$$

for some constants c_m. Comparing the formulas for different values of m, we see easily that

$$c_{lm} = m^{s+1} c_l + c_m = l^{s+1} c_m + c_l ,$$

hence

$$(m^{s+1}-1)c_l = (l^{s+1}-1)c_m .$$

These numbers $m^{s+1} - 1$ cannot all be zero, since $s \neq -1$. Therefore there exists one and only one c with

$$c_m = (m^{s+1}-1)c$$

for every m. It is now easy to check that $F + c$ has the required property, and that c is unique. □

Remark. This lemma definitely fails for $s = -1$. In fact Gauss' formula

$$\Gamma(x) = \frac{m^{x-1/2}}{(2\pi)^{(m-1)/2}} \prod_{k=0}^{m-1} \Gamma\left(\frac{x+k}{m}\right)$$

implies that the logarithmic derivative $F(x) = \Gamma'(x)/\Gamma(x)$ satisfies

$$F(x) = m^{-1} \sum_{k=0}^{m-1} F\left(\frac{x+k}{m}\right) + \log m .$$

Differentiating, we see that $F'(x)$ satisfies the Kubert identities $(*_{-1})$. (In fact $F'(x) = \zeta_2(x)$.) But there is no constant c so that $F + c$ satisfies $(*_0)$. See Appendix 2 for details.

Proof of Theorem 1 for $s \neq -1, -2, \ldots$. Given any continuous $f : (0, 1) \to \mathbf{C}$ satisfying $(*_s)$ we can integrate n times, using Lemma 5, to obtain a continuous function F satisfying $(*_{s+n})$ with $Re(s+n) > 1$. Then

$$F(x) = a l_{s+n}(x) + b l_{s+n}(1-x)$$

by Lemmas 3, 4, as above. Differentiating n times, and using (8), we see that $f(x)$ equals a linear combination of $l_s(x)$ and $l_s(1-x)$. These last two functions are linearly independent; for otherwise applying Lemma 5 n times we would obtain a contradiction. □

The proof for negative integer values of s will require a precise description of the behavior of $f(x)$ as $x \to 0$.

LEMMA 6. *If* $f : (0, 1) \to \mathbf{C}$ *is continuous and satisfies* $(*_s)$ *with* $Re(s) < 1$, *then there exists a constant* A *so that* $f(x) - Ax^{s-1}$ *tends to a finite limit as* $x \to 0$.

Proof. We will first show that the function $g(x) = f(x)/x^{s-1}$ tends to a limit A as $x \to 0$. Let $c_m = f(1/m) + f(2/m) + \ldots + f((m-1)/m)$. Then

$$f(x) = m^{s-1}\big(f(x/m) + f((x+1)/m) + \ldots + f((x+m-1)/m)\big)$$
$$= m^{s-1}\big(f(x/m) + c_m + o(1)\big)$$

as $x \to 0$. Hence

$$g(x) = g(x/m) + O(x^{1-s}),$$

and it follows easily that the sequence of functions $g(x), g(x/m), g(x/m^2), \ldots$ converges uniformly to a limit $A_m(x)$. Evidently this limit function is defined and continuous for all $x > 0$, and satisfies

$$A_m(x) = A_m(x/m).$$

Further, for any $m, n > 1$ we have

$$g(x) = A_m(x) + o(1) = A_n(x) + o(1)$$

as $x \to 0$. Therefore

$$A_m(x) = A_n(x) + o(1) = A_n(x/n) + o(1) = A_m(x/n) + o(1).$$

Substituting x/m^k for x and letting $k \to \infty$, we see that

$$A_m(x) = A_m(x/n).$$

But clearly any continuous function on the positive reals which satisfies all of these periodicity conditions must be constant. Therefore $A = A_m(x)$ is independent of m and x.

Now take $m = 2$, and define $f(0)$ by the equation $f(0) = 2^{s-1}\big(f(0) + f(1/2)\big)$. (Compare §4.) Subtracting this from $f(x) = 2^{s-1}\big(f(x/2) + f((x+1)/2)\big)$ and dividing by x^{s-1} we obtain

$$\frac{f(x) - f(0)}{x^{s-1}} = \frac{f(x/2) - f(0)}{(x/2)^{s-1}} + o(x^{1-s})$$

as $x \to 0$. Taking the corresponding statements for $x/2, x/4, \ldots$, it follows that

$$\frac{f(x) - f(0)}{x^{s-1}} = A + o(x^{1-s}),$$

or in other words

$$f(x) = Ax^{s-1} + f(0) + o(1)$$

as $x \to 0$. □

To illustrate this lemma, note that the Hurwitz zeta function

$$\zeta_{1-s}(x) = x^{s-1} + (x+1)^{s-1} + \ldots$$

is equal to the sum of x^{s-1} and a function $\zeta_{1-s}(x+1)$ which is continuous as $x \to 0$.

Proof of Theorem 1 for $Re(s) < 0$. Since $f(x) - Ax^{s-1}$ tends to a finite limit as $x \to 0$, it follows that $f(x) - A\zeta_{1-s}(x)$ also tends to a finite limit as $x \to 0$. Applying a similar argument to the function $f(1-x)$, we find a constant B so that $f(x) - B\zeta_{1-s}(1-x)$ tends to a limit as $x \to 1$. Hence the difference

$$f(x) - A\zeta_{1-s}(x) - B\zeta_{1-s}(1-x)$$

extends to a continuous function on the closed unit interval. According to Lemma 4, this function must be constant. Since $s \neq 0$, it follows that it is identically zero. Thus

$$f(x) = A\zeta_{1-s}(x) + B\zeta_{1-s}(1-x);$$

where the two functions on the right are linearly independent since one is continuous and one is discontinuous as $x \to 0$. □

In fact the functions $\zeta_{1-s}(x)$ and $\zeta_{1-s}(1-x)$ are linearly independent for all $s \neq 0, 1, 2, \ldots$, as one can check by repeated differentiation.

§4. Extending from (0, 1) to R/Z

We will prove the following. Let s be a complex constant.

LEMMA 7. *If a function* $f : (0, 1) \to \mathbf{C}$ *satisfies the Kubert identities* $(*_s)$ *with* $s \neq 1$, *then it extends uniquely to a function* $\mathbf{R}/\mathbf{Z} \to \mathbf{C}$ *satisfying* $(*_s)$.

Here no mention is made of continuity. If $Re(s) > 1$ and if f happens to be continuous, then we have seen that the extension is also continuous. *However, if* $Re(s) \leqslant 1$ *then the extension cannot be continuous, except in the trivial case of a constant function with* $s = 0$.

Proof. We must choose $f(0)$ so as to satisfy all of the equations

$$f(0) = m^{s-1}\big(f(0) + f(1/m) + \ldots + f((m-1)/n)\big).$$

Setting

$$c_m = f(1/m) + \ldots + f\big((m-1)/m\big),$$

we can write this as

$$(m^{1-s}-1)f(0) = c_m.$$

But $(*_s)$ implies that

$$c_n = m^{s-1}(c_{mn}-c_m)$$

hence

$$c_{mn} = m^{1-s}\,c_n + c_m = n^{1-s}\,c_m + c_n$$

and

$$(m^{1-s}-1)c_n = (n^{1-s}-1)c_m.$$

Since $s \neq 1$, these factors $m^{1-s} - 1$ cannot all be zero. It now follows easily that $f(0)$ exists and is unique. □

For the functions $f(x)$ studied in §2, it is interesting to note that $f(0)$ is always an appropriate value of the Riemann zeta function. Thus for the version $f(x) = l_s(x)$ of the polylogarithm function, the appropriate choice is

$$f(0) = \zeta(s) .$$

In fact, if $Re(s) > 1$, then $l_s(x)$ is continuous on $\mathbf{R}/\mathbf{Z}$ with $l_s(0) = \zeta(s)$, so the required identity

$$(m^{1-s}-1)\zeta(s) = l_s(1/m) + \dots + l_s((m-1)/m)$$

holds by continuity as $x \to 0$. It follows by analytic continuation that this formula is true for all $s \neq 1$. (Since the right side is holomorphic for all s, this identity provides an alternative proof that $\zeta(s)$ extends to an holomorphic function for $s \neq 1$.)

Similarly, if $f(x) = \zeta_{1-s}(x)$ for $0 < x < 1$, then by continuity as $x \to 1$ the appropriate choice is

$$f(0) = \zeta(1-s) .$$

Note that Lemma 7 is definitely false in the exceptional case $s = 1$. In the case of the even function

$$f(x) = \log | 2 \sin \pi x | = \log | 1 - e^{2\pi i x} | ,$$

which satisfies $(*_1)$ in the open unit interval, the identity

$$(10) \qquad f(1/m) + f(2/m) + \dots + f((m-1)/m) = \log m \neq 0$$

shows that it is not possible to define $f(0)$ so as to satisfy $(*_1)$ at zero. This identity is proved by substituting $t = 1$ in the equation

$$1 + t + \dots + t^{m-1} = \prod_{1}^{m-1} (t-\xi^k)$$

where $\xi = e^{2\pi i/m}$, and then taking the logarithm of the absolute value of both sides.

On the other hand, for the Bernoulli polynomial

$$f(x) = x - 1/2 \qquad \text{for} \qquad 0 < x < 1 ,$$

the value $f(0)$ can be defined arbitrarily and $(*_1)$ will always be satisfied.

§5. UNIVERSAL KUBERT FUNCTIONS

The results in this section are either due to Kubert, or are minor variations on results of Kubert.

