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has the form $\exists x \psi (x x_1 \ldots x_n)$ and that $\| \psi [b b_1 \ldots b_n] \|$ is clopen for fixed $b_1, \ldots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters $b_1 \ldots, b_n$. Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \} .$$

By our inductive assumption, u is an open subset of X. Choose, by Zorn's lemma, a maximal family $F = \{ (b_i, c_i) \mid i \in I \}$ such that $b_i \in B$, c_i is a clopen subset of u, $c_i \subseteq \| \psi [b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \phi$. It follows that c, the closure of $\bigcup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA, hence X is extremally disconnected and c is clopen. By completeness of B, there is some $b \in B$ such that $b \cdot e(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi [b] \|$. So, for $\beta \in B$, $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha (x)$ be the $\mathscr{L}_{BA}$-formula stating that x is an atom and let $\beta (x)$, $\gamma (x)$ be the $\mathscr{L}_{BA}$-formulas $\alpha (x) \vee x = 0$ resp. $\forall y (\alpha (y) \rightarrow y \leqslant x)$. Put $M = \{ f \in B \mid \| \beta [f] \| = 1 \|$ and let b be the supremum of M in B. We show that $b(p)$ is, for each $p \in X$, the supremum of the atoms of B_p.

First suppose $s \in B_p$ is an atom of B_p. There is some $f \in M$ such that $f(p) = s$ (note that $\| \alpha [f] \|$ is clopen for each $f \in B$). So $f \leqslant b$ and $s = f(p) \leqslant b(p)$. — On the other hand, suppose $t \in B_p$ and $s \leqslant t$ for every atom s of B_p. Choose $g \in B$ such that $g(p) = t$. Then $p \in c = \| \gamma [g] \|$. For $f \in M$, $e(c) \cdot f \leqslant g$, since $q \in c$ implies that $f(q)$ is zero or an atom of B_q and thus $f(q) \leqslant g(q)$. By the definition of b, $e(c) \cdot b \leqslant g$. This implies (by $p \in c$) $b(p) \leqslant g(p) = t$.

4. Decidability and completions of $Th(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{ \sigma \}$ the theory of separated BAs, where T_{BA} is the theory of BAs and σ was defined in section 3. We give a short review of the completions of T_{sBA}. Let, for $n \in \omega$, φ_n be the $\mathscr{L}_{BA}$-sentence stating that there are exactly n atoms and ψ the $\mathscr{L}_{BA}$-sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg (\varphi_0 \vee \ldots \vee \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an $\mathscr{L}_{BA}$-theory T_{ni} by

$$T_{n0} = T_{sBA} \cup \{ \varphi_n, \neg \psi \}$$
$$T_{n1} = T_{sBA} \cup \{ \varphi_n, \psi \}$$

for $n \in \omega$, and

$$T_{\omega 0} = T_{sBA} \cup \{\chi_n \mid n \in \omega\} \cup \{\neg \psi\}$$
$$T_{\omega 1} = T_{sBA} \cup \{\chi_n \mid n \in \omega\} \cup \{\psi\} .$$

Put $\tau = \{T_{ni} \mid n \in \omega + 1, i \in 2\}$. It is then clear that each separated BA satisfies exactly one of the theories in τ, and for each $t \in \tau$ there is a cBA satisfying t. Moreover, any two models of any $t \in \tau$ are elementarily equivalent by 5.5.10 in [1]. Thus the theories $t \in \tau$ are just the completions of T_{sBA} and can be thought of as being the elementary equivalence types of separated BAs or $cBAs$. Moreover, an $\mathscr{L}_{BA}$-sentence holds in every separated BA iff it holds in every cBA. The following proposition is essential for the main theorems of this section:

4.1. PROPOSITION. *Let $s, t \in \tau$. Then there is a structure (B, A) in* $\mathbf{K}$ *such that A is a model of s and each stalk B_p is a model of t.*

