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3. TRUTH VALUES IN A FOR STATEMENTS ABOUT (B, A)

For the rest of this paper, let $\mathcal{L}_{BA} = \{+, \cdot, -, 0, 1\}$ the language of BAs and $\mathcal{L} = \mathcal{L}_{BA} \cup \{U\}$. Let T_{BAU} be the theory in \mathcal{L} such that the models of T_{BAU} have the form $(B, +, \cdot, -, 0, 1, A)$ where (B, \dots) is a BA and A is a subalgebra of B . We abbreviate a model (B, \dots, A) of T_{BAU} by $\mathcal{M} = (B, A)$. We assume the construction and notations of section 1. For each \mathcal{L} -formula $\varphi(x_1 \dots x_n)$ and $b_1, \dots, b_n \in B$, we defined

$$\|\varphi[b_1 \dots b_n]\| = \{p \in X \mid B_p \models \varphi[b_1(p) \dots b_n(p)]\}$$

where B_p abbreviates $(B_p, 2)$ and 2 is the two-element BA . Our first claim is that if $c = \|\varphi[b_1 \dots b_n]\|$ is a clopen subset of X for every φ , then $e(c) \in A$ is first-order definable in $\mathcal{M} = (B, A)$ from the parameters $b_1, \dots, b_n \in B$:

3.1. LEMMA. There is an effective procedure assigning to each formula $\varphi(x_1 \dots x_n)$ of \mathcal{L} a formula $s_\varphi(yx_1 \dots x_n)$ of \mathcal{L} (where y is a variable not occurring in φ) such that for $\mathcal{M} \models T_{BAU}$, properties (i) and (ii) are equivalent and (ii) implies (iii):

- (i) $\|\varphi[b_1 \dots b_n]\|$ is clopen for every $\varphi(x_1 \dots x_n)$ in \mathcal{L} and $b_1, \dots, b_n \in B$;
- (ii) $\mathcal{M} \models \forall x_1 \dots \forall x_n \exists y s_\varphi(yx_1 \dots x_n)$ for every $\varphi(x_1 \dots x_n)$ in \mathcal{L} ;
- (iii) if $b_1, \dots, b_n \in B$, then $a = e(c)$ where $c = \|\varphi[b_1 \dots b_n]\|$ is the unique element b of B such that $\mathcal{M} \models s_\varphi[bb_1 \dots b_n]$.

Proof. The inductive definition of s_φ will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being φ atomic or φ existential. In both cases the fact that $\|\varphi[\dots]\|$ is clopen will be expressed by stating " $a (= e(\|\varphi[\dots]\|))$ is the largest element of A such that $e^{-1}(a) \subseteq \|\varphi[\dots]\|$ ". This includes, if φ has the form $\exists x\psi$, the maximum principle for the Boolean valuation

$$\psi, b_1 \dots b_n \rightarrow \|\psi[b_1 \dots b_n]\|$$

of \mathcal{M} in C : there is some $b \in B$ such that

$$\|\psi[b'b_1 \dots b_n]\| \leq \|\psi[bb_1 \dots b_n]\|$$

for every $b' \in B$, and hence $\|\psi[bb_1 \dots b_n]\| = \|\exists x\psi[xb_1 \dots b_n]\|$. We now proceed to define the formulas s_φ .

- a) Suppose φ is an atomic formula of \mathcal{L}_{BA} , i.e. φ has the form $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$ where t_1, t_2 are terms in \mathcal{L}_{BA} . Let $s_\varphi(yx_1 \dots x_n)$ be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \cdot t_2 \rightarrow y' \leq y).$$

- b) Suppose φ has the form $U(t(x_1 \dots x_n))$ where t is a term in \mathcal{L}_{BA} . Let ψ, χ be the atomic \mathcal{L}_{BA} -formulas " $t = 1$ " resp. " $t = 0$ ". Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

- c) Suppose φ has the form $\neg \psi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1 x_1 \dots x_n)].$$

- d) Suppose φ has the form $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$. Let s_φ be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

- e) Suppose φ has the form $\exists x \psi(xx_1 \dots x_n)$. Let s_φ be the formula

$$\exists x s_\psi(yxx_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y'x'x_1 \dots x_n) \rightarrow y' \leq y].$$

Let σ be the \mathcal{L}_{BA} -formula stating that the supremum of the atoms of a BA exists; σ^U is the relativization of σ to the one-place predicate U of \mathcal{L} . The models of $T_{BA} \cup \{\sigma\}$ are called separated BA s in [3]. Let T be the \mathcal{L} -theory

$$T = T_{BAU} \cup \{ \forall x_1 \dots \forall x_n \exists y s_\varphi(yx_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathcal{L} \} \\ \cup \{ \sigma^U, s_\sigma(1) \}.$$