Let $A \subset \mathbf{Q}/\mathbf{Z}$ be a subgroup, and let s be a fixed integer. A function

$$f : A \to V$$

to a rational vector space will be called a *Kubert function* if it satisfies

$$(*'_s) \qquad f(ma) = m^{s-1} \sum_{0}^{m-1} f(a+k/m)$$

for every integer m such that $1/m$ belongs to A. It will be convenient to say that f is *universal* if every $\mathbf{Q}$-linear relation between the values $f(a)$ follows from these Kubert relations.

Let $U_s(A)$ be the additive group with one generator $u(a)$ for each element of A, and with defining relations $(*'_s)$. *Then evidently* f *is universal if and only if the induced mapping*

$$u(a) \mapsto f(a)$$

from $U_s(A) \otimes \mathbf{Q}$ *to* V *is injective.*

We are primarily interested in the case where A is the entire group $\mathbf{Q}/\mathbf{Z}$. However, it is very useful to consider finite subgroups of $\mathbf{Q}/\mathbf{Z}$, and requires no extra work to consider arbitrary subgroups.

Note that every automorphism of A gives rise to an automorphism of $U_s(A)$. We will use the notation $\mathrm{Hom}(A, A)^{\bullet}$ for the automorphism group of A, identifying it with the group of invertible elements in the ring $\mathrm{Hom}(A, A)$ consisting of all homomorphisms from A to itself.

THEOREM 2. *The complex vector space* $U_s(A) \otimes \mathbf{C}$ *splits, under the action of the automorphism group of* A, *into a direct sum of* 1*-dimensional eigenspaces, with just one eigenspace corresponding to each continuous character*

$$\chi : \mathrm{Hom}(A, A)^{\bullet} \to \mathbf{C}^{\bullet}.$$

Furthermore, any inclusion $A \subset A' \subset \mathbf{Q}/\mathbf{Z}$ *gives rise to an embedding* $U_s(A) \otimes \mathbf{C} \subset U_s(A') \otimes \mathbf{C}$.

Proofs will be given at the end of this section.

If $A = A_m$ is the cyclic group of order m, note that $\mathrm{Hom}(A, A)$ can be identified with the ring $\mathbf{Z}/m\mathbf{Z}$, and $\mathrm{Hom}(A, A)^{\bullet}$ is an abelian group of order $\varphi(m)$. *In general,* $\mathrm{Hom}(A, A)^{\bullet}$ *is to be topologized as the inverse limit of these groups*

$$\mathrm{Hom}(A_m, A_m)^{\bullet} = (\mathbf{Z}/m\mathbf{Z})^{\bullet}$$

as A_m *varies over all finite subgroups of* A. Similarly, the character group of $\mathrm{Hom}(A, A)^{\bullet}$ is the direct limit of the corresponding Dirichlet character groups $\mathrm{Hom}((\mathbf{Z}/m\mathbf{Z})^{\bullet}, \mathbf{C}^{\bullet})$.

One interesting consequence of Theorem 2 is the following statement, which is reminiscent of Galois theory.

COROLLARY. *If* $A \subset A' \subset \mathbf{Q}/\mathbf{Z}$, *then* $U_s(A) \otimes \mathbf{Q}$ *can be identified with the subspace of* $U_s(A') \otimes \mathbf{Q}$ *which is fixed by all automorphisms of* A' *over* A.

A proof is easily supplied. □

Here is another consequence.

LEMMA 8. *If* $A = A_m$ *is cyclic of order* m, *then the rational vector space* $U_s(A_m) \otimes \mathbf{Q}$ *has dimension* $\varphi(m)$. *For* $m > 2$ *this splits as the direct sum of even and odd parts with respect to the involution*

$$u(a) \mapsto u(-a)\,,$$

where each of these summands has dimension $\varphi(m)/2$.

Proof. This follows immediately from the corresponding statement for $U_s(A) \otimes \mathbf{C}$. The two summands have equal dimension since there are as many even characters $(\chi(-1) = 1)$ as odd characters $(\chi(-1) = -1)$ modulo m. □

If $s \neq 1$, then Lemma 8 could also be derived from the following more explicit statement.

LEMMA 9. *If* $s \neq 1$, *and if* $A = A_m$ *is cyclic of order* m, *then* $U_s(A) \otimes \mathbf{Q}$ *has a basis consisting of the* $\varphi(m)$ *elements* $u(k/m)$ *with* k *relatively prime modulo* m.

However, this statement definitely fails for $s = 1$.

Another complication when $s = 1$ is that Lemma 7 fails, so that we must also consider "punctured" Kubert functions, which are not defined at zero.

Definition. Let $U_s(A-0)$ be the universal group with one generator $u(a)$ for each $a \neq 0$ in A, and with defining relations

$$u(ma) = m^{s-1} \sum_{0}^{m-1} u(a + k/m)$$

for all m and a with $ma \neq 0$ and $1/m \in A$.

If $s \neq 1$, then the proof of Lemma 7 can be used to show that the kernel and cokernel of the natural maps

$$U_s(A_m - 0) \to U_s(A_m)$$

are finite groups of order prime to m. Taking the direct limit over m, it follows that

$$U_s(\mathbf{Q}/\mathbf{Z} - 0) \cong U_s(\mathbf{Q}/\mathbf{Z}).$$

However, for $s = 1$ the situation is different.

LEMMA 10. *The kernel of the natural homomorphism*

$$U_1(A - 0) \to U_1(A)$$

is a free abelian group freely generated by the elements

$$u(1/p) + u(2/p) + \dots + u((p-1)/p),$$

as p *ranges over all primes with* $1/p \in A$. *The cokernel of this homomorphism is free cyclic, generated by* $u(0)$.

A proof is easily supplied, using formula (10) of §4 to prove that there are no relations between these generators. □

The precise structure of $U_s(A)$ can be given as follows.

LEMMA 11. *If* $s \leqslant 1$, *or if* A *is finite, then the group* $U_s(A)$ *is free abelian. In any case,* $U_s(A)$ *is torsion free, and any inclusion* $A \subset A'$ *gives rise to an embedding of* $U_s(A)$ *into* $U_s(A')$.

If $s \geqslant 2$, it is interesting to note that $U_s(\mathbf{Q}/\mathbf{Z})$ is actually a vector space over the rational numbers. For this lemma asserts that it is torsion free, and the relations $(*_s)$ clearly imply that it is divisible.

The proof of Theorem 2 will be based on the following. Let s be any complex number and let $\chi : \mathrm{Hom}(A, A)^\bullet \to \mathbf{C}^\bullet$ be a continuous character.

LEMMA 12. *There is one and, up to a constant multiple, only one function*

$$f = f_\chi : A \to \mathbf{C}$$

satisfying $(*_s')$ *and satisfying* $f(ua) = \chi(u)f(a)$ *for every* u *in* $\mathrm{Hom}(A, A)^\bullet$ *and every* a *in* A.

Proof. To fix our ideas, let us consider only the case $A = \mathbf{Q}/\mathbf{Z}$, so that $\mathrm{Hom}(A, A) = \varprojlim \mathbf{Z}/m\mathbf{Z}$ is the profinite completion $\hat{\mathbf{Z}}$ of the integers. The general case is completely analogous.

Since χ is continuous, there exists an integer $m \neq 0$ so that χ is identically equal to 1 on the congruence class $1 + m\hat{\mathbf{Z}}$ intersected with $\hat{\mathbf{Z}}^\bullet$. The collection of all m with this property forms an ideal $\mathscr{F}$ called the *conductor* of χ. Evidently χ is equal to the composition

$$\hat{\mathbf{Z}}^{\bullet} \to (\mathbf{Z}/\mathscr{F})^{\bullet} \to \mathbf{C}^{\bullet}$$

for some Dirichlet character modulo $\mathscr{F}$, and $\mathscr{F}$ is the unique largest ideal with this property. We will use the same symbol χ for this character on $(\mathbf{Z}/\mathscr{F})^{\bullet}$. *If k is any integer relatively prime to $\mathscr{F}$, it follows that $\chi(k)$ is a well defined root of unity.*

Any fraction in $\mathbf{Q}/\mathbf{Z}$ with denominator n can be written as u/n for some unit u in $\hat{\mathbf{Z}}^{\bullet}$. In view of the identity

$$f(u/n) = \chi(u) f(1/n),$$

we need only compute the values $f(1/n)$ in order to determine f completely.

Note that the unit u in this equation is well defined modulo $n\hat{\mathbf{Z}}$. If n belongs to the ideal $\mathscr{F}$, then it follows that the root of unity $\chi(u)$ is uniquely determined. However, if $n \notin \mathscr{F}$, then we can choose $u \equiv 1 \bmod n$ with $\chi(u) \neq 1$. *This proves that $f(1/n) = 0$ whenever n is not in the ideal $\mathscr{F}$.*

The proof will show that f is some constant multiple of the expression

$$f(1/n) = n^{-s} \prod_{p \mid n} (p - p^s \bar{\chi}(p))/(p-1) \qquad \text{for} \qquad n > 0, n \in \mathscr{F}.$$

Here $\bar{\chi}(p)$ is a well defined root of unity if the prime p is a unit modulo $\mathscr{F}$, and is to be set equal to zero otherwise.

First consider the Kubert identity

$$(\bigstar) \qquad p^{1-s} f\left(\frac{1}{n}\right) = \sum_{0}^{p-1} f\left(\frac{1+kn}{pn}\right)$$

for $n \in \mathscr{F}$.