Proof. By the above remarks, choose $cBAs$ A and F which are models of s resp. t. Let $A * F$ be the free product of A and F. Thus A is relatively complete in $A * F$ and each stalk $(A * F)_p$, where p is an ultrafilter of A, is easily seen to be isomorphic to F, hence a model of t. Unfortunately, $A * F$ is incomplete unless A or F is finite. So let $B = (A * F)^*$ be the completion of $A * F$; note that $A * F$ is a dense subalgebra of B. $(B, A) \in \mathbf{K}$, since the inclusion maps from A to $A * F$ and from $A * F$ to B are complete. For $p \in X$ (the Stone space of A), B_p is a separated BA by 3.2 but in general a proper extension of $(A * F)_p$. We show, with the notations of section 1, that B_p is elementarily equivalent to F. For the following proof of this, recall that, for $f \in F \setminus \{0\}$ and $p \in X$, $\pi_p(f) = f(p) \neq 0$ since F is independent from A in $A * F \subseteq B$. Thus, the restriction of $\pi_p : B \to B_p$ to F is a monomorphism. The elementary equivalence of B_p and F is established by the following four claims.

Claim 1. For each atom f of F, $f(p)$ is an atom of B_p (hence, if F has at least n atoms, where $n \in \omega$, then B_p has at least n atoms): clearly, $f(p) > 0$ for $p \in X$. Assume that

$$u = \{p \in X \mid f(p) \text{ is not an atom of } B_p\}$$

is non-empty. By 3.2, u is a clopen subset of X. Choose, by the maximum principle stated in section 3, $b \in B$ such that $b(p) = 0$ for $p \notin u$ and $0 < b(p) < f(p)$ for $p \in u$. Since $b > 0$, choose $a \in A$ and $g \in F$ such that $0 < a \cdot g \leqslant b$; let $p \in X$ such that $a(p) \cdot g(p) \neq 0$. Thus $p \in u$, $a(p) = 1$, and

$0 < g(p) \leqslant b(p) < f(p)$. It follows that $0 < g < f$, contradicting the fact that f was an atom of F.

Claim 2. If B_p has at least n atoms, where $1 \leqslant n < \omega$, then F has at least n atoms: assume that M is a subset of $At(B_p)$, the set of atoms of B_p, such that M has exactly n elements but $At(F)$ has at most $n - 1$ elements. We prove:

(a) Let $x \in M$. Then there is $f_x \in At(F)$ such that $f_x(p) = x$.

Claim 2 follows from *(a)*, since the assignment of f_x to x is injective. And *(a)* will follow from

(b) Let $x \in M$, u a clopen neighbourhood of p such that, w.l.o.g., for $q \in u$, B_q has at least one atom. Let $b \in B$ such that, for $q \notin u$, $b(q) = 0$ and for $q \in u$, $b(q)$ is an atom of B_q, and $b(p) = x$. Then there are $q \in u$ and $f \in At(F)$ such that $f(q) = b(q)$. (Hence $At(F)$ is non-empty).

Proof of *(b)*. By $b > 0$, choose $a \in A$, $f \in F$ such that $0 < a \cdot f \leqslant b$. Since $b(q) = 0$ for $q \notin u$, there is some $q \in u$ such that $a(q) \cdot f(q) \neq 0$, which implies $0 < f(q) \leqslant b(q) \,.\, f(q) = b(q)$, since $b(q)$ is an atom of B_q. Finally $f \in At(F)$, since a splitting of f in F into two non-zero disjoint elements would give rise to a splitting of $b(q)$ in B_q.

Proof of *(a)*. Let $x \in M$ and choose u and b as in *(b)*. Assume *(a)* is false. Then, for each $f \in At(F)$, $f(p) \neq x = b(p)$; by finiteness of $At(F)$, there is a clopen neighbourhood v of p such that, for $q \in v$ and $f \in At(F)$, $b(q) \neq f(q)$. Let $c \in B$ such that $c(q) = 0$ for $q \notin v$ and $c(q) = b(q)$ for $q \in v$. This contradicts *(b)*, applied to v and c instead of u and b.

