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has the form $\exists x \psi (xx_1 \dots x_n)$ and that $\| \psi [bb_1 \dots b_n] \|$ is clopen for fixed $b_1, \dots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters $b_1 \dots, b_n$. Let

$$u = \cup \{ \| \psi [\beta] \| | \beta \in B \}.$$

By our inductive assumption, u is an open subset of X. Choose, by Zorn's lemma, a maximal family $F = \{(b_i, c_i) \mid i \in I\}$ such that $b_i \in B$, c_i is a clopen subset of u, $c_i \subseteq \|\psi[b_i]\|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that c, the closure of $\bigcup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA, $i \in I$

hence X is extremally disconnected and c is clopen. By completeness of B, there is some $b \in B$ such that $b \cdot e(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \|\psi[b]\|$. So, for $\beta \in B$, $\|\psi[\beta]\| \subseteq u \subseteq c \subseteq \|\psi[b]\| = \|\exists x \psi(x)\|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha(x)$ be the \mathcal{L}_{BA} -formula stating that x is an atom and let $\beta(x)$, $\gamma(x)$ be the \mathcal{L}_{BA} -formulas $\alpha(x) \vee x = 0$ resp. $\forall y (\alpha(y) \to y \leqslant x)$. Put $M = \{f \in B \mid \|\beta[f]\| = 1 \|$ and let b be the supremum of M in B. We show that b(p) is, for each $p \in X$, the supremum of the atoms of B_p .

First suppose $s \in B_p$ is an atom of B_p . There is some $f \in M$ such that f(p) = s (note that $\| \alpha [f] \|$ is clopen for each $f \in B$). So $f \leqslant b$ and $s = f(p) \leqslant b(p)$. — On the other hand, suppose $t \in B_p$ and $s \leqslant t$ for every atom s of B_p . Choose $g \in B$ such that g(p) = t. Then $p \in c = \| \gamma [g] \|$. For $f \in M$, $e(c) \cdot f \leqslant g$, since $g \in C$ implies that f(g) is zero or an atom of B_q and thus $f(g) \leqslant g(g)$. By the definition of $g \in C$, this implies (by $g \in C$) $g \in C$ and $g \in C$ implies that $g(g) \in C$ i

4. Decidability and completions of Th(K)

Call $T_{sBA} = T_{BA} \cup \{\sigma\}$ the theory of separated BAs, where T_{BA} is the theory of BAs and σ was defined in section 3. We give a short review of the completions of T_{sBA} . Let, for $n \in \omega$, φ_n be the \mathcal{L}_{BA} -sentence stating that there are exactly n atoms and ψ the \mathcal{L}_{BA} -sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg (\varphi_0 \lor ... \lor \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an \mathcal{L}_{BA} -theory T_{ni} by

$$T_{n0} = T_{sBA} \cup \{ \varphi_n, \neg \psi \}$$

$$T_{n1} = T_{sBA} \cup \{ \varphi_n, \psi \}$$

for $n \in \omega$, and

$$T_{\omega 0} = T_{sBA} \cup \left\{ \chi_n \mid n \in \omega \right\} \cup \left\{ \neg \psi \right\}$$

$$T_{\omega 1} = T_{sBA} \cup \left\{ \chi_n \mid n \in \omega \right\} \cup \left\{ \psi \right\}.$$

Put $\tau = \{T_{ni} \mid n \in \omega + 1, i \in 2\}$. It is then clear that each separated BA satisfies exactly one of the theories in τ , and for each $t \in \tau$ there is a cBA satisfying t. Moreover, any two models of any $t \in \tau$ are elementarily equivalent by 5.5.10 in [1]. Thus the theories $t \in \tau$ are just the completions of T_{sBA} and can be thought of as being the elementary equivalence types of separated BAs or cBAs. Moreover, an \mathcal{L}_{BA} -sentence holds in every separated BA iff it holds in every cBA. The following proposition is essential for the main theorems of this section:

4.1. PROPOSITION. Let $s, t \in \tau$. Then there is a structure (B, A) in K such that A is a model of s and each stalk B_p is a model of t.