Case 1. If $p \mid n$, then each $1 + kn$ is a unit modulo pn, with $\chi(1+kn) = 1$. Hence this equation reduces simply to

$$p^{-s} f\left(\frac{1}{n}\right) = f\left(\frac{1}{pn}\right).$$

Case 2. If n is not a multiple of p, then there is exactly one k_0 between 1 and $p - 1$ so that $1 + k_0 n$ is some multiple, say lp, of p. Then

$$f\left(\frac{1+k_0 n}{np}\right) = f\left(\frac{l}{n}\right) = \chi(l) f\left(\frac{1}{n}\right),$$

where $\chi(l) = \bar{\chi}(p)$ since $lp \equiv 1 \bmod \mathscr{F}$. Thus the Kubert identity takes the form

$$(p^{1-s} - \bar{\chi}(p)) f\left(\frac{1}{n}\right) = (p-1) f\left(\frac{1}{pn}\right).$$

Evidently this completes the proof that f is uniquely defined up to multiplication by a constant.

To prove that the function f defined in this way satisfies all of the Kubert identities, we must also consider the case where n does *not* belong to the ideal $\mathscr{F}$, so that $f(1/n) = 0$. If pn does belong to $\mathscr{F}$, then the units $1 + kn$ modulo pn, in the argument above, range precisely over the kernel of the homomorphism

$$(\mathbf{Z}/pn\mathbf{Z})^{\bullet} \to (\mathbf{Z}/n\mathbf{Z})^{\bullet} .$$

Since χ is non-trivial on this kernel, by the definition of $\mathscr{F}$, it follows that

$$\sum \chi(1+kn) = 0 ,$$

taking the sum over all k between 0 and $p - 1$ with $1 + kn$ prime to p. Thus both sides of the required equation ($\bigstar$) are zero. Since every other Kubert identity follows from one of these by applying an automorphism to $\mathbf{Q}/\mathbf{Z}$, this completes the proof. $\square$

Proof of Theorem 2. If $A = A_m$ is a finite group of order m, then $U_s(A) \otimes \mathbf{C}$ is finite dimensional, so it certainly splits under the action of the commutative group $\mathrm{Hom}(A, A)^{\bullet}$ into a direct sum of 1-dimensional spaces. According to Lemma 12, there is exactly one of these spaces for each character χ mod m, so the conclusion follows.

The general case now follows by passing to a direct limit over finite subgroups of A. (For any integer n, note that there are only finitely many characters χ whose conductor contains n, hence only finitely many χ with $f_{\chi}(1/n) \neq 0$.) This completes the proof. $\square$

Proof of Lemma 9. It will be convenient to consider the various vector spaces $U_s(A_m) \otimes \mathbf{Q}$ as subspaces of $U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$. This is permissible by the Corollary above (or by Lemma 11)).

Let W_m be the rational vector space spanned by all elements

$$u(a) \in U_s(\mathbf{Q}/\mathbf{Z}) \otimes \mathbf{Q}$$

such that a has denominator precisely m, and hence generates the cyclic group A_m. We will show that $W_m \subset W_{pm}$. Assuming this for the moment, it follows inductively that

$$W_m = U_s(A_m) \otimes \mathbf{Q} .$$

Hence the $\varphi(m)$ generators of W_m must be linearly independent, as was to be proved.

Suppose then that a generates A_m. If $p \mid m$, then the Kubert identity

$$u(a) = p^{s-1} \sum_{0}^{p-1} u((a+k)/p) ,$$

where each $(a+k)/p$ has denominator precisely pm, proves that $u(a) \equiv 0 \bmod W_{pm}$. On the other hand, if p is prime to m, then the relation

$$u(pa) - p^{s-1}u(a) = p^{s-1}\sum_{1}^{p-1} u(a+k/p)$$

proves that

$$u(pa) \equiv p^{s-1}u(a) \bmod W_{pm}.$$

Choosing $r \geqslant 1$ so that $p^r \equiv 1 \bmod m$, it follows that

$$u(a) = u(p^r a) \equiv p^{r(s-1)}u(a) \bmod W_{pm}.$$

Since $s \neq 1$, this proves that $u(a) \equiv 0 \bmod W_{pm}$, as required. □

Proof of Lemma 11. For any $a \in \mathbf{Q}/\mathbf{Z}$ let a_p be the p-primary component of a. Thus $a = \sum a_p$, where the denominator of a_p is a power of p. Represent each a_p as a rational in the interval $0 \leqslant a_p < 1$.

Definition. We will say that a is *reduced* if $0 \leqslant a_p < 1 - p^{-1}$ for every prime p.

Then for $s \leqslant 1$ we will prove explicitly that $U_s(A)$ is a free abelian group, with one free generator $u(a)$ for each reduced element a of A. Evidently it suffices to check that $U_s(A)$ is generated by these elements. For a simple counting argument shows that the number of reduced elements in any finite subgroup $A_m = m^{-1}\mathbf{Z}/\mathbf{Z}$ is equal to the rank

$$\varphi(m) = m\prod_{p|m}(1-p^{-1})$$

of $U_s(A_m)$.

Suppose that a is not reduced, say $1 - p^{-1} \leqslant a_p < 1$ for some prime p. Then the identity

$$p^{1-s}u(pa) = u(a) + u(a-1/p) + \ldots + u(a-(p-1)/p)$$

shows that $u(a)$ is a linear combination of $u(pa)$, where pa has strictly smaller denominator than a, and elements $a - k/p$ which are reduced at the prime p and have q-primary component unchanged for $q \neq p$. A straightforward induction now completes the proof in the case $s \leqslant 1$.

If $s \geqslant 2$, this argument shows only that the reduced generators form a basis for the rational vector space $U_s(A) \otimes \mathbf{Q}$. To prove that $U_s(A_m)$ is free abelian, we will show that the tensor product $U_s(A_m) \otimes \mathbf{Z}_q$ is generated by $\varphi(m)$ elements for any prime q. This will show that there cannot be any torsion.

As free generators, we will choose all elements $u(a)$ where $a = \sum a_p$ is "reduced" at all primes p other than q. *However, we require that the q-primary component a_q should have denominator equal to the highest power of q dividing m.*

The proof that these elements generate over $\mathbf{Z}_q$ proceeds as above for $p \neq q$, and proceeds as in the proof of Lemma 9 when $p = q$. Details are easily supplied. □

§6. On Q-linear relations

S. Chowla and P. Chowla have suggested the following conjecture in a private communication to the author. Let $a_1, a_2, \ldots$ be a sequence of integers which is periodic, $a_n = a_{n+p}$, for some prime p. Then

(11) $$\textstyle\sum_1^\infty a_n/n^2 \neq 0$$

except in the special case

$$a_1 = \ldots = a_{p-1} = a_p/(1-p^2)\,.$$

If we use the Hurwitz function

$$\zeta_2(k/p) = p^2(k^{-2} + (k+p)^{-2} + \ldots)\,,$$

then the inequality (11) can be written as

$$\textstyle\sum_1^p a_k\, \zeta_2(k/p) \neq 0\,;$$

and the exceptional case corresponds to the Kubert relation

$$\zeta_2(1) = p^{-2} \textstyle\sum_1^p \zeta_2(k/p)\,.$$

Thus the Chowlas' conjecture is true if and only if the real numbers

$$\zeta_2(1/p), \ldots, \zeta_2((p-1)/p)$$

are linearly independent over the rational numbers. More generally, for any $m \geqslant 2$ one might conjecture that the $\varphi(m)$ real numbers $\zeta_2(k/m)$, where k varies over all relatively prime integers between 1 and $m-1$, are **Q**-linearly independent. Using Lemma 9, a completely equivalent statement would be the following.

Conjecture: Every **Q**-linear relation between the real numbers $\zeta_2(x)$, where x is rational with $0 < x \leqslant 1$ is a consequence of the Kubert relations $(*_{-1})$.

In fact, since $\zeta_2(x+1) \equiv \zeta_2(x) \bmod \mathbf{Q}$ for positive rational x, it might be more natural to sharpen this conjecture by taking the values of ζ_2 modulo **Q**. *In other words, it is conjectured that the mapping*

$$\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/\mathbf{Q}$$

induced by ζ_2 *is a "universal" function satisfying* $(*_{-1})$. It follows easily from Theorem 3 below that the corresponding conjecture for the even part,

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x\,,$$

of ζ_2 is indeed true; but the odd part of ζ_2 seems difficult to work with.

One can make analogous and equally plausible conjectures for the Hurwitz functions $\zeta_3, \zeta_4, \ldots$. In Appendix 2 we will describe analogous conjectures for certain functions closely related to the gamma function.

Bass [2], studying multiplicative relations between cyclotomic units, has proved the following result. Let

$$f_0(x) = \log|1 - e^{2\pi i x}| = \log(2 \sin \pi x)$$

for $0 < x < 1$. Note that $f_0(1-x) = f_0(x)$.

THEOREM OF BASS. *Every* $\mathbf{Q}$*-linear relation between the numbers* $f_0(x)$ *for rational* $x \in (0, 1)$ *is a consequence of the Kubert relations* $(*_1)$, *together with evenness.*

A proof will be indicated at the end of this section.

Note that this is the exceptional case in which Lemma 7 does not apply, so that $f_0(0)$ cannot be defined.