Claim 3. If F has a non-zero atomless element f (which means that $F \restriction f$ is atomless), then each B_p has a non-zero atomless element x: let $x = \pi_p(f)$. $x > 0$, since π_p is one-one on F. $F \restriction f$ and hence, by Claim 2, $(B \restriction f)_p$ is atomless. So $B_p \restriction x = \pi_p(B \restriction f) = (B \restriction f)_p$ is atomless.

Claim 4. If B_p has a non-zero atomless element x, then F has a non-zero atomless element f: assume that F is atomic. Let

$$u = \{q \in X \mid B_q \text{ is not atomic}\}\,.$$

u is a clopen neighbourhood of p. By the maximum principle, choose $b \in B$ such that $b(q) = 0$ for $q \notin u$, $b(q)$ is a non-zero atomless element of

B_q for $q \in u$, $b(p) = x$. Choose $a \in A$, $g \in F$ such that $0 < a \cdot g \leqslant b$; w.l.o.g., g is an atom of F. Choose $q \in X$ such that $a(q) \cdot g(q) \neq 0$. Thus $q \in u$ and $g(q) \leqslant b(q)$. By Claim 1, $g(q)$ is an atom of B_q, contradicting the choice of $b(q)$.

4.2. REMARK. Suppose that, for every i in an index set I, $\mathcal{M}_i = (B_i, A_i)$ is an element of **K**. Then $\mathcal{M} = (B, A)$, where $B = \prod_{i \in I} B_i$ and $A = \prod_{i \in I} A_i$, is in **K**. Let $\varphi(x_1 \ldots x_k)$ be an $\mathcal{L}$-formula and $b_1, \ldots, b_k \in B$, $b_j = (b_{ij})_{i \in I}$. Put $a_i = e(\| \varphi[b_{i1} \ldots b_{ik}] \|^{\mathcal{M}_i})$. Then

$$e(\| \varphi[b_1 \ldots b_k] \|^{\mathcal{M}}) = (a_i)_{i \in I}.$$

Proof. By induction on the complexity of φ.

We shall need the following Feferman-Vaught theorem about sheaves over Boolean spaces from [2]:

4.3. THEOREM (Comer). *Let $\mathcal{L}$ be an arbitrary language. There is an effective assignment*

$$\varphi(x_1 \ldots x_k) \mapsto (\Phi; \vartheta_1, \ldots, \vartheta_m)$$

for $\mathcal{L}$-formulas $\varphi(x_1 \ldots x_k)$ such that

a) $\vartheta_1, \ldots, \vartheta_m$ *are $\mathcal{L}$-formulas having at most the free variables $x_1 \ldots x_k$, and*

$$\models (\bigvee_{1 \leqq i \leqq m} \vartheta_i) \wedge \bigwedge_{1 \leqq i < j \leqq m} \neg(\vartheta_i \wedge \vartheta_j)$$

b) Φ *is an $\mathcal{L}_{BA}$-formula having at most the free variables $y_1 \ldots y_m$;*

c) *for each sheaf $\mathcal{S} = (S, \pi, X . \mu)$ of $\mathcal{L}$-structures such that X is a Boolean space and $\| \psi[f_1 \ldots f_n] \|$ is a clopen subset of X for every $\psi(x_1 \ldots x_n)$ in $\mathcal{L}$ and $f_1, \ldots, f_n \in \Gamma(\mathcal{S})$: if $b_1, \ldots, b_k \in \Gamma(\mathcal{S})$, then*

$$\Gamma(\mathcal{S}) \models \varphi[b_1 \ldots b_k] \quad \textit{iff} \quad C \models \Phi[c_1 \ldots c_m],$$

where C is the BA of clopen subsets of X and $c_i = \| \vartheta_i[b_1 \ldots b_k] \|$.

For two separated *BAs* A and A', let I be the set of partial functions f from A to A' such that $\mathrm{dom}(f) = \{a_1, \ldots, a_n\}$ is a finite partition of A (where some of the a_i may be zero), $rge(f) = \{a_1', \ldots, a_n'\}$ where $a_i' = f(a_i)$ is a partition of A', and every $A \restriction a_i$ is elementarily equivalent to $A' \restriction a_i'$. If A, A' are $\aleph_1$-saturated or σ-complete, the following conditions are equivalent:

a) $A \equiv A'$;

b) I is non-empty;

c) I has the back-and-forth property.

