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has the form  $\exists x \psi (xx_1 \dots x_n)$  and that  $\| \psi [bb_1 \dots b_n] \|$  is clopen for fixed  $b_1, \dots, b_n \in B$  and arbitrary  $b \in B$ . For the rest of the proof, we omit the parameters  $b_1 \dots, b_n$ . Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \} .$$

By our inductive assumption,  $u$  is an open subset of  $X$ . Choose, by Zorn's lemma, a maximal family  $F = \{ (b_i, c_i) \mid i \in I \}$  such that  $b_i \in B$ ,  $c_i$  is a clopen subset of  $u$ ,  $c_i \subseteq \| \psi [b_i] \|$ ,  $i \neq j$  implies  $c_i \cap c_j = \phi$ . It follows that  $c$ , the closure of  $\cup_{i \in I} c_i$ , includes  $u$  (by maximality of  $F$ ).  $A$  is a  $cBA$ ,

hence  $X$  is extremally disconnected and  $c$  is clopen. By completeness of  $B$ , there is some  $b \in B$  such that  $b \cdot e(c_i) = b_i$  for  $i \in I$ . Thus, for  $i \in I$ ,  $c_i \subseteq \| \psi [b] \|$ . So, for  $\beta \in B$ ,  $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$ .

Finally we show that  $B_p$  is separated for each  $p \in X$ . Let  $\alpha(x)$  be the  $\mathcal{L}_{BA}$ -formula stating that  $x$  is an atom and let  $\beta(x)$ ,  $\gamma(x)$  be the  $\mathcal{L}_{BA}$ -formulas  $\alpha(x) \vee x = 0$  resp.  $\forall y (\alpha(y) \rightarrow y \leq x)$ . Put  $M = \{ f \in B \mid \| \beta [f] \| = 1 \|$  and let  $b$  be the supremum of  $M$  in  $B$ . We show that  $b(p)$  is, for each  $p \in X$ , the supremum of the atoms of  $B_p$ .

First suppose  $s \in B_p$  is an atom of  $B_p$ . There is some  $f \in M$  such that  $f(p) = s$  (note that  $\| \alpha [f] \|$  is clopen for each  $f \in B$ ). So  $f \leq b$  and  $s = f(p) \leq b(p)$ . — On the other hand, suppose  $t \in B_p$  and  $s \leq t$  for every atom  $s$  of  $B_p$ . Choose  $g \in B$  such that  $g(p) = t$ . Then  $p \in c = \| \gamma [g] \|$ . For  $f \in M$ ,  $e(c) \cdot f \leq g$ , since  $q \in c$  implies that  $f(q)$  is zero or an atom of  $B_q$  and thus  $f(q) \leq g(q)$ . By the definition of  $b$ ,  $e(c) \cdot b \leq g$ . This implies (by  $p \in c$ )  $b(p) \leq g(p) = t$ .

#### 4. DECIDABILITY AND COMPLETIONS OF $Th(\mathbf{K})$

Call  $T_{sBA} = T_{BA} \cup \{ \sigma \}$  the theory of separated  $BA$ s, where  $T_{BA}$  is the theory of  $BA$ s and  $\sigma$  was defined in section 3. We give a short review of the completions of  $T_{sBA}$ . Let, for  $n \in \omega$ ,  $\varphi_n$  be the  $\mathcal{L}_{BA}$ -sentence stating that there are exactly  $n$  atoms and  $\psi$  the  $\mathcal{L}_{BA}$ -sentence stating that there is a non-zero atomless element. Let  $\chi_n = \neg (\varphi_0 \vee \dots \vee \varphi_{n-1})$ ; so  $\chi_n$  says that there are at least  $n$  atoms. Define, for  $n \in \omega + 1$  and  $i \in 2 = \{ 0, 1 \}$ , an  $\mathcal{L}_{BA}$ -theory  $T_{ni}$  by

$$\begin{aligned} T_{n0} &= T_{sBA} \cup \{ \varphi_n, \neg \psi \} \\ T_{n1} &= T_{sBA} \cup \{ \varphi_n, \psi \} \end{aligned}$$

