Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 28 (1982)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON BOOLEAN ALGEBRAS WITH DISTINGUISHED SUBALGEBRAS

Autor: Koppelberg, Sabine

**Kapitel:** 3. Truth values in for statements about (B, A)

**DOI:** https://doi.org/10.5169/seals-52239

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

#### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF: 23.11.2025** 

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

## 3. Truth values in A for statements about (B, A)

For the rest of this paper, let  $\mathcal{L}_{BA} = \{+, \cdot, -, 0, 1\}$  the language of BAs and  $\mathcal{L} = \mathcal{L}_{BA} \cup \{U\}$ . Let  $T_{BAU}$  be the theory in  $\mathcal{L}$  such that the models of  $T_{BAU}$  have the form  $(B, +, \cdot, -, 0, 1, A)$  where (B, ...) is a BA and A is a subalgebra of B. We abbreviate a model (B, ..., A) of  $T_{BAU}$  by  $\mathcal{M} = (B, A)$ . We assume the construction and notations of section 1. For each  $\mathcal{L}$ -formula  $\varphi(x_1 ... x_n)$  and  $b_1, ..., b_n \in B$ , we defined

$$\| \varphi [b_1 ... b_n] \| = \{ p \in X | B_p \models \varphi [b_1 (p) ... b_n (p)] \}$$

where  $B_p$  abbreviates  $(B_p, 2)$  and 2 is the two-element BA. Our first claim is that if  $c = \| \varphi [b_1 \dots b_n] \|$  is a clopen subset of X for every  $\varphi$ , then  $e(c) \in A$  is first-order definable in  $\mathcal{M} = (B, A)$  from the parameters  $b_1, \dots, b_n \in B$ :

- 3.1. Lemma. There is an effective procedure assigning to each formula  $\varphi(x_1 \dots x_n)$  of  $\mathscr{L}$  a formula  $s_{\varphi}(yx_1 \dots x_n)$  of  $\mathscr{L}$  (where y is a variable not occurring in  $\varphi$ ) such that for  $\mathscr{M} \models T_{BAU}$ , properties (i) and (ii) are equivalent and (ii) implies (iii):
  - (i)  $\| \varphi [b_1 ... b_n] \|$  is clopen for every  $\varphi (x_1 ... x_n)$  in  $\mathcal{L}$  and  $b_1, ..., b_n \in B$ ;
- (ii)  $\mathcal{M} \models \forall x_1 \dots \forall x_n \exists y s_{\varphi} (yx_1 \dots x_n)$  for every  $\varphi (x_1 \dots x_n)$  in  $\mathcal{L}$ ;
- (iii) if  $b_1, ..., b_n \in B$ , then a = e(c) where  $c = \| \varphi [b_1 ... b_n] \|$  is the unique element b of B such that  $\mathcal{M} \models s_{\varphi} [bb_1 ... b_n]$ .

*Proof.* The inductive definition of  $s_{\varphi}$  will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being  $\varphi$  atomic or  $\varphi$  existential. In both cases the fact that  $\|\varphi[...]\|$  is clopen will be expressed by stating " $a (= e (\|\varphi[...]\|)$  is the largest element of A such that  $e^{-1}(a) \subseteq \|\varphi[...]\|$ ". This includes, if  $\varphi$  has the form  $\exists x \psi$ , the maximum principle for the Boolean valuation

$$\psi, b_1 \dots b_n \to \| \psi [b_1 \dots b_n] \|$$

of  $\mathcal{M}$  in C: there is some  $b \in B$  such that

$$\|\psi[b'b_1 \dots b_n]\| \leqslant \|\psi[bb_1 \dots b_n]\|$$

for every  $b' \in B$ , and hence  $\|\psi[bb_1 \dots b_n]\| = \|\exists x\psi[xb_1 \dots b_n]\|$ . We now proceed to define the formulas  $s_{\varphi}$ .