Moreover, if $f \in I$ is as above and A, A' are arbitrary separated *BAs*, then $(A, a_1, \ldots, a_n) \equiv (A', a_1', \ldots, a_n')$.

Let T_{sBA2} be the $\mathscr{L}$-theory

$$T_{sBA2} = T_{sBA} \cup \{\forall x (U(x) \leftrightarrow x = 0 \vee x = 1)\} .$$

Since T_{BA} is decidable, T_{sBA} and T_{sBA2} are decidable.

4.4. THEOREM. *There is an effective procedure deciding for every $\mathscr{L}$-sentence φ whether $T \vdash \varphi$. Moreover, $T \vdash \varphi$ if and only if φ holds in every model $\mathscr{M}$ in* **K**.

Proof. Let φ be given. Construct $(\Phi(y_1 \ldots y_m); \vartheta_1, \ldots, \vartheta_m)$ by 4.3. For every i such that $1 \leqslant i \leqslant m$, decide whether $T_{sBA2} \vdash \neg \vartheta_i$. W.l.o.g., assume that $T_{sBA2} \cup \{\vartheta_i\}$ is consistent for $1 \leqslant i \leqslant r$ and inconsistent for $r + 1 \leqslant i \leqslant m$. By $\vdash \vartheta_1 \vee \ldots \vee \vartheta_m$, we have $1 \leqslant r$ (it is possible that $r = m$). Next, construct the formula

$$\Phi'(y_1 \ldots y_m) = \Big(\bigwedge_{r+1 \leqq i \leqq m} (y_i = 0) \rightarrow \Phi(y_1 \ldots y_m)\Big) .$$

We show the equivalence of

a) $T \vdash \varphi$;

b) $\mathscr{M} \models \varphi$ for every $\mathscr{M} \in \mathbf{K}$;

c) $T_{sBA} \vdash \forall y_1 \ldots \forall y_m \, \Phi'(y_1 \ldots y_m)$.

Then, by decidability of T_{sBA}, T is decidable and 4.4 is proved. *a)* implies *b)* by 3.2. To prove that *c)* implies *a)*, assume there is $\mathscr{M} \models T$ such that $\mathscr{M} \not\models \varphi$, e.g. $\mathscr{M} = (B, A)$. Put $a_i = e(\|\vartheta_i\|^{\mathscr{M}})$. By 4.3 and $\mathscr{M} \not\models \varphi$, we see $A \not\models \Phi[a_1 \ldots a_m]$. By our choice of $r \leqslant m$, we get $a_{r+1} = \ldots = a_m = 0$. Thus $A \not\models \Phi'[a_1 \ldots a_m]$ and c) is false. Now assume c) does not hold; we show that b) is false. Let A' be a separated *BA* and $a_1', \ldots, a_m' \in A'$ such that $a_{r+1}' = \ldots = a_m' = 0$ and $A' \not\models \Phi[a_1' \ldots a_m']$. W.l.o.g., $a_i' \neq 0$ for $1 \leqslant i \leqslant r$. By choice of r, there are $t_1, \ldots, t_r \in \tau$ such that $t_i \models \vartheta_i$ for $1 \leqslant i \leqslant r$.

Let, for these i, s_i be the element of τ such that $A' \restriction a_i' \models s_i$. By 4.1, there are $\mathcal{M} = (B, A) \in \mathbf{K}$ and $a_1, \ldots, a_r \in A$ such that $1 = a_1 + \ldots + a_r$, $a_i \cdot a_j = 0$ for $1 \leqslant i < j \leqslant r$, $A \restriction a_i \models s_i$ and $(B \restriction a_i)_p \models t_i$ for those $p \in X$ satisfying $a_i(p) = 1$. So $e(\| \vartheta_i \|^{\mathcal{M}}) = a_i$ by 4.2. Put $a_{r+1} = \ldots = a_m = 0$. It follows that $(A, a_1, \ldots, a_m) \equiv (A', a_1', \ldots, a_m')$, $A \not\models \Phi[a_1 \ldots a_m]$ and $\mathcal{M} \not\models \varphi$ by 4.3.