for  $n \in \omega$ , and

$$T_{\omega 0} = T_{sBA} \cup \{ \chi_n \mid n \in \omega \} \cup \{ \neg \psi \}$$

$$T_{\omega 1} = T_{sBA} \cup \{ \chi_n \mid n \in \omega \} \cup \{ \psi \}.$$

Put  $\tau = \{ T_{ni} \mid n \in \omega + 1, i \in 2 \}$ . It is then clear that each separated  $BA$  satisfies exactly one of the theories in  $\tau$ , and for each  $t \in \tau$  there is a  $cBA$  satisfying  $t$ . Moreover, any two models of any  $t \in \tau$  are elementarily equivalent by 5.5.10 in [1]. Thus the theories  $t \in \tau$  are just the completions of  $T_{sBA}$  and can be thought of as being the elementary equivalence types of separated  $BAs$  or  $cBAs$ . Moreover, an  $\mathcal{L}_{BA}$ -sentence holds in every separated  $BA$  iff it holds in every  $cBA$ . The following proposition is essential for the main theorems of this section:

4.1. PROPOSITION. *Let  $s, t \in \tau$ . Then there is a structure  $(B, A)$  in  $\mathbf{K}$  such that  $A$  is a model of  $s$  and each stalk  $B_p$  is a model of  $t$ .*

*Proof.* By the above remarks, choose  $cBAs$   $A$  and  $F$  which are models of  $s$  resp.  $t$ . Let  $A * F$  be the free product of  $A$  and  $F$ . Thus  $A$  is relatively complete in  $A * F$  and each stalk  $(A * F)_p$ , where  $p$  is an ultrafilter of  $A$ , is easily seen to be isomorphic to  $F$ , hence a model of  $t$ . Unfortunately,  $A * F$  is incomplete unless  $A$  or  $F$  is finite. So let  $B = (A * F)^*$  be the completion of  $A * F$ ; note that  $A * F$  is a dense subalgebra of  $B$ .  $(B, A) \in \mathbf{K}$ , since the inclusion maps from  $A$  to  $A * F$  and from  $A * F$  to  $B$  are complete. For  $p \in X$  (the Stone space of  $A$ ),  $B_p$  is a separated  $BA$  by 3.2 but in general a proper extension of  $(A * F)_p$ . We show, with the notations of section 1, that  $B_p$  is elementarily equivalent to  $F$ . For the following proof of this, recall that, for  $f \in F \setminus \{0\}$  and  $p \in X$ ,  $\pi_p(f) = f(p) \neq 0$  since  $F$  is independent from  $A$  in  $A * F \subseteq B$ . Thus, the restriction of  $\pi_p : B \rightarrow B_p$  to  $F$  is a monomorphism. The elementary equivalence of  $B_p$  and  $F$  is established by the following four claims.

*Claim 1.* For each atom  $f$  of  $F$ ,  $f(p)$  is an atom of  $B_p$  (hence, if  $F$  has at least  $n$  atoms, where  $n \in \omega$ , then  $B_p$  has at least  $n$  atoms): clearly,  $f(p) > 0$  for  $p \in X$ . Assume that

$$u = \{ p \in X \mid f(p) \text{ is not an atom of } B_p \}$$

is non-empty. By 3.2,  $u$  is a clopen subset of  $X$ . Choose, by the maximum principle stated in section 3,  $b \in B$  such that  $b(p) = 0$  for  $p \notin u$  and  $0 < b(p) < f(p)$  for  $p \in u$ . Since  $b > 0$ , choose  $a \in A$  and  $g \in F$  such that  $0 < a \cdot g \leq b$ ; let  $p \in X$  such that  $a(p) \cdot g(p) \neq 0$ . Thus  $p \in u$ ,  $a(p) = 1$ , and

$0 < g(p) \leq b(p) < f(p)$ . It follows that  $0 < g < f$ , contradicting the fact that  $f$  was an atom of  $F$ .

*Claim 2.* If  $B_p$  has at least  $n$  atoms, where  $1 \leq n < \omega$ , then  $F$  has at least  $n$  atoms: assume that  $M$  is a subset of  $At(B_p)$ , the set of atoms of  $B_p$ , such that  $M$  has exactly  $n$  elements but  $At(F)$  has at most  $n - 1$  elements. We prove:

(a) Let  $x \in M$ . Then there is  $f_x \in At(F)$  such that  $f_x(p) = x$ .