Proof. By the above remarks, choose $cBAs\ A$ and F which are models of s resp. t. Let A*F be the free product of A and F. Thus A is relatively complete in A*F and each stalk $(A*F)_p$, where p is an ultrafilter of A, is easily seen to be isomorphic to F, hence a model of t. Unfortunately, A*F is incomplete unless A or F is finite. So let $B=(A*F)^*$ be the completion of A*F; note that A*F is a dense subalgebra of B. (B,A) $\in \mathbf{K}$, since the inclusion maps from A to A*F and from A*F to B are complete. For $p \in X$ (the Stone space of A), B_p is a separated BA by 3.2 but in general a proper extension of $(A*F)_p$. We show, with the notations of section 1, that B_p is elementarily equivalent to F. For the following proof of this, recall that, for $f \in F \setminus \{0\}$ and $p \in X$, $\pi_p(f) = f(p) \neq 0$ since F is independent from A in $A*F \subseteq B$. Thus, the restriction of $\pi_p: B \to B_p$ to F is a monomorphism. The elementary equivalence of B_p and F is established by the following four claims.

Claim 1. For each atom f of F, f(p) is an atom of B_p (hence, if F has at least n atoms, where $n \in \omega$, then B_p has at least n atoms): clearly, f(p) > 0 for $p \in X$. Assume that

$$u = \{ p \in X | f(p) \text{ is not an atom of } B_p \}$$

is non-empty. By 3.2, u is a clopen subset of X. Choose, by the maximum principle stated in section 3, $b \in B$ such that b(p) = 0 for $p \notin u$ and 0 < b(p) < f(p) for $p \in u$. Since b > 0, choose $a \in A$ and $g \in F$ such that $0 < a \cdot g \leq b$; let $p \in X$ such that $a(p) \cdot g(p) \neq 0$. Thus $p \in u$, a(p) = 1, and

 $0 < g(p) \le b(p) < f(p)$. It follows that 0 < g < f, contradicting the fact that f was an atom of F.

Claim 2. If B_p has at least n atoms, where $1 \le n < \omega$, then F has at least n atoms: assume that M is a subset of $At(B_p)$, the set of atoms of B_p , such that M has exactly n elements but At(F) has at most n-1 elements. We prove:

(a) Let $x \in M$. Then there is $f_x \in At(F)$ such that $f_x(p) = x$.

Claim 2 follows from (a), since the assignment of f_x to x is injective. And (a) will follow from

(b) Let $x \in M$, u a clopen neighbourhood of p such that, w.l.o.g., for $q \in u$, B_q has at least one atom. Let $b \in B$ such that, for $q \notin u$, b(q) = 0 and for $q \in u$, b(q) is an atom of B_q , and b(p) = x. Then there are $q \in u$ and $f \in At(F)$ such that f(q) = b(q). (Hence At(F) is non-empty).

Proof of (b). By b > 0, choose $a \in A$, $f \in F$ such that $0 < a \cdot f \le b$. Since b(q) = 0 for $q \notin u$, there is some $q \in u$ such that $a(q) \cdot f(q) \neq 0$, which implies $0 < f(q) \le b(q)$. f(q) = b(q), since b(q) is an atom of B_q . Finally $f \in At(F)$, since a splitting of f in F into two non-zero disjoint elements would give rise to a splitting of b(q) in B_q .

Proof of (a). Let $x \in M$ and choose u and b as in (b). Assume (a) is false. Then, for each $f \in At(F)$, $f(p) \neq x = b(p)$; by finiteness of At(F), there is a clopen neighbourhood v of p such that, for $q \in v$ and $f \in At(F)$, $b(q) \neq f(q)$. Let $c \in B$ such that c(q) = 0 for $q \notin v$ and c(q) = b(q) for $q \in v$. This contradicts (b), applied to v and c instead of u and b.