In the next theorem, we characterize elementary equivalence of models of T. Call the following sentences in $\mathcal{L}_{BA}$ basic sentences: $\varphi_n \wedge \psi$, $\varphi_n \wedge \neg \psi$, $\chi_n \wedge \psi$, $\chi_n \wedge \neg \psi$ (where $n \in \omega$). It follows by the analysis of the completions of T_{sBA} given in the beginning of this section that for each $\mathcal{L}_{BA}$-sentence ϑ there are basic sentences $\beta_1, \ldots, \beta_n$ such that

$$T_{sBA} \vdash (\vartheta \leftrightarrow \bigvee_{i=1}^{n} \beta_i) \wedge \bigwedge_{1 \leqq i < j \leqq n} \neg (\beta_i \wedge \beta_j) .$$

This fact is easily extended to T_{sBA2}: by replacing each atomic formula $U(t)$ where t is a term in $\mathcal{L}_{BA}$ by "$t = 0 \vee t = 1$", we see that for each $\mathcal{L}$-sentence ϑ there are basic sentences $\beta_1, \ldots, \beta_n$ satisfying

$$T_{sBA2} \vdash (\vartheta \leftrightarrow \bigvee_{i=1}^{n} \ ) \wedge \bigwedge_{1 \leqq i < j \leqq n} \neg (\beta_i \wedge \beta_j) .$$

Now, if β, γ are basic sentences, let $\sigma_{\beta\gamma}$ be the following $\mathcal{L}$-sentence:

$$\sigma_{\beta\gamma} = \exists y \, (\gamma^y \wedge s_\beta(y)) ,$$

where $s_\beta(y)$ is the $\mathcal{L}$-formula assigned to β in 3.1 and γ^y is the result of relativizing the quantifiers $\exists x \varphi \ldots$ in γ to $\exists x \, (U(x) \wedge x \leqslant y \wedge \varphi^y \ldots)$. A model (B, A) of T satisfies $\sigma_{\beta\gamma}$ iff $A \restriction a \models \gamma$, where $a = e(c)$ and $c = \| \beta \|$.

4.5. Theorem. *Let $\mathcal{M} = (B, A)$, $\mathcal{M}' = (B', A')$ be models of T. Then $\mathcal{M}$ is elementarily equivalent to $\mathcal{M}'$ if and only if, for any basic sentences β, γ,*

$$\mathcal{M} \models \sigma_{\beta\gamma} \quad \text{iff} \quad \mathcal{M}' \models \sigma_{\beta\gamma} .$$

Proof. The only-if-part is clear. Suppose that $\mathcal{M}$ and $\mathcal{M}'$ satisfy the same sentences of the form $\sigma_{\beta\gamma}$. Let φ be an $\mathcal{L}$-sentence and $\mathcal{M} \models \varphi$; we want to show that $\mathcal{M}' \models \varphi$. Let $(\Phi(y_1 \ldots y_m); \vartheta_1, \ldots, \vartheta_m)$ be the sequence assigned to φ by 4.3; every ϑ_i is an $\mathcal{L}$-sentence. Put $a_i = e(\| \vartheta_i \|^{\mathcal{M}})$; by 4.3 and $e : C \to A$ being an isomorphism, we have that $\{a_1, \ldots, a_m\}$

is a partition of A and $A \models \Phi\,[a_1 \dots a_m]$. In the same way, put $a_i' = e'(\|\vartheta_i\|^{\mathcal{M}'})$; $\{a_1', \dots, a_m'\}$ is a partition of A'. It is sufficient to show that $(A, a_1, \dots, a_m) \equiv (A', a_1', \dots, a_m')$, for this implies $A' \models \Phi\,[a_1' \dots a_m']$ and finally $\mathcal{M}' \models \varphi$ by 4.3.