Claim 2 follows from (a), since the assignment of  $f_x$  to  $x$  is injective. And (a) will follow from

(b) Let  $x \in M$ ,  $u$  a clopen neighbourhood of  $p$  such that, w.l.o.g., for  $q \in u$ ,  $B_q$  has at least one atom. Let  $b \in B$  such that, for  $q \notin u$ ,  $b(q) = 0$  and for  $q \in u$ ,  $b(q)$  is an atom of  $B_q$ , and  $b(p) = x$ . Then there are  $q \in u$  and  $f \in At(F)$  such that  $f(q) = b(q)$ . (Hence  $At(F)$  is non-empty).

Proof of (b). By  $b > 0$ , choose  $a \in A$ ,  $f \in F$  such that  $0 < a \cdot f \leq b$ . Since  $b(q) = 0$  for  $q \notin u$ , there is some  $q \in u$  such that  $a(q) \cdot f(q) \neq 0$ , which implies  $0 < f(q) \leq b(q)$ .  $f(q) = b(q)$ , since  $b(q)$  is an atom of  $B_q$ . Finally  $f \in At(F)$ , since a splitting of  $f$  in  $F$  into two non-zero disjoint elements would give rise to a splitting of  $b(q)$  in  $B_q$ .

Proof of (a). Let  $x \in M$  and choose  $u$  and  $b$  as in (b). Assume (a) is false. Then, for each  $f \in At(F)$ ,  $f(p) \neq x = b(p)$ ; by finiteness of  $At(F)$ , there is a clopen neighbourhood  $v$  of  $p$  such that, for  $q \in v$  and  $f \in At(F)$ ,  $b(q) \neq f(q)$ . Let  $c \in B$  such that  $c(q) = 0$  for  $q \notin v$  and  $c(q) = b(q)$  for  $q \in v$ . This contradicts (b), applied to  $v$  and  $c$  instead of  $u$  and  $b$ .

*Claim 3.* If  $F$  has a non-zero atomless element  $f$  (which means that  $F \upharpoonright f$  is atomless), then each  $B_p$  has a non-zero atomless element  $x$ : let  $x = \pi_p(f)$ .  $x > 0$ , since  $\pi_p$  is one-one on  $F$ .  $F \upharpoonright f$  and hence, by Claim 2,  $(B \upharpoonright f)_p$  is atomless. So  $B_p \upharpoonright x = \pi_p(B \upharpoonright f) = (B \upharpoonright f)_p$  is atomless.

*Claim 4.* If  $B_p$  has a non-zero atomless element  $x$ , then  $F$  has a non-zero atomless element  $f$ : assume that  $F$  is atomic. Let

$$u = \{q \in X \mid B_q \text{ is not atomic}\}.$$

$u$  is a clopen neighbourhood of  $p$ . By the maximum principle, choose  $b \in B$  such that  $b(q) = 0$  for  $q \notin u$ ,  $b(q)$  is a non-zero atomless element of

$B_q$  for  $q \in u$ ,  $b(p) = x$ . Choose  $a \in A$ ,  $g \in F$  such that  $0 < a \cdot g \leq b$ ; w.l.o.g.,  $g$  is an atom of  $F$ . Choose  $q \in X$  such that  $a(q) \cdot g(q) \neq 0$ . Thus  $q \in u$  and  $g(q) \leq b(q)$ . By Claim 1,  $g(q)$  is an atom of  $B_q$ , contradicting the choice of  $b(q)$ .