Claim 3. If F has a non-zero atomless element f (which means that $F \upharpoonright f$ is atomless), then each B_p has a non-zero atomless element x: let $x = \pi_p(f)$. x > 0, since π_p is one-one on F. $F \upharpoonright f$ and hence, by Claim 2, $(B \upharpoonright f)_p$ is atomless. So $B_p \upharpoonright x = \pi_p(B \upharpoonright f) = (B \upharpoonright f)_p$ is atomless.

Claim 4. If B_p has a non-zero atomless element x, then F has a non-zero atomless element f: assume that F is atomic. Let

$$u = \{ q \in X \mid B_q \text{ is not atomic} \}$$
.

u is a clopen neighbourhood of p. By the maximum principle, choose $b \in B$ such that b(q) = 0 for $q \notin u$, b(q) is a non-zero atomless element of

 B_q for $q \in u$, b(p) = x. Choose $a \in A$, $g \in F$ such that $0 < a \cdot g \le b$; w.l.o.g., g is an atom of F. Choose $q \in X$ such that $a(q) \cdot g(q) \ne 0$. Thus $q \in u$ and $g(q) \le b(q)$. By Claim 1, g(q) is an atom of B_q , contradicting the choice of b(q).

4.2. Remark. Suppose that, for every i in an index set I, $\mathcal{M}_i = (B_i, A_i)$ is an element of K. Then $\mathcal{M} = (B, A)$, where $B = \prod_{i \in I} B_i$ and $A = \prod_{i \in I} A_i$, is in K. Let $\varphi(x_1 \dots x_k)$ be an \mathcal{L} -formula and $b_1, \dots, b_k \in B$, $b_j = (b_{ij})_{i \in I}$. Put $a_i = e(\|\varphi[b_{i1} \dots b_{ik}]\|^{\mathcal{M}_i})$. Then

$$e\left(\parallel\varphi\left[b_1\ldots b_k\right]\parallel^{\mathcal{M}}\right) \,=\, (a_i)_{i\in I}\,.$$

Proof. By induction on the complexity of φ .

We shall need the following Feferman-Vaught theorem about sheaves over Boolean spaces from [2]:

4.3. Theorem (Comer). Let \mathcal{L} be an arbitrary language. There is an effective assignment

$$\varphi(x_1 \dots x_k) \mapsto (\Phi; \vartheta_1, \dots, \vartheta_m)$$

for \mathcal{L} -formulas $\varphi(x_1 \dots x_k)$ such that

a) $\theta_1, ..., \theta_m$ are \mathcal{L} -formulas having at most the free variables $x_1 ... x_k$, and

$$\models (\bigvee_{1 \leq i \leq m} \vartheta_i) \land \bigwedge_{1 \leq i < j \leq m} \neg (\vartheta_i \land \vartheta_j)$$

- b) Φ is an \mathcal{L}_{BA} -formula having at most the free variables $y_1 \dots y_m$;
- c) for each sheaf $\mathcal{G} = (S, \pi, X, \mu)$ of \mathcal{L} -structures such that X is a Boolean space and $\|\psi[f_1...f_n]\|$ is a clopen subset of X for every $\psi(x_1...x_n)$ in \mathcal{L} and $f_1,...,f_n \in \Gamma(\mathcal{G})$: if $b_1,...,b_k \in \Gamma(\mathcal{G})$, then

$$\Gamma\left(\mathscr{S}\right) \models \varphi\left[b_{1} \dots b_{k}\right] \quad iff \quad C \models \Phi\left[c_{1} \dots c_{m}\right],$$

where C is the BA of clopen subsets of X and $c_i = \| \vartheta_i [b_1 ... b_k] \|$.