The last two axioms of T express, for a model $\mathcal{M} = (B, A)$ of T_{BAU} , that A and each stalk B_p are separated BA s. Let \mathbf{K} be the class of \mathcal{L} -structures $\mathcal{M} = (B, A)$ where B is a cBA and A is relatively complete in B . We shall prove in section 4 that T is an axiomatization of the first-order theory of \mathbf{K} . The easy part of this is:

3.2. THEOREM. *Each structure \mathcal{M} in \mathbf{K} is a model of T .*

Proof. Let $\mathcal{M} = (B, A) \in \mathbf{K}$, i.e. B is complete and A is relatively complete in B . Hence $\mathcal{M} \models T_{BAU}$ and A is a separated BA . By 1.1, $\| \varphi[b_1 \dots b_n] \|$ is clopen for every atomic formula φ of \mathcal{L} and arbitrary $b_1, \dots, b_n \in B$. If $\| \varphi[b_1 \dots b_n] \|$ and $\| [\psi[b_1 \dots b_n]] \|$ are clopen subsets of X , so are $\| \neg \varphi[b_1 \dots b_n] \|$ and $\| (\varphi \vee \psi)[b_1 \dots b_n] \|$. Hence we assume that φ

has the form $\exists x \psi (xx_1 \dots x_n)$ and that $\| \psi [bb_1 \dots b_n] \|$ is clopen for fixed $b_1, \dots, b_n \in B$ and arbitrary $b \in B$. For the rest of the proof, we omit the parameters b_1, \dots, b_n . Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \}.$$

By our inductive assumption, u is an open subset of X . Choose, by Zorn's lemma, a maximal family $F = \{ (b_i, c_i) \mid i \in I \}$ such that $b_i \in B$, c_i is a clopen subset of u , $c_i \subseteq \| \psi [b_i] \|$, $i \neq j$ implies $c_i \cap c_j = \emptyset$. It follows that c , the closure of $\bigcup_{i \in I} c_i$, includes u (by maximality of F). A is a cBA ,

hence X is extremally disconnected and c is clopen. By completeness of B , there is some $b \in B$ such that $b \cdot e(c_i) = b_i$ for $i \in I$. Thus, for $i \in I$, $c_i \subseteq \| \psi [b] \|$. So, for $\beta \in B$, $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$.

Finally we show that B_p is separated for each $p \in X$. Let $\alpha(x)$ be the \mathcal{L}_{BA} -formula stating that x is an atom and let $\beta(x)$, $\gamma(x)$ be the \mathcal{L}_{BA} -formulas $\alpha(x) \vee x = 0$ resp. $\forall y (\alpha(y) \rightarrow y \leq x)$. Put $M = \{ f \in B \mid \| \beta [f] \| = 1 \|$ and let b be the supremum of M in B . We show that $b(p)$ is, for each $p \in X$, the supremum of the atoms of B_p .

First suppose $s \in B_p$ is an atom of B_p . There is some $f \in M$ such that $f(p) = s$ (note that $\| \alpha [f] \|$ is clopen for each $f \in B$). So $f \leq b$ and $s = f(p) \leq b(p)$. — On the other hand, suppose $t \in B_p$ and $s \leq t$ for every atom s of B_p . Choose $g \in B$ such that $g(p) = t$. Then $p \in c = \| \gamma [g] \|$. For $f \in M$, $e(c) \cdot f \leq g$, since $q \in c$ implies that $f(q)$ is zero or an atom of B_q and thus $f(q) \leq g(q)$. By the definition of b , $e(c) \cdot b \leq g$. This implies (by $p \in c$) $b(p) \leq g(p) = t$.

4. DECIDABILITY AND COMPLETIONS OF $Th(\mathbf{K})$

Call $T_{sBA} = T_{BA} \cup \{ \sigma \}$ the theory of separated BA s, where T_{BA} is the theory of BA s and σ was defined in section 3. We give a short review of the completions of T_{sBA} . Let, for $n \in \omega$, φ_n be the \mathcal{L}_{BA} -sentence stating that there are exactly n atoms and ψ the \mathcal{L}_{BA} -sentence stating that there is a non-zero atomless element. Let $\chi_n = \neg (\varphi_0 \vee \dots \vee \varphi_{n-1})$; so χ_n says that there are at least n atoms. Define, for $n \in \omega + 1$ and $i \in 2 = \{0, 1\}$, an \mathcal{L}_{BA} -theory T_{ni} by

$$\begin{aligned} T_{n0} &= T_{sBA} \cup \{ \varphi_n, \neg \psi \} \\ T_{n1} &= T_{sBA} \cup \{ \varphi_n, \psi \} \end{aligned}$$