4.2. REMARK. Suppose that, for every  $i$  in an index set  $I$ ,  $\mathcal{M}_i = (B_i, A_i)$  is an element of  $\mathbf{K}$ . Then  $\mathcal{M} = (B, A)$ , where  $B = \prod_{i \in I} B_i$  and  $A = \prod_{i \in I} A_i$ , is in  $\mathbf{K}$ . Let  $\varphi(x_1 \dots x_k)$  be an  $\mathcal{L}$ -formula and  $b_1, \dots, b_k \in B$ ,  $b_j = (b_{ij})_{i \in I}$ . Put  $a_i = e(\|\varphi[b_{i1} \dots b_{ik}]\|_{\mathcal{M}_i})$ . Then

$$e(\|\varphi[b_1 \dots b_k]\|_{\mathcal{M}}) = (a_i)_{i \in I}.$$

*Proof.* By induction on the complexity of  $\varphi$ .

We shall need the following Feferman-Vaught theorem about sheaves over Boolean spaces from [2]:

4.3. THEOREM (Comer). *Let  $\mathcal{L}$  be an arbitrary language. There is an effective assignment*

$$\varphi(x_1 \dots x_k) \mapsto (\Phi; \mathfrak{D}_1, \dots, \mathfrak{D}_m)$$

for  $\mathcal{L}$ -formulas  $\varphi(x_1 \dots x_k)$  such that

- a)  $\mathfrak{D}_1, \dots, \mathfrak{D}_m$  are  $\mathcal{L}$ -formulas having at most the free variables  $x_1 \dots x_k$ , and

$$\models (\bigvee_{1 \leq i \leq m} \mathfrak{D}_i) \wedge \bigwedge_{1 \leq i < j \leq m} \neg(\mathfrak{D}_i \wedge \mathfrak{D}_j)$$

- b)  $\Phi$  is an  $\mathcal{L}_{BA}$ -formula having at most the free variables  $y_1 \dots y_m$ ;
- c) for each sheaf  $\mathcal{S} = (S, \pi, X, \mu)$  of  $\mathcal{L}$ -structures such that  $X$  is a Boolean space and  $\|\psi[f_1 \dots f_n]\|$  is a clopen subset of  $X$  for every  $\psi(x_1 \dots x_n)$  in  $\mathcal{L}$  and  $f_1, \dots, f_n \in \Gamma(\mathcal{S})$ : if  $b_1, \dots, b_k \in \Gamma(\mathcal{S})$ , then

$$\Gamma(\mathcal{S}) \models \varphi[b_1 \dots b_k] \text{ iff } C \models \Phi[c_1 \dots c_m],$$

where  $C$  is the BA of clopen subsets of  $X$  and  $c_i = \|\mathfrak{D}_i[b_1 \dots b_k]\|$ .

For two separated BAs  $A$  and  $A'$ , let  $I$  be the set of partial functions  $f$  from  $A$  to  $A'$  such that  $\text{dom}(f) = \{a_1, \dots, a_n\}$  is a finite partition of  $A$  (where some of the  $a_i$  may be zero),  $\text{rge}(f) = \{a'_1, \dots, a'_n\}$  where  $a'_i = f(a_i)$  is a partition of  $A'$ , and every  $A \upharpoonright a_i$  is elementarily equivalent

to  $A' \vdash a_i'$ . If  $A, A'$  are  $\aleph_1$ -saturated or  $\sigma$ -complete, the following conditions are equivalent:

- a)  $A \equiv A'$ ;
- b)  $I$  is non-empty;
- c)  $I$  has the back-and-forth property.

Moreover, if  $f \in I$  is as above and  $A, A'$  are arbitrary separated  $BA$ s, then  $(A, a_1, \dots, a_n) \equiv (A', a_1', \dots, a_n')$ .

Let  $T_{sBA2}$  be the  $\mathcal{L}$ -theory

$$T_{sBA2} = T_{sBA} \cup \{ \forall x (U(x) \leftrightarrow x = 0 \vee x = 1) \}.$$

Since  $T_{BA}$  is decidable,  $T_{sBA}$  and  $T_{sBA2}$  are decidable.