Bass' theorem is equivalent, using the results of §5, to the following classical statement. Fixing some integer $m \geqslant 3$, let $\xi = e^{2\pi i/m}$, and let V_m be the multiplicative group generated by the elements

$$1 - \xi, 1 - \xi^2, \ldots, 1 - \xi^{m-1}$$

in the cyclotomic field $\mathbf{Q}[\xi]$. Elements of the intersection $V_m \cap \mathbf{Z}[\xi]^{\bullet}$ are called *circular units* (or cyclotomic units).

COROLLARY. *This group* $V_m \cap \mathbf{Z}[\xi]^{\bullet}$ *of circular units has finite index in the group* $\mathbf{Z}[\xi]^{\bullet}$ *consisting of all units of the cyclotomic field.*

Compare Hilbert [8], as well as Sinnott [25].

Proof. Let $m = q_1 \ldots q_n$ be the factorization of m into powers of distinct primes. By Lemmas 8 and 10, Bass' theorem is equivalent to the statement that the additive group generated by the elements

$$f_0(k/m) = \log|1 - \xi^k|$$

has rank $\varphi(m)/2 + n - 1$. Since each generator of V_m is equal to a real number multiplied by a root of unity, this is equivalent to the statement that V_m has rank $\varphi(m)/2 + n - 1$. However it is not difficult to check that V_m splits as the direct sum of the group of circular units and a free abelian group generated by the elements $1 - e^{2\pi i/q_j}$. Hence Bass' theorem is also equivalent to the statement that the group of circular units has rank $\varphi(m)/2 - 1$. According to the Dirichlet unit theorem, this implies that it has finite index in the group of all units of $\mathbf{Z}[\xi]$. □

The author [21] has conjectured that the function $\mathbf{Q}/\mathbf{Z} \to \mathbf{R}$ defined by

$$x \mapsto \Lambda(\pi x) = -\int_0^{\pi x} \log|2 \sin\theta|\, d\theta$$

is a universal odd function satisfying $(*_2)$. This seems very difficult. However, W. Sinnott has pointed out to the author this the situation for the derivatives of $\log 2 \sin\theta$ is much easier to analyze.

Let $f_t(x)$ be the t-th derivative of $\log|2\sin\theta|$, evaluated at $\theta = \pi x$. For example $f_1(x) = \cot(\pi x)$, $f_2(x) = -csc^2(\pi x)$. Note that $f_1(1-x) = (-1)^t f_t(x)$. The values at $x = 0$ are to be defined as in §4.

THEOREM 3. *For each fixed* $t = 1, 2, ...,$ *the function*

$$f_t : \mathbf{Q}/\mathbf{Z} \to \mathbf{R}$$

is a universal even or odd function satisfying $(*_{1-t})$.

That is every $\mathbf{Q}$-linear relation between the values $f_t(x)$ for x in $\mathbf{Q}/\mathbf{Z}$ follows from $(*_{1-t})$, together with evenness or oddnes according as t is even or odd.

Fixing some integer $m \geqslant 3$, let $\xi = e^{2\pi i/m}$. If t is even, the proof will show that the values

$$f_t(1/m), ..., f_t((m-1)/m)$$

span the real part of the cyclotomic field $\mathbf{Q}[\xi]$. Similarly, if t is odd, the values $if_t(k/m)$ span the totally imaginary subspace of $\mathbf{Q}[\xi]$. In either case, these values span a rational vector space of dimension $\varphi(m)/2$, as required by Lemma 8.

Compare Ewing [7] for an analogous discussion of the values of $csc(\pi x)$ and its derivatives at rational x.

The proof will depend upon well known properties of Dirichlet L-functions. Fixing some positive integer m, let

$$\chi : (\mathbf{Z}/m\mathbf{Z})^{\cdot} \to \mathbf{C}^{\cdot}$$

be an arbitrary Dirichlet character modulo m. We allow the degenerate case $m = 1$ with the understanding that the only character modulo 1 is the constant function $\chi_0(k) = 1$. Recall that such a character is *primitive* (or has conductor generated by m) if it cannot be factored through the projection

$$(\mathbf{Z}/m\mathbf{Z})^{\cdot} \to (\mathbf{Z}/d\mathbf{Z})^{\cdot}$$

for any divisor $d < m$. As usual, we set $\chi(k) = 0$ if k is a non-unit modulo m.

The associated L-function is defined by

$$L(s, \chi) = \sum_1^{\infty} \chi(k)/k^s$$

for $Re(s) > 1$. In terms of the Hurwitz function

$$\zeta_s(k/m)/m^s = k^{-s} + (k+m)^{-s} + \ldots$$

we can clearly write this as a finite sum

$$L(s, \chi) = \sum_1^m \chi(k)\, \zeta_s(k/m)/m^s\,. \tag{12}$$

It follows that $L(s, \chi)$ *extends to a function which is holomorphic in* s *for all complex* s, *whenever* $\chi \neq \chi_0$. For it is easy to check that the difference $\zeta_s(x) - (s-1)^{-1}$ is holomorphic in s; and the $(s-1)^{-1}$ terms cancel whenever $\chi \neq \chi_0$.

On the other hand, for the trivial character χ_0, evidently $L(s, \chi_0)$ is equal to the Riemann zeta function, with a pole at $s = 1$.

Now let us restrict to integer values of s.

LEMMA 13. *For primitive* $\chi \neq \chi_0$, *and for integer values of* s, *the function* $L(s, \chi)$ *is zero if and only if* $s \leqslant 0$ *and* $\chi(-1) = (-1)^s$.

For $s > 1$, the statement that $L(s, \chi) \neq 1$ is fairly easy to prove, while for $s = 1$ it is a basic result of Dirichlet. See for example [5] or [23]. For $s \leqslant 0$, this lemma is proved using the functional equation relating $L(s, \chi)$ and $L(1-s, \bar{\chi})$. (Compare [10].) Details of this last argument may be found in Appendix 1. □

In the case of the trivial character χ_0, this lemma remains true except for anomalous behavior at $s = 0$ (where $\zeta(s)$ is non-zero) and $s = 1$ (where $\zeta(s)$ has a pole).

These Dirichlet L-functions can also be expressed as finite linear combinations of polylogarithms, via Fourier analysis, as follows. Let $\xi = e^{2\pi i/m}$.

LEMMA 14. *If* $\chi \neq \chi_0$ *is primitive modulo* m, *then*

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau\,,$$

where

$$\tau = \tau(\chi) = \sum_1^m \chi(k)\xi^k$$

is a complex constant with absolute value $\sqrt{m}$.

In the case of the trivial character χ_0, this lemma remains true provided that $l_s(1)$ is interpreted as in §4.

Proof of Lemma 14. Since both sides are holomorphic in s for all complex s, it will suffice to consider the case $Re(s) > 1$. First note that the "Fourier transform" of the complex valued function χ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is equal to $\tau\bar{\chi}$; that is

(13)
$$\sum_{j \bmod m} \chi(j)\xi^{jk} = \tau\bar{\chi}(k).$$

If k is a unit modulo m, this follows from the equation $\chi(j) = \bar{\chi}(k)\chi(jk)$, while if k is a non-unit modulo m then, using the hypothesis that χ is primitive, it is not difficult to check that both sides of this equation are zero. Now dividing both sides by k^s and summing over all positive integers k, we obtain

$$\sum_{j \bmod m} \chi(j)\mathscr{L}_s(\xi^j) = \tau L(s, \bar{\chi}).$$

Since $\mathscr{L}_s(\xi^j) = l_s(j/m)$, this implies the required equation.

To compute $|\tau|$ combine (13) with the complex conjugate equation to obtain

$$m\chi(n) = \sum_j \chi(j) \sum_k \xi^{k(j-n)} = \sum_k \xi^{-kn} \sum_j \chi(j)\xi^{kj}$$
$$= \sum_k \xi^{-kn}\, \tau\bar{\chi}(k) = \tau\bar{\tau}\chi(n);$$

hence $m = \tau\bar{\tau}$ as asserted. □

Remark. Similar arguments prove that the Fourier transform of the Hurwitz function $\zeta_s(j/m)$ on the finite ring $\mathbf{Z}/m\mathbf{Z}$ is a multiple of $l_s(k/m)$. More generally, one can show that any function on $\mathbf{Z}/m\mathbf{Z}$ satisfies $(*_s)$ if and only if its Fourier transform satisfies $(*_{1-s})$.

Proof of Theorem 3. We will work with the polylogarithm function

$$\mathscr{L}_s(\xi^k) = l_s(k/m)$$

where $\xi = e^{2\pi i/m}$. If $s = 1 - t$ is a non-positive integer, recall from §2 that $\mathscr{L}_s(z)$ is a rational function with rational coefficients. Hence $l_s(k/m)$ takes values in the cyclotomic field $\mathbf{Q}[\xi]$.

The Galois group G of $\mathbf{Q}[\xi]$ over $\mathbf{Q}$ can be identified with $(\mathbf{Z}/m\mathbf{Z})^{\bullet}$. Evidently the mapping

$$U_s(A_m) \to \mathbf{Q}[\xi]$$

induced by l_s is G-equivariant, in the sense that the automorphism $u(k/m) \mapsto u(gk/m)$ of $U_s(A_m)$ corresponds to the automorphism $f(\xi) \mapsto f(\xi^g)$ of $\mathbf{Q}[\xi]$ for every g in $G \cong (\mathbf{Z}/m\mathbf{Z})^{\bullet}$. Tensoring both sides with the complex numbers, each splits into a direct sum of 1-dimensional eigenspaces under the action of G. Hence, to compute the rank of this map, we need only decide how many eigenspaces are mapped non-trivially.