For every ϑ_i, choose basic sentences $\beta_{i1}, \dots, \beta_{in_i}$ such that

$$T_{sBA2} \vdash (\vartheta_i \leftrightarrow \bigvee_j \beta_{ij}) \wedge \bigwedge_{j<l} \neg\,(\beta_{ij} \wedge \beta_{il})\,.$$

Put $\alpha_{ij} = e(\|\beta_{ij}\|^{\mathcal{M}})$, $\alpha_{ij}' = e'(\|\beta_{ij}\|^{\mathcal{M}'})$ for $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n_i$. Then a_i is the disjoint sum of the α_{ij} $(1 \leqslant j \leqslant n_i)$, a_i' is the disjoint sum of the α'_{ij} $(1 \leqslant j \leqslant n_i)$. For every i, j,

$$A \upharpoonright \alpha_{ij} \equiv A' \upharpoonright \alpha_{ij}' :$$

let γ be any basic sentence of $\mathcal{L}_{BA}$ and assume $A \upharpoonright \alpha_{ij} \models \gamma$; we want to show that $A' \upharpoonright \alpha_{ij}' \models \gamma$. But $A \upharpoonright \alpha_{ij} \models \gamma$ means that $\mathcal{M} \models \sigma_{\beta_{ij}\gamma}$. By our main assumption, $\mathcal{M}' \models \sigma_{\beta_{ij}\gamma}$ and $A' \upharpoonright \alpha'_{ij} \models \gamma$.

We have shown that the partial function f mapping α_{ij} to α_{ij}' is an element of the set of back-and-forth-isomorphisms defined after 4.3. Hence,

$$(A, \alpha_{11}, \dots, \alpha_{mn_m}) \equiv (A', \alpha_{11}', \dots, \alpha_{mn_m}')$$

and

$$(A, a_1, \dots, a_m) \equiv (A', a_1', \dots, a_m')\,.$$

We shall finally describe the completions of T by giving a one-one correspondance between a set P (consisting of pairs of mappings from $\omega \times 2$ to $(\omega+1) \times 2)$ and these completions. For $m, m' \in \omega + 1$ and $j, j' \in 2$, define

$$(m, j) + (m', j') = (m'', j'')$$

where m'' is the cardinal sum of m and m' and j'' is the maximum of j and j'. Let

$$P = \{(\alpha, \rho) \mid \alpha, \rho : \omega \times 2 \to (\omega+1) \times 2 \text{ and, for } (n, i) \in \omega \times 2,\ \rho(n, i) = \rho(n+1, i) + \alpha(n, i)\}\,.$$

In the following definition, we refer to the $\mathcal{L}_{BA}$-theories T_{ni} defined in the beginning of this section. For $(\alpha, \rho) \in P$, let $T_{\alpha\rho}$ the $\mathcal{L}$-theory

$$\begin{aligned} T_{\alpha\rho} = T &\cup \{\exists x\,(\sigma_{(\varphi_n \wedge \neg\psi)}(x) \wedge \gamma^x) \mid n \in \omega,\ \gamma \in T_{\alpha(n,0)}\} \\ &\cup \{\exists x\,(\sigma_{(\chi_n \wedge \neg\psi)}(x) \wedge \gamma^x) \mid n \in \omega,\ \gamma \in T_{\rho(n,0)}\} \\ &\cup \{\exists x\,(\sigma_{(\varphi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega,\ \gamma \in T_{\alpha(n,1)}\} \\ &\cup \{\exists x\,(\sigma_{(\chi_n \wedge \psi)}(x) \wedge \gamma^x) \mid n \in \omega,\ \gamma \in T_{\rho(n,1)}\}\,. \end{aligned}$$

If $\mathscr{M} = (B, A)$ is a model of T, then $\mathscr{M} \models T_{\alpha\rho}$ iff, for $a_1 = e(\| \varphi_n \wedge \neg \psi \|^{\mathscr{M}})$ $A \restriction a_1 \models T_{\alpha(n,0)}, \ldots,$ for $a_4 = e(\| \chi_n \wedge \psi \|^{\mathscr{M}})$, $A \restriction a_4 \models T_{\rho(n,1)}$.

4.6. THEOREM. $\{T_{\alpha\rho} \mid (\alpha, \rho) \in P\}$ *is the set of completions of* T. *Moreover, each* $T_{\alpha\rho}$ *has a model in* $\mathbf{K}$.