For two separated $BAs\ A$ and A', let I be the set of partial functions f from A to A' such that dom $(f) = \{a_1, ..., a_n\}$ is a finite partition of A (where some of the a_i may be zero), $rge(f) = \{a_1', ..., a_n'\}$ where $a_i' = f(a_i)$ is a partition of A', and every $A \mid a_i$ is elementarily equivalent

to $A' \upharpoonright a_i'$. If A, A' are \aleph_1 -saturated or σ -complete, the following conditions are equivalent:

- a) $A \equiv A'$;
- b) I is non-empty;
- c) I has the back-and-forth property.

Moreover, if $f \in I$ is as above and A, A' are arbitrary separated BAs, then $(A, a_1, ..., a_n) \equiv (A', a_1', ..., a_n')$.

Let T_{sBA2} be the \mathscr{L} -theory

$$T_{sBA2} = T_{sBA} \cup \left\{ \forall x \left(U(x) \leftrightarrow x = 0 \lor x = 1 \right) \right\}.$$

Since T_{BA} is decidable, T_{sBA} and T_{sBA2} are decidable.

4.4. Theorem. There is an effective procedure deciding for every \mathcal{L} -sentence φ whether $T \vdash \varphi$. Moreover, $T \vdash \varphi$ if and only if φ holds in every model \mathcal{M} in \mathbf{K} .

Proof. Let φ be given. Construct $(\Phi(y_1 \dots y_m); \vartheta_1, \dots, \vartheta_m)$ by 4.3. For every i such that $1 \leqslant i \leqslant m$, decide whether $T_{sBA2} \vdash \neg \vartheta_i$. W.l.o.g., assume that $T_{sBA2} \cup \{\vartheta_i\}$ is consistent for $1 \leqslant i \leqslant r$ and inconsistent for $r+1 \leqslant i \leqslant m$. By $\vdash \vartheta_1 \vee \dots \vee \vartheta_m$, we have $1 \leqslant r$ (it is possible that r=m). Next, construct the formula

$$\Phi'\left(y_1\ldots y_m\right) = \left(\bigwedge_{r+1\leq i\leq m} (y_i=0) \to \Phi\left(y_1\ldots y_m\right)\right).$$

We show the equivalence of

- a) $T \vdash \varphi$;
- b) $\mathcal{M} \models \varphi$ for every $\mathcal{M} \in \mathbf{K}$;
- c) $T_{sBA} \vdash \forall y_1 \dots \forall y_m \Phi'(y_1 \dots y_m)$.

Then, by decidability of T_{sBA} , T is decidable and 4.4 is proved. a) implies b) by 3.2. To prove that c) implies a), assume there is $\mathcal{M} \models T$ such that $\mathcal{M} \not\models \varphi$, e.g. $\mathcal{M} = (B, A)$. Put $a_i = e (\parallel \vartheta_i \parallel^{\mathcal{M}})$. By 4.3 and $\mathcal{M} \not\models \varphi$, we see $A \not\models \Phi [a_1 \dots a_m]$. By our choice of $r \leqslant m$, we get $a_{r+1} = \dots = a_m = 0$. Thus $A \not\models \Phi' [a_1 \dots a_m]$ and c) is false. Now assume c) does not hold; we show that c0 is false. Let c1 be a separated c2 and c3. W.l.o.g., c4 such that c4 is c5. By choice of c6, there are c6 and c7 is such that c8 is c9. By choice of c9, there are c9 and c9 is for c9 is c9.

Let, for these i, s_i be the element of τ such that $A' \upharpoonright a_i' \models s_i$. By 4.1, there are $\mathcal{M} = (B, A) \in \mathbf{K}$ and $a_1, ..., a_r \in A$ such that $1 = a_1 + ... + a_r, a_i \cdot a_j = 0$ for $1 \leqslant i < j \leqslant r$, $A \upharpoonright a_i \models s_i$ and $(B \upharpoonright a_i)_p \models t_i$ for those $p \in X$ satisfying $a_i(p) = 1$. So $e(\|\vartheta_i\|^{\mathcal{M}}) = a_i$ by 4.2. Put $a_{r+1} = ... = a_m = 0$. It follows that $(A, a_1, ..., a_m) \equiv (A', a_1', ..., a_m'), A \not\models \Phi[a_1 ... a_m]$ and $\mathcal{M} \not\models \varphi$ by 4.3.