For each character χ mod m, let $\chi' : (\mathbf{Z}/d\mathbf{Z})^{\bullet} \to \mathbf{C}^{\bullet}$ be the associated primitive character, where $d \mid m$ generates the conductor of χ. Evidently the sum

$$\sum_{k \bmod d} l_s(k/d) \otimes \chi'(k)$$

belongs to the $\bar{\chi}$-eigenspace under the action of G on $\mathbf{Q}[\xi] \otimes \mathbf{C}$. By Lemmas 13 and 14, its image $\sum \chi'(k) l_s(k/d)$ in $\mathbf{C}$ is zero if and only if $\chi(-1) = (-1)^s$; except for the single anomalous case when $s = 0$ and $\chi = \chi_0$. Thus the rank of this mapping

$$U_s(A_m) \to \mathbf{Q}[\xi]$$

is at least $\varphi(m)/2$ for $s < 0$, and at least $1 + \varphi(m)/2$ when $s = 0$.

It follows that the image $l_s(A_m)$ *spans the real part of the cyclotomic field* $\mathbf{Q}[\xi]$ *when* $s = 1 - t < 0$ *is odd, and the totally imaginary part of* $\mathbf{Q}[\xi]$ *when* s *is even.* Here l_s is related to the real valued functions f_t of Theorem 3 by the identity

$$l_{1-t}(x) + f_t(x)/(2i)^t = 0$$

for $t \geqslant 2$; which follows from (8) and (9). Similarly, for $t = 1$, the image of the function

$$if_1(k/m) = 2l_0(k/m) + 1$$

spans the totally imaginary subspace of $\mathbf{Q}[\xi]$.

Since the dimension $\varphi(m)/2$ of this image is the maximum allowed by Lemma 8, this completes the proof of Theorem 3. □

Proof of Bass' Theorem. Recall that V_m is the multiplicative group in $\mathbf{Q}[\xi]$ spanned by the $1 - \xi^k$. Evidently the Galois group G of $\mathbf{Q}[\xi]$ operates on V_m. Since each generator is the product of a real number and a root of unity, G operates also on the additive group $\log |V_m|$, generated by the images

$$f_0(k/m) = \log |1 - \xi^k| .$$

Note that $f_0(x)$ is precisely the even part $-\big(l_1(x) + l_1(-x)\big)/2$ of the function $-l_1(x) = \log(1 - e^{2\pi i x})$.

As in the proof of Theorem 3, we can consider the map

$$U_1(A_m - 0) \to \log |V_m|$$

induced by f_0, and split both sides into eigenspaces under the action of $G \cong (\mathbf{Z}/m\mathbf{Z})^{\bullet}$. For each even character $\chi \neq \chi_0$, with conductor generated by $d \mid m$, the corresponding L-function

$$\sum_{k \bmod d} \chi'(k) f_0(k/d) = - \sum_{k \bmod d} \chi'(k) l_1(k/d) = -\tau L(1, \bar{\chi}')$$

is non-zero according to Dirichlet. Thus we obtain a contribution of $-1 + \varphi(m)/2$ to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing m. Lemmas 8 and 10 of §5 now complete the proof. □

Appendix 1

Relations between polylogarithm and Hurwitz function

For every complex number s, it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function $l_s(x)$ and of the function $\zeta_{1-s}(x)$ or $\beta_s(x) = -s\zeta_{1-s}(x)$. This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of s, the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1+i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0\,.$$

Integrating, we see that

$$l_1(x) - l_1(1-x) + 2\pi i\, \beta_1(x)/1! = 0$$

$$l_2(x) + l_2(1-x) + (2\pi i)^2\beta_2(x)/2! = 0$$

and so on, for $0 < x < 1$. For even values of the subscript, specializing to $x = 0$ as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k}b_{2k}/(2k)! = 0\,.$$

In particular, it follows that $\zeta(0) = -\frac{1}{2}$, and that the numbers $b_2, -b_4, b_6, -b_8, \ldots$ are strictly positive. On the other hand, differentiating the formula for $l_0(x)$, we obtain

$$l_{-1}(x) = -csc^2(\pi x)/4\,.$$

This is an even function satisfying $(*_{-1})$, so it must be some multiple of $\zeta_2(x) + \zeta_2(1-x)$. Comparing asymptotic behavior as $x \to 0$, we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1!\,.$$

Differentiating, we see that

$$-\zeta_3(x) + \zeta_3(1-x) = (2\pi i)^3 l_{-2}(x)/2!$$
$$\zeta_4(x) + \zeta_4(1-x) = (2\pi i)^4 l_{-3}(x)/3!$$

and so on.

For $s \neq 0, 1, 2, \ldots$ we know from §3 that there is some relation of the form

$$l_s(x) = A_s\zeta_{1-s}(x) + B_s\zeta_{1-s}(1-x) \tag{14}$$

for $0 < x < 1$; where A_s and B_s are certain uniquely determined constants. Expressing each of these functions of x as the sum of an even part and an odd part, we see that

$$\begin{cases} l_s^{\text{even}}(x) = (A_s+B_s)\zeta_{1-s}^{\text{even}}(x) \\ l_s^{\text{odd}}(x) = (A_s-B_s)\zeta_{1-s}^{\text{odd}}(x) . \end{cases} \tag{15}$$

Evidently the functions $s \mapsto A_s \pm B_s$ *are meromorphic, taking finite non-zero values for all* $s \in \mathbf{C} - \mathbf{Z}$. Differentiating with respect to x, we see that

$$A_s \pm B_s = s(A_{s+1} \mp B_{s+1})/(2\pi i) . \tag{16}$$

For integral values of s, using the discussion above, we easily obtain the following table of values, where $0! = 1$.

s	...	-2	-1	0	1	2	3	...
$A_s + B_s$	...	0	$\frac{2 \cdot 1!}{(2\pi i)^2}$	0	∞	$\frac{(2\pi i)^2}{2 \cdot 1!}$	∞	...
$A_s - B_s$	...	$-\frac{2 \cdot 2!}{(2\pi i)^3}$	0	$-\frac{2 \cdot 0!}{2\pi i}$	$\frac{2\pi i}{2 \cdot 0!}$	∞	$\frac{(2\pi i)^3}{2 \cdot 2!}$	...

Now suppose that we specialize to $x = 0$, by the procedure of §4. Then equation (14) reduces to a form

$$\zeta(s) = (A_s+B_s)\zeta(1-s)$$

of Riemann's functional equation. It follows that

$$(A_s+B_s)(A_{1-s}+B_{1-s}) = 1 ,$$

and hence using (16) that

$$(A_s-B_s)(A_{1-s}-B_{1-s}) = -1 .$$

This discussion gives all of the information about $A_s \pm B_s$ which we will need. However, it is possible to compute precise values as follows. Let $\zeta_{1-s}(e^{2\pi i}x)$ be the result of analytic continuation in a loop circling the origin. Then evidently

$$\zeta_{1-s}(e^{2\pi i}x) - \zeta_{1-s}(x) = (e^{2\pi is} - 1)x^{s-1}.$$

Using the integral formula (6), computation shows that

$$l_s(e^{2\pi i}x) - l_s(x) = -(2\pi i)^s x^{s-1}/\Gamma(s).$$

Comparing these two expressions, and noting that $\zeta_{1-s}(1-x)$ is holomorphic throughout a neighborhood of $x = 0$, we can solve for A_s. The result after some manipulation is

$$A_s = \frac{i(2\pi)^s e^{-\pi is/2}}{2\Gamma(s)\sin(\pi s)}.$$

Now comparing the behavior of l_s and ζ_{1-s} under complex conjugation we see easily that

$$B_s = \overline{A_{\bar{s}}} = \frac{-i(2\pi)^s e^{\pi is/2}}{2\Gamma(s)\sin(\pi s)}.$$

In particular, it follows that

$$A_s + B_s = \frac{(2\pi)^s}{2\Gamma(s)\cos(\pi s/2)}, \qquad A_s - B_s = \frac{i(2\pi)^s}{2\Gamma(s)\sin(\pi s/2)}.$$

As an application of formula (15), let us prove the corresponding functional equation for a Dirichlet L-function. Recall from Lemma 14 that for any primitive Dirichlet character χ modulo m the function

$$L(s, \chi) = \sum_1^m \chi(k)\zeta_s(k/m)/m^s$$

satisfies

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau.$$

Here we may just as well use either the even or the odd parts of ζ_s and l_s according as $\chi(-1)$ is $+1$ or -1. Therefore, it follows from (15) that

$$L(s, \bar{\chi}) = (A_s \pm B_s) \sum_1^m \chi(k)\zeta_{1-s}(k/m)/\tau$$
$$= m^s(A_s \pm B_s) L(1-s, \chi)/\tau.$$

Thus we have proved the functional equation

(17) $$L(s, \bar{\chi}) = m^{1-s}(A_s + \chi(-1)B_s) L(1-s, \chi)/\tau(\chi).$$

Here the factor m^{1-s}/τ is never zero or infinite, while $A_s \pm B_s$ is zero or infinite only at certain integer values, as indicated in the table above.

