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3. p-DEFINABILITY

The concept of p-definability was introduced in [13] to characterize a large class of polynomials. Among naturally occurring polynomials of p-bounded degree it appears to contain a large majority. In this section we shall start to explore the extent of the class by considering various equivalent definitions of it. We start with the one given in [13] in its most simplified form.

Definition 1. A family P is *p-definable* over F iff *either* (a) $\exists Q$ over F of p-bounded formula size such that for all i

$$P_i = \sum_{\substack{(b_1, \ldots, b_i) \\ \in \{0,1\}^i}} \left[Q_i(b_1, \ldots, b_i) \prod_{b_k = 1} x_k \right] \qquad (\dagger)$$

or (b) P is the p-projection of a p-definable family.

If two polynomials P_i, Q_i are related as in part (a) of the definition we say that Q_i *defines* P_i. This relationship is to be interpreted as follows: P_i may or may not be a tractable polynomial but at least its coefficients are, i.e. there is a tractable Q_i whose values at the points $\{0, 1\}^i$ are just the 2^i coefficients of P_i.

The permanent and determinant are widely recognised as being among the conceptually simplest polynomials. This is reflected here by the fact that part (a) of the above definition is sufficient to specify them. For example $\mathrm{Perm}_{n \times n}\{x_{ij} \mid 1 \leqslant i, j \leqslant n\}$ is defined by

$$Q_{n \times n} = \left(\prod_{i=1}^{n} \sum_{j=1} y_{ij} \right) \left(\prod_{\substack{j = m \\ i \neq k}} (1 - y_{ij}\, y_{km}) \right).$$

Part (a) of the definition on its own, however, would be artificial and restrictive. Certainly only multilinear polynomials would be allowed. Also HC can be defined using (a) and (b) together (see Appendix 2) but apparently not with (a) alone.

Definition 1 is somewhat opaque. For example, it does not make clear even whether it covers all p-computable families. To resolve such questions the following formulation is useful.

Definition 2. A family P is *p-definable* over F iff *either* (a) $\exists Q$ over F that is p-computable such that for all i for some j $(0 \leqslant j \leqslant i)$

$$P_i(x_1, \ldots, x_i) = \sum_{\substack{b_{j+1}, \ldots, b_i \\ \in \{0,1\}^{i-j}}} \left[Q_i(x_1, \ldots, x_j, b_{j+1}, \ldots, b_i) \prod_{b_k = 1} x_k \right]$$

or (b) P is the p-projection of a p-definable family.

Later we shall see that this is indeed equivalent to Definition 1.

Remark 1. Every p-computable P is p-definable, for in Definition 2 we can take $Q_i = P_i$ and $j = i$.

Consider now a mathematically still simpler formulation that will be useful for proving closure properties.

Definition 3. A family P is *p-definable* over F if there is a p-computable Q and a polynomial t such that for all m there is an $i \leqslant t(m)$ such that

$$P_m(x_1, \ldots, x_m) = \sum_{\substack{b_{m+1}, \ldots, b_i \\ \in \{0,1\}^{i-m}}} Q_i(x_1, \ldots, x_m, b_{m+1}, \ldots, b_i) \quad .$$

PROPOSITION 1. *Definitions 2 and 3 are equivalent.*

Proof. Clearly P_m defined in Definition 3 can be translated into Definition 2 by taking the same defining Q_i, choosing $j = m$ and taking the projection $x_k = 1$ for $k = j + 1, \ldots, i$.

In the converse direction consider P_i as in Definition 2 (a). It clearly equals

$$\sum_{\substack{b_{j+1}, \ldots, b_i \\ \in \{0,1\}^{i-j}}} Q_i(x_1, \ldots, x_j, b_{j+1}, \ldots, b_i) \prod_{r = j+1}^{i} (x_r b_r + (1 - b_r))$$

which is of the form required in Definition 3 (but with a different Q_i !) $\square$

For completeness and further simplicity we may also consider:

Definition 4. As Definition 3 but with Q restricted to p-bounded formula size.

PROPOSITION 2. *Definitions 1 and 4 are equivalent.*

Proof. Clearly Definition 1 implies Definition 4 exactly as Definition 2 implies Definition 3 (see proof of Proposition 1 above.)

To see the converse we use the form used in [13]. This is conveniently called Definition 1* as it is intermediate between Definitions 1 and 2. It is identical to Definition 1 except that line (†) is replaced by:

$$P_i = \sum_{\substack{b_{j+1}, \ldots, b_i \\ \in \{0,1\}^{i-j}}} \left[Q_i(x_1, \ldots, x_j, b_{j+1}, \ldots, b_i) \prod_{b_k = 1} x_k \right] \quad \text{for some } j\,.$$

Suppose now that a family P is p-definable in the sense of Definition 4. Then the argument in Proposition 1 showing that Definition 3 implies Definition 2 establishes that P is p-definable in the sense of Definition 1*. But Theorem 3 in [13] shows that any P so definable is the p-projection of HC and our Appendix 2 shows that HC is p-definable in the sense of Definition 1. The result follows. □

In Appendix 1 it will be shown that Definition 3 implies Definition 4. Together with Propositions 1 and 2 this will establish:

THEOREM 1. *Definitions 1, 2, 3 and 4 are all equivalent.*

4. CLOSURE PROPERTIES

A p-definable family P is *complete* over F if every family that is p-definable over F is the p-projection of P. It is known that several famous polynomials such as the permanent, hamiltonian circuits, the monomer-dimer polynomial and certain reliability problems are all complete for appropriate fields [6, 13]. In fact the projections required to establish these facts are all *strict projections* (i.e. no two indeterminates map to the same indeterminate). Hence these superficially dissimilar polynomials are related in the closest possible way: each one can be obtained from any other by fixing some indeterminates and renaming the others.

In the light of the simplicity of its completeness class the robustness of the notion of p-definability is perhaps remarkable. It can be explored conveniently by listing the operations under which it is closed.

First we consider the operation of substitution. The polynomials to be substituted can be viewed conveniently as an array.

Definitions. R is a *family array* over F if it is a set $\{ R^{m,n} \mid n \leqslant m \}$ of polynomials over F where $R^{m,n}$ has m indeterminates. It has *p-bounded degree* if for some p-bounded t $\deg(R^{m,n}) \leqslant t(m)$.

The various definitions of p-definability have analogues that are equivalent to each other for family arrays. For the current purpose it is best to adapt the fourth one:

Definition. Family array R is *p-definable* iff there is a p-bounded t such that for all m, n there is a T with formula size less than $t(m)$ such that

$$R^{m,n} = \sum_{\mathbf{b}} T(\mathbf{x}, \mathbf{b}) .$$