4.4. THEOREM. *There is an effective procedure deciding for every  $\mathcal{L}$ -sentence  $\varphi$  whether  $T \vdash \varphi$ . Moreover,  $T \vdash \varphi$  if and only if  $\varphi$  holds in every model  $\mathcal{M}$  in  $\mathbf{K}$ .*

*Proof.* Let  $\varphi$  be given. Construct  $(\Phi(y_1 \dots y_m); \mathfrak{g}_1, \dots, \mathfrak{g}_m)$  by 4.3. For every  $i$  such that  $1 \leq i \leq m$ , decide whether  $T_{sBA2} \vdash \neg \mathfrak{g}_i$ . W.l.o.g., assume that  $T_{sBA2} \cup \{ \mathfrak{g}_i \}$  is consistent for  $1 \leq i \leq r$  and inconsistent for  $r + 1 \leq i \leq m$ . By  $\vdash \mathfrak{g}_1 \vee \dots \vee \mathfrak{g}_m$ , we have  $1 \leq r$  (it is possible that  $r = m$ ). Next, construct the formula

$$\Phi'(y_1 \dots y_m) = \left( \bigwedge_{r+1 \leq i \leq m} (y_i = 0) \rightarrow \Phi(y_1 \dots y_m) \right).$$

We show the equivalence of

- a)  $T \vdash \varphi$ ;
- b)  $\mathcal{M} \models \varphi$  for every  $\mathcal{M} \in \mathbf{K}$ ;
- c)  $T_{sBA} \vdash \forall y_1 \dots \forall y_m \Phi'(y_1 \dots y_m)$ .

Then, by decidability of  $T_{sBA}$ ,  $T$  is decidable and 4.4 is proved. *a)* implies *b)* by 3.2. To prove that *c)* implies *a)*, assume there is  $\mathcal{M} \models T$  such that  $\mathcal{M} \not\models \varphi$ , e.g.  $\mathcal{M} = (B, A)$ . Put  $a_i = e(\|\mathfrak{g}_i\|^{\mathcal{M}})$ . By 4.3 and  $\mathcal{M} \not\models \varphi$ , we see  $A \not\models \Phi[a_1 \dots a_m]$ . By our choice of  $r \leq m$ , we get  $a_{r+1} = \dots = a_m = 0$ . Thus  $A \not\models \Phi'[a_1 \dots a_m]$  and *c)* is false. Now assume *c)* does not hold; we show that *b)* is false. Let  $A'$  be a separated  $BA$  and  $a_1', \dots, a_m' \in A'$  such that  $a_{r+1}' = \dots = a_m' = 0$  and  $A' \not\models \Phi[a_1' \dots a_m']$ . W.l.o.g.,  $a_i' \neq 0$  for  $1 \leq i \leq r$ . By choice of  $r$ , there are  $t_1, \dots, t_r \in \tau$  such that  $t_i \models \mathfrak{g}_i$  for  $1 \leq i \leq r$ .

Let, for these  $i, s_i$  be the element of  $\tau$  such that  $A' \upharpoonright a_i' \models s_i$ . By 4.1, there are  $\mathcal{M} = (B, A) \in \mathbf{K}$  and  $a_1, \dots, a_r \in A$  such that  $1 = a_1 + \dots + a_r$ ,  $a_i \cdot a_j = 0$  for  $1 \leq i < j \leq r$ ,  $A \upharpoonright a_i \models s_i$  and  $(B \upharpoonright a_i)_p \models t_i$  for those  $p \in X$  satisfying  $a_i(p) = 1$ . So  $e(\|\mathfrak{G}_i\|^\mathcal{M}) = a_i$  by 4.2. Put  $a_{r+1} = \dots = a_m = 0$ . It follows that  $(A, a_1, \dots, a_m) \equiv (A', a_1', \dots, a_m')$ ,  $A \not\models \Phi[a_1 \dots a_m]$  and  $\mathcal{M} \not\models \varphi$  by 4.3.

In the next theorem, we characterize elementary equivalence of models of  $T$ . Call the following sentences in  $\mathcal{L}_{BA}$  basic sentences:  $\varphi_n \wedge \psi$ ,  $\varphi_n \wedge \neg \psi$ ,  $\chi_n \wedge \psi$ ,  $\chi_n \wedge \neg \psi$  (where  $n \in \omega$ ). It follows by the analysis of the completions of  $T_{sBA}$  given in the beginning of this section that for each  $\mathcal{L}_{BA}$ -sentence  $\mathfrak{G}$  there are basic sentences  $\beta_1, \dots, \beta_n$  such that

$$T_{sBA} \vdash (\mathfrak{G} \leftrightarrow \bigvee_{i=1}^n \beta_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \wedge \beta_j).$$