The proof of Lemma 13 can now easily be completed as follows. If $s \leqslant 0$ is an integer, then $L(1-s, \chi) \neq 0$, so it follows that $L(s, \bar{\chi})$ equals zero if and only if $A_s \pm B_s$ is zero, as indicated in the table. □

Appendix 2

Some relatives of the gamma function

This appendix will describe certain functions $\gamma_1(x)$, $\gamma_2(x)$, ... which satisfy a modified form of the Kubert identities, with a polynomial correction term. (See (22) below.) They are defined as partial derivatives of the Hurwitz function by the formula

$$\gamma_{1-t}(x) = \partial\zeta_t(x)/\partial t\,. \tag{18}$$

We will show that γ_1 is related to the classical gamma function via Lerch's identity

$$\gamma_1(x) = \log(\Gamma(x)/\sqrt{2\pi})\,. \tag{19}$$

(Compare [27, p. 60].) As a bonus, we will give a self-contained exposition of the basic properties of the gamma function, based on formulas (18) and (19).

Let us begin with equation (18), which defines $\gamma_s(x)$ as an analytic function of both variables for all $s \neq 0$ and all $x > 0$. Recall that the Hurwitz function $\zeta_t(x) = x^{-t} + (x+1)^{-t} + \ldots$ (analytically extended in t for $t \neq 1$) satisfies

$$\zeta_t(x+1) = \zeta_t(x) - x^{-t}\,.$$

Differentiating with respect to t, and then substituting $t = 1 - s$, we obtain

$$\gamma_s(x+1) = \gamma_s(x) + x^{s-1}\log x\,. \tag{20}$$

In particular,

$$\gamma_1(x+1) = \gamma_1(x) + \log x\,.$$

Note that

$$\zeta_t'(x) = -t\zeta_{t+1}(x)$$

hence

$$\zeta_t''(x) = t(t+1)\zeta_{t+2}(x)\,,$$

where the prime stands for the derivative with respect to x. By analytic continuation, this last equation holds also at $t = 0$. Differentiating with respect to t at $t = 0$, we obtain

$$\gamma_1''(x) = \zeta_2(x) . \tag{21}$$

In particular, it follows that $\gamma_1''(x) > 0$ *for all* $x > 0$.

Let us define the gamma function as follows. (Compare Artin [1].)

LEMMA 15 (Bohr and Mollerup). *There is one and only one twice continuously differentiable function* $\Gamma(x) > 0$ *for* $x > 0$ *which satisfies*

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1, \qquad \text{and} \qquad (\log \Gamma(x))'' \geqslant 0 .$$

Proof. Evidently it suffices to show that there is one and, up to an additive constant, only one C^2-function

$$f(x) = \log \Gamma(x) + c$$

for $x > 0$ which satisfies the two conditions

$$f(x+1) = f(x) + \log x$$

and

$$f''(x) \geqslant 0 .$$

Existence is clear, since the equation $\gamma_1(x)$ satisfies both of these conditions. To prove uniqueness, let us differentiate twice to obtain

$$f''(x+1) = f''(x) - 1/x^2 ,$$

hence

$$f''(x+n+1) = f''(x) - x^{-2} - (x+1)^{-2} - \ldots - (x+n)^{-2} \geqslant 0 .$$

Taking the limit as $n \to \infty$, it follows that

$$f''(x) \geqslant \zeta_2(x) .$$

On the other hand, note that the difference $f(x) - \gamma_1(x)$ is periodic, of period 1. Hence its second derivative $f''(x) - \zeta_2(x)$ is periodic, and has average $\int_0^1 (f''(x) - \zeta_2(x))dx$ equal to zero. Clearly it follows that $f''(x) = \zeta_2(x)$ everywhere. Integrating twice, we see that

$$f(x) = \gamma_1(x) + ax + b .$$

Subtracting the corresponding equation for $f(x+1)$, we see that $a = 0$, which completes the proof. □

This argument shows that

$$\gamma_1(x) = \log(\Gamma(x)/C)$$

for some constant C, whose precise value will be computed later.

Remark: The customary definition of the gamma function is the expression

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

which was used in §2 and Appendix 1. Here is an outline proof that this expression does indeed satisfy the conditions of Lemma 15. Integration by parts shows that $\Gamma(x+1) = x\Gamma(x)$. Note that a twice differentiable positive function satisfies $(\log f(x))'' \geqslant 0$ if and only if the matrix

$$\begin{bmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{bmatrix}$$

is positive semi-definite, for all x. But the collection of all 2×2 positive semi-definite matrices forms a convex cone. It follows that the sum $f(x) + g(x)$ of any two functions which satisfy this condition will also satisfy it. Similarly the integral

$$\begin{bmatrix} \Gamma(x) & \Gamma'(x) \\ \Gamma'(x) & \Gamma''(x) \end{bmatrix} = \int_0^\infty \begin{bmatrix} 1 & \log t \\ \log t & (\log t)^2 \end{bmatrix} e^{-t}t^{x-1}dt$$

is a positive semi-definite matrix. Hence $(\log \Gamma(x))'' \geqslant 0$ as required. □

Now consider the Kubert identity

$$m^t\zeta_t(x) = \sum_0^{m-1} \zeta_t((x+k)/m)\,.$$

If we differentiate both sides with respect to t, then substitute $t = 1 - s$ and $\zeta_t = -\beta_s/s$, we obtain

$$\gamma_s(x) = (\log m)\beta_s(x)/s + m^{s-1}\sum_0^m \gamma_s((x+k)/m)\,. \tag{22}$$

Thus γ_s *satisfies* the Kubert identity $(*_s)$, *except for a correction term involving the Bernoulli polynomial* $\beta_s(x)$, *for* $s = 1, 2, 3, \ldots$.

If we work modulo the logarithms of positive rational numbers, then the function

$$\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/\mathbf{Q} \log \mathbf{Q}^+$$

induced by γ_s actually satisfies $(*_s)$. *It seems natural to conjecture that this is a universal Kubert function on* $\mathbf{Q}/\mathbf{Z}$ *for integers* $s \geqslant 1$.

For $s = 1$, the "even" part of this conjecture can easily be proved using Bass' theorem, together with the classical identity

$$\gamma_1(x) + \gamma_1(1-x) + \log(2 \sin \pi x) = 0$$

for $0 < x < 1$, which is proved below, and the fact that $\gamma_1(1) = \log(1/\sqrt{2\pi})$ where π is transcendental. For the odd part of γ_1, Rohrlich has conjectured universality even if we work modulo the logarithms of *all* algebraic numbers. See [17, p. 66].

In the case $s = 1$, formula (22) takes the form

(23) $$\gamma_1(x) = (\log m)\left(x - \frac{1}{2}\right) + \textstyle\sum_0^{m-1} \gamma_1((x+k)/m).$$

Hence the derivative $\gamma_1'(x) = \Gamma'(x)/\Gamma(x)$ satisfies

(24) $$\gamma_1'(x) = \log m + m^{-1} \textstyle\sum_0^{m-1} \gamma_1'((x+k)/m).$$

Note that $\gamma_1'(x+1) = \gamma_1'(x) + 1/x \equiv \gamma_1'(x) \bmod \mathbf{Q}$, if x is positive and rational. *We may conjecture that* γ_1' *induces a universal function* $\mathbf{Q}/\mathbf{Z} \to \mathbf{R}/(\mathbf{Q} + \mathbf{Q} \log \mathbf{Q}^+)$ *satisfying* $(*_0)$. (It can be shown that $\gamma_1'(1)$ is equal to the negative of Euler's constant. Thus even at $x = 1$ the number theoretic properties of $\gamma_1'(x)$ are not known.)

As a typical application of (23), taking $x = 1$ we obtain the equation

$$\gamma_1(1/m) + \gamma_1(2/m) + \ldots + \gamma_1((m-1)/m) = \log(1/\sqrt{m}).$$

In particular, $\gamma_1(1/2) = \log(1/\sqrt{2})$.

As a further application of (23), we will prove the classical formula

(25) $$\gamma_1(x) + \gamma_1(1-x) + \log(2 \sin \pi x) = 0$$

for $0 < x < 1$. If we add (23) to the corresponding formula for $\gamma_1(1-x)$, then the correction terms cancel out. Hence the sum $\gamma_1(x) + \gamma_1(1-x)$ satisfies the Kubert identities $(*_1)$ in their original form. By Theorem 1, this implies that

$$\gamma_1(x) + \gamma_1(1-x) = c \log(2 \sin \pi x)$$

for some constant c. One way to evaluate c would be to differentiate twice:

$$\zeta_2(x) + \zeta_2(1-x) = -c\pi^2/\sin^2 \pi x,$$

and to note that both $\zeta_2(x)$ and $\pi^2/\sin^2 \pi x$ are asymptotic to $1/x^2$ as $x \to 0$. (Compare Appendix 1.) Another would be to substitute $x = 1/2$, noting that $\gamma_1(1/2) = -\frac{1}{2} \log 2$ while $\log(2 \sin \pi/2) = \log 2$. Using either method, one finds that $c = -1$, proving equation (25). □

Next let us prove Lerch's identity (19). We showed during the proof of Lemma 15 that $\gamma_1(x) = \log(\Gamma(x)/C)$ for some constant $C > 0$. Exponentiating (25), we obtain

$$\frac{\Gamma(x)}{C}\frac{\Gamma(1-x)}{C} 2 \sin \pi x = 1 .$$

Since

$$\Gamma(x) \sim x^{-1}, \qquad \Gamma(1-x) \sim 1, \qquad \text{and} \qquad 2 \sin \pi x \sim 2\pi x$$

as $x \to 0$, it follows that $C = \sqrt{2\pi}$, as required. □

This argument also proves the classical *Euler functional equation*

$$\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x .$$

Taking $x = 1/2$, it proves that $\Gamma(1/2) = \sqrt{\pi}$.