In the next theorem, we characterize elementary equivalence of models of T. Call the following sentences in \mathcal{L}_{BA} basic sentences: $\varphi_n \wedge \psi$, $\varphi_n \wedge \neg \psi$, $\chi_n \wedge \psi$, $\chi_n \wedge \psi$ (where $n \in \omega$). It follows by the analysis of the completions of T_{sBA} given in the beginning of this section that for each \mathcal{L}_{BA} -sentence ϑ there are basic sentences $\beta_1, ..., \beta_n$ such that

$$T_{sBA} \vdash (\emptyset \leftrightarrow \bigvee_{i=1}^{n} \beta_i) \land \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \land \beta_j) .$$

This fact is easily extended to T_{sBA2} : by replacing each atomic formula U(t) where t is a term in \mathcal{L}_{BA} by " $t = 0 \lor t = 1$ ", we see that for each \mathcal{L} -sentence θ there are basic sentences $\beta_1, ..., \beta_n$ satisfying

$$T_{sBA2} \vdash (\vartheta \leftrightarrow \bigvee_{i=1}^{n}) \land \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \land \beta_j).$$

Now, if β , γ are basic sentences, let $\sigma_{\beta\gamma}$ be the following \mathcal{L} -sentence:

$$\sigma_{\beta\gamma} = \exists y (\gamma^{y} \wedge s_{\beta}(y)),$$

where $s_{\beta}(y)$ is the \mathcal{L} -formula assigned to β in 3.1 and γ^{y} is the result of relativizing the quantifiers $\exists x \varphi ...$ in γ to $\exists x (U(x) \land x \leqslant y \land \varphi^{y} ...)$. A model (B, A) of T satisfies $\sigma_{\beta\gamma}$ iff $A \upharpoonright a \models \gamma$, where a = e(c) and $c = \|\beta\|$.

4.5. Theorem. Let $\mathcal{M} = (B, A)$, $\mathcal{M}' = (B', A')$ be models of T. Then \mathcal{M} is elementarily equivalent to \mathcal{M}' if and only if, for any basic sentences β , γ ,

$$\mathscr{M} \models \sigma_{\beta\gamma} \quad \text{iff} \quad \mathscr{M}' \models \sigma_{\beta\gamma} \,.$$

Proof. The only-if-part is clear. Suppose that \mathcal{M} and \mathcal{M}' satisfy the same sentences of the form $\sigma_{\beta\gamma}$. Let φ be an \mathcal{L} -sentence and $\mathcal{M} \models \varphi$; we want to show that $\mathcal{M}' \models \varphi$. Let $(\Phi(y_1 \dots y_m); \vartheta_1, \dots, \vartheta_m)$ be the sequence assigned to φ by 4.3; every ϑ_i is an \mathcal{L} -sentence. Put $a_i = e(\|\vartheta_i\|^{\mathcal{M}})$; by 4.3 and $e: C \to A$ being an isomorphism, we have that $\{a_1, \dots, a_m\}$

is a partition of A and $A \models \Phi [a_1 \dots a_m]$. In the same way, put $a_i' = e' (\| \vartheta_i \|^{\mathcal{M}'}); \{ a_1', ..., a_m' \}$ is a partition of A'. It is sufficient to show that $(A, a_1, ..., a_m) \equiv (A', a_1', ..., a_m')$, for this implies $A' \models \Phi [a_1' \dots a_m']$ and finally $\mathcal{M}' \models \varphi$ by 4.3.