This fact is easily extended to  $T_{sBA2}$ : by replacing each atomic formula  $U(t)$  where  $t$  is a term in  $\mathcal{L}_{BA}$  by " $t = 0 \vee t = 1$ ", we see that for each  $\mathcal{L}$ -sentence  $\mathfrak{G}$  there are basic sentences  $\beta_1, \dots, \beta_n$  satisfying

$$T_{sBA2} \vdash (\mathfrak{G} \leftrightarrow \bigvee_{i=1}^n \beta_i) \wedge \bigwedge_{1 \leq i < j \leq n} \neg (\beta_i \wedge \beta_j).$$

Now, if  $\beta, \gamma$  are basic sentences, let  $\sigma_{\beta\gamma}$  be the following  $\mathcal{L}$ -sentence:

$$\sigma_{\beta\gamma} = \exists y (\gamma^y \wedge s_\beta(y)),$$

where  $s_\beta(y)$  is the  $\mathcal{L}$ -formula assigned to  $\beta$  in 3.1 and  $\gamma^y$  is the result of relativizing the quantifiers  $\exists x\varphi \dots$  in  $\gamma$  to  $\exists x(U(x) \wedge x \leq y \wedge \varphi^y \dots)$ . A model  $(B, A)$  of  $T$  satisfies  $\sigma_{\beta\gamma}$  iff  $A \upharpoonright a \models \gamma$ , where  $a = e(c)$  and  $c = \|\beta\|$ .

4.5. THEOREM. Let  $\mathcal{M} = (B, A)$ ,  $\mathcal{M}' = (B', A')$  be models of  $T$ . Then  $\mathcal{M}$  is elementarily equivalent to  $\mathcal{M}'$  if and only if, for any basic sentences  $\beta, \gamma$ ,

$$\mathcal{M} \models \sigma_{\beta\gamma} \text{ iff } \mathcal{M}' \models \sigma_{\beta\gamma}.$$

*Proof.* The only-if-part is clear. Suppose that  $\mathcal{M}$  and  $\mathcal{M}'$  satisfy the same sentences of the form  $\sigma_{\beta\gamma}$ . Let  $\varphi$  be an  $\mathcal{L}$ -sentence and  $\mathcal{M} \models \varphi$ ; we want to show that  $\mathcal{M}' \models \varphi$ . Let  $(\Phi(y_1 \dots y_m); \mathfrak{G}_1, \dots, \mathfrak{G}_m)$  be the sequence assigned to  $\varphi$  by 4.3; every  $\mathfrak{G}_i$  is an  $\mathcal{L}$ -sentence. Put  $a_i = e(\|\mathfrak{G}_i\|^\mathcal{M})$ ; by 4.3 and  $e : C \rightarrow A$  being an isomorphism, we have that  $\{a_1, \dots, a_m\}$

is a partition of  $A$  and  $A \models \Phi [a_1 \dots a_m]$ . In the same way, put  $a'_i = e' (\| \mathcal{G}_i \| \mathcal{M}')$ ;  $\{a'_1, \dots, a'_m\}$  is a partition of  $A'$ . It is sufficient to show that  $(A, a_1, \dots, a_m) \equiv (A', a'_1, \dots, a'_m)$ , for this implies  $A' \models \Phi [a'_1 \dots a'_m]$  and finally  $\mathcal{M}' \models \varphi$  by 4.3.

For every  $\mathcal{G}_i$ , choose basic sentences  $\beta_{i1}, \dots, \beta_{in_i}$  such that

$$T_{sBA2} \vdash (\mathcal{G}_i \leftrightarrow \bigvee_j \beta_{ij} \wedge \bigwedge_{j < l} \neg (\beta_{ij} \wedge \beta_{il}).$$