Similarly, exponentiating (23), we obtain the classical *Gauss multiplication formula*

$$\frac{\Gamma(x)}{\sqrt{2\pi}} = m^{x-1/2} \prod_0^{m-1} \frac{\Gamma((x+k)/m)}{\sqrt{2\pi}} .$$

As an example, taking $x = 1$ and $m = 2$, we obtain another proof that $\Gamma(1/2) = \sqrt{\pi}$.

Note that each γ_{s+1} is essentially just an indefinite integral of γ_s, up to a constant factor and a polynomial summand. More precisely, differentiating the equation

$$\zeta_t'(x) = -t\zeta_{t+1}(x)$$

with respect to t and setting $s = -t$, we find that

$$(26) \qquad \gamma_{s+1}'(x) = \partial\gamma_{s+1}(x)/\partial x = s\gamma_s(x) + \beta_s(x)/s .$$

The function $\exp(\gamma_s(x))$ can be thought of as a kind of higher order gamma function, satisfying

$$\exp(\gamma_s(n+1) - \gamma_s(1)) = 1^{1^{s-1}} 2^{2^{s-1}} \dots n^{n^{s-1}} .$$

(Compare Shintani [24].)

As a final remark, let us apply these methods to derive the Stirling asymptotic series for $\gamma_1(x)$ as $x \to \infty$. Using (26), together with (3) and (20), we have

$$\int_x^{x+1} \gamma_1(u)du = x \log x - x .$$

As in the discussion of Bernoulli polynomials in §2, the left side of this equation can be expanded as a Taylor series

$$\frac{e^D - I}{D} \gamma_1(x) = \sum_0^{\infty} D^n\gamma_1(x)/(n+1)! ,$$

which converges whenever $\gamma_1(x)$ is analytic throughout a unit disk centered at x, or in other words whenever $x > 1$. Here D stands for d/dx. Recall from §2 that the inverse operator is given formally by

$$\frac{D}{e^D - I} = \sum_0^\infty b_n D^n/n!\,.$$

Hence, applying this inverse operator to both sides of the equation

$$\frac{e^D - I}{D}\gamma_1(x) = x \log x - x\,,$$

we might hope that

$$\gamma_1(x) \overset{?}{=} \frac{D}{e^D - I}(x \log x - x) = \sum_0^\infty b_n D^n(x \log x - x)/n!\,.$$

Unfortunately, this series does not converge. However, if we truncate, setting

$$s_N(x) = \sum_0^N b_n D^n(x \log x - x)/n!$$

for some integer $N \geqslant 1$, then we will prove that

$$\gamma_1(x) = s_N(x) + O(x^{-N})$$

as $x \to \infty$. This is the required asymptotic series. More explicitly, we can write it as

$$\gamma_1(x) = (x \log x - x) - \frac{1}{2}\log x + \sum_2^N \frac{b_n x^{1-n}}{n(n-1)} + O(x^{-N})\,. \tag{27}$$

(For a more precise description of the error term, see [1, p. 31]. Using (19) this yields the corresponding asymptotic formula for $\Gamma(x)$.)

To prove this formula, substitute the identity

$$x \log x - x = \sum_0^\infty \frac{D^m}{(m+1)!}\gamma_1(x)$$

in the definition of $s_N(x)$ to obtain a double series

$$s_N(x) = \sum_{n=0}^N \sum_{m=0}^\infty \frac{b_n D^n}{n!}\frac{D^m}{(m+1)!}\gamma_1(x)\,,$$

which converges absolutely whenever $x > 1$. If we collect terms involving the same total power of D, then evidently all the terms involving $D^1, D^2, \ldots, D^N$ must cancel. Since

$$D^n\gamma_1(x) = \pm(n-1)!\zeta_n(x)$$

for $n \geq 2$, it follows that the resulting series has the form

$$s_N(x) = \gamma_1(x) + \sum_{N+1}^{\infty} a_n \zeta_n(x)$$

for suitable constants a_n. Setting

$$E(x) = \sum_{N+1}^{\infty} a_n x^{-n},$$

we can write the error term as

$$s_N(x) - \gamma_1(x) = E(x) + E(x+1) + \ldots .$$

Note that all of these series converge absolutely for $x > 1$. Evidently

$$E(x) = O(x^{-N-1})$$

as $x \to \infty$, for any fixed N, so

$$s_N(x) - \gamma_1(x) = O(x^{-N})$$

as required. □

This argument yields similar asymptotic series for related functions such as $\zeta_s(x)$, $\gamma_s(x)$, and $\gamma_s'(x)$. Such estimates work also for complex values of x, as long as x stays well away from the negative real axis.

Appendix 3

Volume and the Dehn invariant in hyperbolic 3-space

We will describe some constructions in hyperbolic space involving the dilogarithm function $\mathcal{L}_2(z)$ and its Kubert identity (7). Further details may be found in the paper "Scissors Congruences, II" by J. L. Dupont and C.-H. Sah (*J. Pure Appl. Algebra* 25 (1982), 159-195).

Using the upper half-space model for hyperbolic 3-space, consider a totally asymptotic 3-simplex Δ. In other words, we assume that the vertices a, b, c, d of Δ all lie on the 2-sphere of points at infinity, which we identify with the extended complex plane $\mathbf{C} \cup \infty$. Then Δ is determined up to orientation preserving isometry by the cross ratio

$$z = (a, b; c, d) = (c-a)(d-b)/(c-b)(d-a).$$

[The semicolon is inserted in our cross ratio symbol as a remainder of its symmetry properties, which are similar to those of the four index symbol R_{hijk} in Riemannian geometry.] In particular, the volume of Δ can be expressed as a function of the cross ratio z.

THEOREM (S. Bloch and D. Wigner). *If* z *belongs to the upper half-plane,* $\mathrm{Im}(z) > 0$, *then this volume* $V = V(z)$ *is equal to the imaginary part of the dilogarithm* $\mathscr{L}_2(z)$ *plus a correction term of* $\log|z| \arg(1-z)$. *The correspondence* $z \mapsto V(z)$ *for* $\mathrm{Im}(z) > 0$ *extends to a function which is single valued and real analytic throughout* $\mathbf{C} - \{0, 1\}$, *and continuous throughout* $\mathbf{C} \cup \infty$.

Here we use the branch $-\pi < \arg(1-z) < \pi$ of the many valued function $\arg(1-z)$ in the region $\mathrm{Im}(z) > 0$.

Proof. For the first assertion, it suffices to consider the simplex Δ with vertices ∞, 0, 1, z; where we assume that $\mathrm{Im}(z) > 0$. The image of Δ under vertical projection from the point ∞ to the boundary plane $\mathbf{C}$ is just the Euclidean triangle with vertices 0, 1, z. Let

$$\theta_1 = \arg(z),\ \theta_2 = \arg(1/(1-z)),\ \theta_3 = \arg((z-1)/z)$$

be the angles at these three vertices, equal to corresponding dihedral angles of the hyperbolic simplex Δ. Note that $\Sigma\theta_k = \pi$. We will assume the volume formula

$$V(z) = \Sigma\Lambda(\theta_k)\,, \tag{28}$$

to be summed from 1 to 3, where

$$\Lambda(\theta) = -\int_0 \log(2\sin\theta)d\theta\,.$$

This is proved for example in [21]. Using the law of sines

$$\sin\theta_1 : \sin\theta_2 : \sin\theta_3 = |1-z| : |z| : 1$$

and the equation $\Sigma\, d\theta_k = 0$, we see that

$$dV(z) = -\Sigma \log(2\sin\theta_k)d\theta_k$$

is equal to $-\log|1-z|\,d\theta_1 - \log|z|\,d\theta_2$; or in other words

$$dV(z) = \log|z|\,d\arg(1-z) - \log|1-z|\,d\arg(z)\,. \tag{29}$$

On the other hand $d\mathscr{L}_2(z) = -\log(1-z)d\log(z)$, hence,

$$d\,\mathrm{Im}\,\mathscr{L}_2(z) = -\log|1-z|\,d\arg(z) - \arg(1-z)d\log|z|\,.$$

The required formula

$$V(z) = \mathrm{Im}\,\mathscr{L}_2(z) + \log|z|\arg(1-z) \tag{30}$$

then follows since both sides of this equation have the same total differential, and since both sides tend to the limit zero as z tends to any point of the real interval $(0, 1)$.

As an example of this formula, note the identity

$$V(e^{2i\theta}) = \operatorname{Im} \mathscr{L}_2(e^{2i\theta}) = 2\Lambda(\theta).$$

Since the right side of (29) is a well defined smooth closed 1-form, everywhere on $\mathbf{C} - \{0, 1\}$, we need only check that its integral in a loop around zero or one vanishes, in order to check that $V(z)$ extends as a single valued function. But the expression (30) shows that $V(z)$ extends to a single valued function near zero, and also that $V(z)$ tends to zero as $z \to 0$. Using the identity

$$V(z) + V(1-z) = 0$$

which follows from (29), we see that the same is true for z near 1.

Now consider the fractional linear automorphism of period three

$$z \mapsto 1/(1-z) \mapsto (z-1)/z \mapsto z$$

which carries the upper half-plane to itself. The expression (28) shows that

$$V(z) = V\big(1/(1-z)\big) = V\big((z-1)/z\big).$$

Since $0 \mapsto 1 \mapsto \infty \mapsto 0$, it follows that $V(z)$ also tends to zero as $z \to \infty$. □

Note that $V(z)$ is strictly positive in the upper half-plane for geometrical reasons. The identity

$$V(\bar{z}) = -V(z)$$

shows that $V(z)$ is negative on the lower half-plane and zero on $\mathbf{R} \cup \infty$. Note also the identities

(31) $$V(1-z) = V(1/z) = -V(z),$$

which are equivalent to the statement that the expression $V(a, b; c, d)$ is skew symmetric as a function of four variables.