Proof. If (α, ρ) and (α', ρ') are different elements of P, then $T_{\alpha\rho} \cup T_{\alpha'\rho'}$ is inconsistent (recall that every T_{mj}, where $m \in \omega + 1$, $j \in 2$, is complete in $\mathscr{L}_{BA}$). If $\mathscr{M}$ is a model of T, there is some $(\alpha, \rho) \in P$ such that $\mathscr{M} \models T_{\alpha\rho}$ (e.g., put $a_1 = e(\| \varphi_n \wedge \neg \psi \|^{\mathscr{M}})$ and let $\alpha(n, 0)$ be the pair $(k, j) \in (\omega + 1) \times 2$ such that $A \restriction a_1 \models T_{kj}$, etc.). If $(\alpha, \rho) \in P$ and $\mathscr{M}$, $\mathscr{M}'$ are models of $T_{\alpha\rho}$, then $\mathscr{M}$ and $\mathscr{M}'$ are elementarily equivalent by 4.5, since $T_{\alpha\rho}$ says which sentences of the form $\sigma_{\beta\gamma}$ are satisfied in $\mathscr{M}$ and $\mathscr{M}'$. So it is sufficient to prove that each $T_{\alpha\rho}$ has a model which lies even in $\mathbf{K}$.

For simplicity, we construct $\mathscr{M} \in \mathbf{K}$ satisfying the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$ — for, if $\mathscr{N} \in \mathbf{K}$ satisfies the part of $T_{\alpha\rho}$ which refers to $T_{\alpha(n,1)}$ and $T_{\rho(n,1)}$, then $\mathscr{M} \times \mathscr{N} \in \mathbf{K}$ is a model of $T_{\alpha\rho}$. Abbreviate $\alpha(n, 0)$ by t_n, $\rho(n, 0)$ by s_n. We first construct a complete *BA* A and a sequence $(a_n)_{n \in \omega}$ in A such that the a_n are pairwise disjoint and

$$(*) \quad A \restriction a_n \models t_n, \qquad A \restriction r_n \models s_n$$

where $r_n = -(a_0 + \ldots + a_{n-1})$. Let A be a complete *BA* which is a model of s_0. Suppose $a_0, \ldots, a_{n-1} \in A$ are pairwise disjoint and $a_0, \ldots, a_{n-1}, r_n$ satisfy (*). Since $s_n = s_{n+1} + t_n$, $A \restriction r_n \models s_n$ and A is complete, there are a_n and $r_{n+1} \in A$ such that $r_n = a_n + r_{n+1}$, $a_n \cdot r_{n+1} = 0$, $A \restriction a_n \models t_n$ and $A \restriction r_{n+1} \models s_{n+1}$. — Finally, let $a_\omega = -\sum_{n \in \omega} a_n$. By the proof of 4.1, there is, for $n \in \omega$, $\mathscr{M}_n = (B_n, A_n) \in \mathbf{K}$ such that $A_n = A \restriction a_n$ and each stalk $(B_n)_p$ of the sheaf representation of $\mathscr{M}_n$ is a model of $\varphi_n \wedge \neg\psi$. Moreover there is $\mathscr{M}_\omega = (B_\omega, A_\omega) \in \mathbf{K}$ such that $A_\omega = A \restriction a_\omega$ and each stalk $(B_\omega)_p$ of the sheaf representation of $\mathscr{M}_\omega$ is a model of $T_{\omega 0}$. Put $\mathscr{M} = (B, A)$ where B is a complete *BA* which lies over A as $\prod_{n \in \omega} B_n$ lies over $\prod_{n \in \omega} A_n$. By 4.2, $e(\| \varphi_n \wedge \neg \psi \|^{\mathscr{M}}) = a_n$ and $e(\| \chi_n \wedge \neg \psi \|^{\mathscr{M}}) = r_n$;so $\mathscr{M}$ is a model of the part of $T_{\alpha\rho}$ referring to $T_{\alpha(n,0)}$ and $T_{\rho(n,0)}$.