Put  $\alpha_{ij} = e (\| \beta_{ij} \| \mathcal{M})$ ,  $\alpha'_{ij} = e' (\| \beta_{ij} \| \mathcal{M}')$  for  $1 \leq i \leq m$ ,  $1 \leq j \leq n_i$ . Then  $a_i$  is the disjoint sum of the  $\alpha_{ij}$  ( $1 \leq j \leq n_i$ ),  $a'_i$  is the disjoint sum of the  $\alpha'_{ij}$  ( $1 \leq j \leq n_i$ ). For every  $i, j$ ,

$$A \upharpoonright \alpha_{ij} \equiv A' \upharpoonright \alpha'_{ij} :$$

let  $\gamma$  be any basic sentence of  $\mathcal{L}_{BA}$  and assume  $A \upharpoonright \alpha_{ij} \models \gamma$ ; we want to show that  $A' \upharpoonright \alpha'_{ij} \models \gamma$ . But  $A \upharpoonright \alpha_{ij} \models \gamma$  means that  $\mathcal{M} \models \sigma_{\beta_{ij}\gamma}$ . By our main assumption,  $\mathcal{M}' \models \sigma_{\beta_{ij}\gamma}$  and  $A' \upharpoonright \alpha'_{ij} \models \gamma$ .

We have shown that the partial function  $f$  mapping  $\alpha_{ij}$  to  $\alpha'_{ij}$  is an element of the set of back-and-forth-isomorphisms defined after 4.3. Hence,

$$(A, \alpha_{11}, \dots, \alpha_{mn_m}) \equiv (A', \alpha'_{11}, \dots, \alpha'_{mn_m})$$

and

$$(A, a_1, \dots, a_m) \equiv (A', a'_1, \dots, a'_m).$$

We shall finally describe the completions of  $T$  by giving a one-one correspondance between a set  $P$  (consisting of pairs of mappings from  $\omega \times 2$  to  $(\omega+1) \times 2$ ) and these completions. For  $m, m' \in \omega+1$  and  $j, j' \in 2$ , define

$$(m, j) + (m', j') = (m'', j'')$$

where  $m''$  is the cardinal sum of  $m$  and  $m'$  and  $j''$  is the maximum of  $j$  and  $j'$ . Let

$$P = \{ (\alpha, \rho) \mid \alpha, \rho : \omega \times 2 \rightarrow (\omega+1) \times 2 \text{ and, for } (n, i) \in \omega \times 2, \rho(n, i) = \rho(n+1, i) + \alpha(n, i) \}.$$

In the following definition, we refer to the  $\mathcal{L}_{BA}$ -theories  $T_{ni}$  defined in the beginning of this section. For  $(\alpha, \rho) \in P$ , let  $T_{\alpha\rho}$  the  $\mathcal{L}$ -theory

$$\begin{aligned} T_{\alpha\rho} = & T \cup \{ \exists x (\sigma_{(\varphi_n \wedge \neg \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\alpha(n,0)} \} \\ & \cup \{ \exists x (\sigma_{(\chi_n \wedge \neg \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\rho(n,0)} \} \\ & \cup \{ \exists x (\sigma_{(\varphi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\alpha(n,1)} \} \\ & \cup \{ \exists x (\sigma_{(\chi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega, \gamma \in T_{\rho(n,1)} \}. \end{aligned}$$

If  $\mathcal{M} = (B, A)$  is a model of  $T$ , then  $\mathcal{M} \models T_{\alpha\rho}$  iff, for  $a_1 = e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}})$   $A \upharpoonright a_1 \models T_{\alpha(n,0)}$ , ..., for  $a_4 = e(\|\chi_n \wedge \psi\|^{\mathcal{M}})$ ,  $A \upharpoonright a_4 \models T_{\rho(n,1)}$ .

4.6. THEOREM.  $\{T_{\alpha\rho} \mid (\alpha, \rho) \in P\}$  is the set of completions of  $T$ . Moreover, each  $T_{\alpha\rho}$  has a model in  $\mathbf{K}$ .