For every θ_i , choose basic sentences $\beta_{i1}, ..., \beta_{in_i}$ such that

$$T_{sBA2} \vdash (\vartheta_i \leftrightarrow \bigvee_j \beta_{ij}) \land \bigwedge_{j < l} \neg (\beta_{ij} \land \beta_{il}).$$

Put $\alpha_{ij} = e \ (\|\beta_{ij}\|^{\mathcal{M}})$, $\alpha_{ij}' = e' \ (\|\beta_{ij}\|^{\mathcal{M}'})$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n_i$. Then a_i is the disjoint sum of the $\alpha_{ij} \ (1 \leqslant j \leqslant n_i)$, a_i' is the disjoint sum of the $\alpha'_{ij} \ (1 \leqslant j \leqslant n_i)$. For every i, j,

$$A \upharpoonright \alpha_{ij} \equiv A' \upharpoonright \alpha_{ij}'$$
:

let γ be any basic sentence of \mathscr{L}_{BA} and assume $A \upharpoonright \alpha_{ij} \models \gamma$; we want to show that $A' \upharpoonright \alpha_{ij}' \models \gamma$. But $A \upharpoonright \alpha_{ij} \models \gamma$ means that $\mathscr{M} \models \sigma_{\beta_{ij}\gamma}$. By our main assumption, $\mathscr{M}' \models \sigma_{\beta_{ij}\gamma}$ and $A' \upharpoonright \alpha'_{ij} \models \gamma$.

We have shown that the partial function f mapping α_{ij} to α_{ij}' is an element of the set of back-and-forth-isomorphisms defined after 4.3. Hence,

$$(A, \alpha_{11}, ..., \alpha_{mn_m}) \equiv (A', \alpha_{11}', ..., \alpha_{mn_m}')$$

and

$$(A, a_1, ..., a_m) \equiv (A', a_1', ..., a_m').$$

We shall finally describe the completions of T by giving a one-one correspondence between a set P (consisting of pairs of mappings from $\omega \times 2$ to $(\omega+1)\times 2$) and these completions. For $m,m'\in\omega+1$ and $j,j'\in 2$, define

$$(m,j) + (m',j') = (m'',j'')$$

where m'' is the cardinal sum of m and m' and j'' is the maximum of j and j'. Let

$$P = \left\{ (\alpha, \rho) \mid \alpha, \rho : \omega \times 2 \to (\omega + 1) \times 2 \text{ and, for} \right.$$
$$(n, i) \in \omega \times 2, \rho(n, i) = \rho(n + 1, i) + \alpha(n, i) \right\}.$$

In the following definition, we refer to the \mathcal{L}_{BA} -theories T_{ni} defined in the beginning of this section. For $(\alpha, \rho) \in P$, let $T_{\alpha\rho}$ the \mathcal{L} -theory

$$T_{\alpha\rho} = T \cup \left\{ \exists x \left(\sigma_{(\varphi_{n} \land \neg \psi)}(x) \land \gamma^{x} \right) \middle| n \in \omega, \ \gamma \in T_{\alpha(n,0)} \right\}$$

$$\cup \left\{ \exists x \left(\sigma_{(\chi_{n} \land \neg \psi)}(x) \land \gamma^{x} \right) \middle| n \in \omega, \ \gamma \in T_{\rho(n,0)} \right\}$$

$$\cup \left\{ \exists x \left(\sigma_{(\varphi_{n} \land \psi)}(x) \land \gamma^{x} \right) \middle| n \in \omega, \ \gamma \in T_{\alpha(n,1)} \right\}$$

$$\cup \left\{ \exists x \left(\sigma_{(\chi_{n} \land \psi)}(x) \land \gamma^{x} \right) \middle| n \in \omega, \ \gamma \in T_{\rho(n,1)} \right\} .$$

If $\mathcal{M} = (B, A)$ is a model of T, then $\mathcal{M} \models T_{\alpha\rho}$ iff, for $a_1 = e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}})$ $A \mid a_1 \models T_{\alpha(n,0)}, ...,$ for $a_4 = e(\|\chi_n \wedge \psi\|^{\mathcal{M}}), A \mid a_4 \models T_{\rho(n,1)}.$

4.6. Theorem. $\{T_{\alpha\rho} \mid (\alpha, \rho) \in P\}$ is the set of completions of T. Moreover, each $T_{\alpha\rho}$ has a model in K.