This function $V(z)$ satisfies the *multiplicative Kubert identity*

(32) $$V(z^n) = n\Sigma\, V(wz),$$

to be summed over all n-th roots of unity, $w^n = 1$. This follows easily since both $\mathscr{L}_2(z)$ and $\log|z| \arg(1-z)$ satisfy this same identity for z near zero.

Another important property is the *cocycle equation*

(33) $$\Sigma_0^4(-1)^i\, V(a_0, \ldots, \hat{a}_i, \ldots, a_4) = 0,$$

for any five distinct points $a_0, \ldots, a_4$ in $\mathbf{C} \cup \infty$. Geometrically, this is true since the convex body in hyperbolic space spanned by five vertices can be expressed as a union of simplices with disjoint interiors in two different ways. Analytically, it can be proved using the *Abel functional equation*

$$\mathscr{L}_2(xx'yy') = \mathscr{L}_2(xy') + \mathscr{L}_2(yx') + \mathscr{L}_2(-xx') + \mathscr{L}_2(-yy') + \log^2(x'/y')/2\,,$$

where x' stands for $1/(1-x)$. Still another proof will be sketched later.

Dupont and Sah show that the Kubert identity can be proved as a formal consequence of this cocycle equation. Hence it has a geometric interpretation in terms of cutting and pasting of simplices. As a geometric corollary, they prove that the "scissors congruence group" for hyperbolic 3-space is divisible. That is any hyperbolic polyhedron can be cut up and reassembled into n pieces which are isometric to each other, for any n.

Another geometric invariant associated with a hyperbolic simplex is the *Dehn invariant.* For a finite 3-simplex, this is defined to be the six fold sum

$$\Sigma_{\text{edges}}\ \text{length} \otimes (\text{dihedral angle})$$

in the additive group $\mathbf{R} \otimes (\mathbf{R}/2\pi\mathbf{Z})$, taking the tensor product over $\mathbf{Z}$. For a simplex with one or more vertices in $\mathbf{C} \cup \infty$, the definition is the same except that we must first chop off a horospherical neighborhood of each infinite vertex before measuring edge lengths. The result does not depend on the particular choice of horospheres.

LEMMA (Dupont and Sah). *For a totally asymptotic simplex, with dihedral angles* $\theta_1, \theta_2, \theta_3$ *along the edges meeting at a vertex, this Dehn invariant is equal to* $2\Sigma \log(2 \sin \theta_i) \otimes \theta_i$.

If we express this as a function of the associated cross ratio z, using the law of sines as above, the formula becomes

$$\frac{1}{2}\,\text{Dehn}(z) = \log|1 - z| \otimes \arg(z) - \log|z| \otimes \arg(1-z)\,.$$

This function also satisfies the Kubert identity (32), and it is clear from its geometric definition that it satisfies the symmetry condition (31) and the cocycle equation (33).

To prove this lemma, we first choose one particular horospherical neighborhood of each vertex. It is convenient to choose that horosphere which is tangent to the opposite face. Consider, for example, a simplex with vertices ∞, v_1, v_2, v_3. The preferred horosphere through v_i can be described as a Euclidean sphere, tangent to the boundary plane $\mathbf{C}$ at v_i, and tangent to the orthogonal plane which passes through the other two vertices v_j, v_k. The Euclidean radius r_i of this sphere is equal to the distance of v_i from the line through v_j, v_k. In other words r_i is equal to an altitude of the triangle v_1, v_2, v_3. Hence r_i is inversely proportional to the edge length $|v_j - v_k|$, and inversely proportional to $\sin \theta_i$; say $r_i = c/\sin \theta_i$.

This horosphere intersects the line from v_i to ∞ at Euclidean height $h = 2r_i$. On the other hand, the preferred horosphere through the point ∞ intersects each vertical line at some constant height $h = c'$. If we integrate the hyperbolic length element dh/h along the line from v_i to ∞ between these two intersection points, we obtain

$$\text{(34)} \qquad \text{truncated edge length} = \int_{2r_i}^{c'} dh/h = \log(2 \sin \theta_i) + c''$$

where $c'' = \log(c'/4c)$ is constant. (Here negative lengths must be allowed.) The corresponding contribution to the Dehn invariant is

$$\log(2 \sin \theta_i) \otimes \theta_i + c'' \otimes \theta_i \,.$$

There is an identical contribution from the opposite edge v_j, v_k. *In fact the symmetry property*

$$(a, b\,; c, d) = (c, d\,; a, b)$$

of the cross ratio implies that there is an isometry of Δ *carrying any given edge to the opposite edge.* Now, summing over all six edges, since the $c'' \otimes \theta_i$ terms cancel, we obtain the required formula

$$\text{(35)} \qquad \mathrm{Dehn}(\Delta) = 2\Sigma_1^3 \log(2 \sin \theta_i) \otimes \theta_i \,. \qquad \square$$

Remark. The curious similarity between the two equations (28) and (35) can be explained by a theorem of Schläfli. For a family of simplices Δ in the n-dimensional spherical space of constant curvature $K > 0$, Schläfli's equation can be written as

$$(n-1)K\, dV_n(\Delta) = \Sigma\, V_{n-2}(F) d\theta_F \,,$$

to be summed over all $(n-2)$-dimensional faces F, where $V_{n-2}(F)$ is the $(n-2)$-dimensional volume and θ_F is the dihedral angle along F. In other words, we have

$$(n-1)K\, \partial V_n/\partial\theta_F = V_{n-2}(F) \,.$$

For a proof, also in the case $K < 0$, see Kneser, "Der Simplexinhalt in der nichteuklidischen Geometrie", *Deutsche Math. 1* (1936), 337-340. In the case $n = 3$, $K = -1$, the Schläfli equation takes the form

$$-2\, dV_3(\Delta) = \Sigma_{\mathrm{edges}}\, V_1(E) d\theta_E \,.$$

For a family of 3-simplices with one or more vertices at infinity, this equation remains valid providing that we cut off a horospherical neighborhood of each infinite vertex before measuring edge lengths. It follows that we can prove equation (34) simply by differentiating (28), or conversely that we can prove (28) by integrating (34), using the identity $\Lambda(0) = \Lambda(\pi) = 0$ to fix the constant of integration.

Although the cocycle equation for the Dehn invariant is an immediate consequence of its geometric definition, it may be of interest to give an analytic proof. Let us introduce the skew-symmetric bimultiplicative symbol

$$(x|y) = \log|x| \otimes \arg(y) - \log|y| \otimes \arg(x),$$

for x and y in the multiplicative group $\mathbf{C}^{\bullet}$, with values in the additive group $\mathbf{R} \otimes (\mathbf{R}/2\pi\mathbf{Z})$. Then

$$\frac{1}{2}\,\mathrm{Dehn}(z) = \frac{1}{2}\,\mathrm{Dehn}(a, b; c, d)$$

is equal to $(1-z|z)$. Expressing z and $1 - z = (a, c; b, d)$ as 4-fold products and using the bimultiplicative property, we can expand $(1-z|z)$ as a sum of sixteen terms, of which four cancel. The remaining twelve can be grouped as

$$(1-z|z) = f(b, c, d) - f(a, c, d) + f(a, b, d) - f(a, b, c),$$

where f stands for the skew function

$$f(a, b, c) = (a-b|b-c) + (b-c|c-a) + (c-a|a-b).$$

This proves that the function $\mathrm{Dehn}(a, b; c, d)$ is a coboundary, and hence a cocycle.

We can define a sharpened Dehn invariant by this same formalism, using the expression

$$\log(x) \wedge \log(y),$$

with values in $\wedge^2_{\mathbf{Z}}(\mathbf{C}/2\pi i\mathbf{Z})$ in place of our symbol $(x|y)$. If we split this exterior power into eigenspaces under the action of complex conjugation, then the component of

$$\log(x) \wedge \log(y)$$

in the -1 eigenspace can be identified with $(x|y)$.

The cocycle equation for the volume function $V(a, b; c, d)$ can also be proved by this formalism. We must simply replace $(x|y)$ by the differential form valued symbol

$$\log|x|\, d\arg(y) - \log|y|\, d\arg(x).$$

Details will be omitted.

Dupont and Sah show that the volume function and the sharpened Dehn invariant can be incorporated into a single function ρ, as follows. Let

$$\rho(z) = 1 \wedge L(z) - 1 \wedge L(1-z) + l(z) \wedge l(1-z),$$

with values in $\wedge^2 \mathbf{C}$, where $l(z) = \log(z)/2\pi i$ and

$$L(z) = \mathscr{L}_2(z)/4\pi^2 = \int_0 l(1-z)dl(z).$$

This expression is certainly well defined in the strip $0 < \operatorname{Re}(z) < 1$, and satisfies $\rho(z) + \rho(1-z) = 0$. If we analytically continue each of its constituent functions in a loop around zero or one, then the expression $\rho(z)$ remains unchanged. Hence ρ is well defined as a mapping from $\mathbf{C} - \{0, 1\}$ to $\wedge^2 \mathbf{C}$. They show that ρ also satisfies the symmetry condition (31), the Kubert identity (32), and the cocycle equation (33).

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