*Proof.* If  $(\alpha, \rho)$  and  $(\alpha', \rho')$  are different elements of  $P$ , then  $T_{\alpha\rho} \cup T_{\alpha'\rho'}$  is inconsistent (recall that every  $T_{mj}$ , where  $m \in \omega + 1, j \in 2$ , is complete in  $\mathcal{L}_{BA}$ ). If  $\mathcal{M}$  is a model of  $T$ , there is some  $(\alpha, \rho) \in P$  such that  $\mathcal{M} \models T_{\alpha\rho}$  (e.g., put  $a_1 = e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}})$  and let  $\alpha(n, 0)$  be the pair  $(k, j) \in (\omega + 1) \times 2$  such that  $A \upharpoonright a_1 \models T_{kj}$ , etc.). If  $(\alpha, \rho) \in P$  and  $\mathcal{M}, \mathcal{M}'$  are models of  $T_{\alpha\rho}$ , then  $\mathcal{M}$  and  $\mathcal{M}'$  are elementarily equivalent by 4.5, since  $T_{\alpha\rho}$  says which sentences of the form  $\sigma_{\beta\gamma}$  are satisfied in  $\mathcal{M}$  and  $\mathcal{M}'$ . So it is sufficient to prove that each  $T_{\alpha\rho}$  has a model which lies even in  $\mathbf{K}$ .

For simplicity, we construct  $\mathcal{M} \in \mathbf{K}$  satisfying the part of  $T_{\alpha\rho}$  which refers to  $T_{\alpha(n,0)}$  and  $T_{\rho(n,0)}$  — for, if  $\mathcal{N} \in \mathbf{K}$  satisfies the part of  $T_{\alpha\rho}$  which refers to  $T_{\alpha(n,1)}$  and  $T_{\rho(n,1)}$ , then  $\mathcal{M} \times \mathcal{N} \in \mathbf{K}$  is a model of  $T_{\alpha\rho}$ . Abbreviate  $\alpha(n, 0)$  by  $t_n$ ,  $\rho(n, 0)$  by  $s_n$ . We first construct a complete  $BA$   $A$  and a sequence  $(a_n)_{n \in \omega}$  in  $A$  such that the  $a_n$  are pairwise disjoint and

$$(*) \quad A \upharpoonright a_n \models t_n, \quad A \upharpoonright r_n \models s_n$$

where  $r_n = -(a_0 + \dots + a_{n-1})$ . Let  $A$  be a complete  $BA$  which is a model of  $s_0$ . Suppose  $a_0, \dots, a_{n-1} \in A$  are pairwise disjoint and  $a_0, \dots, a_{n-1}, r_n$  satisfy (\*). Since  $s_n = s_{n+1} + t_n$ ,  $A \upharpoonright r_n \models s_n$  and  $A$  is complete, there are  $a_n$  and  $r_{n+1} \in A$  such that  $r_n = a_n + r_{n+1}$ ,  $a_n \cdot r_{n+1} = 0$ ,  $A \upharpoonright a_n \models t_n$  and  $A \upharpoonright r_{n+1} \models s_{n+1}$ . — Finally, let  $a_\omega = -\sum_{n \in \omega} a_n$ . By the proof of 4.1,

there is, for  $n \in \omega$ ,  $\mathcal{M}_n = (B_n, A_n) \in \mathbf{K}$  such that  $A_n = A \upharpoonright a_n$  and each stalk  $(B_n)_p$  of the sheaf representation of  $\mathcal{M}_n$  is a model of  $\varphi_n \wedge \neg \psi$ . Moreover there is  $\mathcal{M}_\omega = (B_\omega, A_\omega) \in \mathbf{K}$  such that  $A_\omega = A \upharpoonright a_\omega$  and each stalk  $(B_\omega)_p$  of the sheaf representation of  $\mathcal{M}_\omega$  is a model of  $T_{\omega 0}$ . Put  $\mathcal{M} = (B, A)$  where  $B$  is a complete  $BA$  which lies over  $A$  as  $\prod_{n \in \omega} B_n$  lies over  $\prod_{n \in \omega} A_n$ . By 4.2,  $e(\|\varphi_n \wedge \neg \psi\|^{\mathcal{M}}) = a_n$  and  $e(\|\chi_n \wedge \neg \psi\|^{\mathcal{M}}) = r_n$ ; so  $\mathcal{M}$  is a model of the part of  $T_{\alpha\rho}$  referring to  $T_{\alpha(n,0)}$  and  $T_{\rho(n,0)}$ .