Proof. If (α, ρ) and (α', ρ') are different elements of P, then $T_{\alpha\rho} \cup T_{\alpha'\rho'}$ is inconsistent (recall that every T_{mj} , where $m \in \omega + 1$, $j \in 2$, is complete in \mathcal{L}_{BA}). If \mathcal{M} is a model of T, there is some $(\alpha, \rho) \in P$ such that $\mathcal{M} \models T_{\alpha\rho}$ (e.g., put $a_1 = e \mid || \varphi_n \wedge \neg \psi ||^{\mathcal{M}}$) and let $\alpha(n, 0)$ be the pair $(k, j) \in (\omega + 1) \times 2$ such that $A \mid a_1 \models T_{kj}$, etc.). If $(\alpha, \rho) \in P$ and \mathcal{M} , \mathcal{M}' are models of $T_{\alpha\rho}$, then \mathcal{M} and \mathcal{M}' are elementarily equivalent by 4.5, since $T_{\alpha\rho}$ says which sentences of the form $\sigma_{\beta\gamma}$ are satisfied in \mathcal{M} and \mathcal{M}' . So it is sufficient to prove that each $T_{\alpha\rho}$ has a model which lies even in K.

For simplicity, we construct $\mathcal{M} \in \mathbf{K}$ satisfying the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$ — for, if $\mathcal{N} \in \mathbf{K}$ satisfies the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,1)}$ and $T_{\rho(n,1)}$, then $\mathcal{M} \times \mathcal{N} \in \mathbf{K}$ is a model of $T_{\alpha\rho}$. Abbreviate $\alpha(n,0)$ by t_n , $\rho(n,0)$ by s_n . We first construct a complete BAA and a sequence $(a_n)_{n\in\omega}$ in A such that the a_n are pairwise disjoint and

(*)
$$A \upharpoonright a_n \models t_n$$
, $A \upharpoonright r_n \models s_n$

where $r_n = -(a_0 + ... + a_{n-1})$. Let A be a complete BA which is a model of s_0 . Suppose $a_0, ..., a_{n-1} \in A$ are pairwise disjoint and $a_0, ..., a_{n-1}, r_n$ satisfy (*). Since $s_n = s_{n+1} + t_n$, $A \upharpoonright r_n \models s_n$ and A is complete, there are a_n and $r_{n+1} \in A$ such that $r_n = a_n + r_{n+1}$, $a_n \cdot r_{n+1} = 0$, $A \upharpoonright a_n \models t_n$ and $A \upharpoonright r_{n+1} \models s_{n+1}$. — Finally, let $a_{\omega} = -\sum_{n \in \omega} a_n$. By the proof of 4.1, there is, for $n \in \omega$, $\mathcal{M}_n = (B_n, A_n) \in \mathbf{K}$ such that $A_n = A \upharpoonright a_n$ and each stalk $(B_n)_p$ of the sheaf representation of \mathcal{M}_n is a model of $\varphi_n \land \neg \psi$. Moreover there is $\mathcal{M}_{\omega} = (B_{\omega}, A_{\omega}) \in \mathbf{K}$ such that $A_{\omega} = A \upharpoonright a_{\omega}$ and each stalk $(B_{\omega})_p$ of the sheaf representation of \mathcal{M}_{ω} is a model of $T_{\omega 0}$. Put $\mathcal{M} = (B, A)$ where B is a complete BA which lies over A as $\prod_{n \in \omega} B_n$ lies over $\prod_{n \in \omega} A_n$. By 4.2, $e(\lVert \varphi_n \land \neg \psi \rVert^{\mathcal{M}}) = a_n$ and $e(\lVert \chi_n \land \neg \psi \rVert^{\mathcal{M}}) = r_n$; so \mathcal{M} is a model of the part of $T_{\alpha\rho}$ referring to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$.