a) Suppose  $\varphi$  is an atomic formula of  $\mathcal{L}_{BA}$ , i.e.  $\varphi$  has the form  $t_1 (x_1 \dots x_n) = t_2 (x_1 \dots x_n)$  where  $t_1, t_2$  are terms in  $\mathcal{L}_{BA}$ . Let  $s_{\varphi} (yx_1 \dots x_n)$  be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' t_2 \rightarrow y' \leqslant y).$$

b) Suppose  $\varphi$  has the form  $U(t(x_1 ... x_n))$  where t is a term in  $\mathcal{L}_{BA}$ . Let  $\psi$ ,  $\chi$  be the atomic  $\mathcal{L}_{BA}$ -formulas "t=1" resp. "t=0". Let  $s_{\varphi}$  be the formula

$$\exists y_1 \; \exists y_2 \; [y = y_1 + y_2 \; \wedge \; s_{\psi} \; (y_1 x_1 \; \dots \; x_n) \; \wedge \; s_{\chi} \; (y_2 x_1 \; \dots \; x_n)] \; .$$

- c) Suppose  $\varphi$  has the form  $\neg \psi (x_1 \dots x_n)$ . Let  $s_{\varphi}$  be the formula  $\exists y_1 [y = -y_1 \land s_{\psi} (y_1 x_1 \dots x_n)]$ .
- d) Suppose  $\varphi$  has the form  $\psi$   $(x_1 \dots x_n) \vee \chi$   $(x_1 \dots x_n)$ . Let  $s_{\varphi}$  be the formula  $\exists y_1 \exists y_2 [y = y_1 + y_2 \land s_{\psi} (y_1 x_1 \dots x_n) \land s_{\chi} (y_2 x_1 \dots x_n)].$
- e) Suppose  $\varphi$  has the form  $\exists x \ \psi \ (xx_1 \dots x_n)$ . Let  $s_{\varphi}$  be the formula  $\exists xs_{\psi} \ (yxx_1 \dots x_n) \land \forall x' \forall y' \ [s_{\psi} \ (y'x'x_1 \dots x_n) \rightarrow y' \leqslant y]$ .

Let  $\sigma$  be the  $\mathcal{L}_{BA}$ -formula stating that the supremum of the atoms of a BA exists;  $\sigma^U$  is the relativization of  $\sigma$  to the one-place predicate U of  $\mathcal{L}$ . The models of  $T_{BA} \cup \{\sigma\}$  are called separated BAs in [3]. Let T be the  $\mathcal{L}$ -theory

$$T = T_{BAU} \cup \left\{ \forall x_1 \dots \forall x_n \exists y \ s_{\varphi} (yx_1 \dots x_n) \mid \varphi (x_1 \dots x_n) \text{ in } \mathcal{L} \right\}$$
$$\cup \left\{ \sigma^U, s_{\sigma} (1) \right\}.$$

The last two axioms of T express, for a model  $\mathcal{M} = (B, A)$  of  $T_{BAU}$ , that A and each stalk  $B_p$  are separated BAs. Let K be the class of  $\mathcal{L}$ -structures  $\mathcal{M} = (B, A)$  where B is a cBA and A is relatively complete in B. We shall prove in section 4 that T is an axiomatization of the first-order theory of K. The easy part of this is:

# 3.2. Theorem. Each structure $\mathcal{M}$ in $\mathbf{K}$ is a model of T.

*Proof.* Let  $\mathcal{M} = (B, A) \in \mathbf{K}$ , i.e. B is complete and A is relatively complete in B. Hence  $\mathcal{M} \models T_{BAU}$  and A is a separated BA. By 1.1,  $\| \varphi [b_1 \dots b_n] \|$  is clopen for every atomic formula  $\varphi$  of  $\mathcal{L}$  and arbitrary  $b_1, \dots, b_n \in B$ . If  $\| \varphi [b_1 \dots b_n] \|$  and  $\| [\psi [b_1 \dots b_n] \|$  are clopen subsets of X, so are  $\| \neg \varphi [b_1 \dots b_n] \|$  and  $\| (\varphi \lor \psi) [b_1 \dots b_n] \|$ . Hence we assume that  $\varphi$ 

has the form  $\exists x \psi (xx_1 \dots x_n)$  and that  $\| \psi [bb_1 \dots b_n] \|$  is clopen for fixed  $b_1, \dots, b_n \in B$  and arbitrary  $b \in B$ . For the rest of the proof, we omit the parameters  $b_1 \dots, b_n$ . Let

$$u = \cup \{ \| \psi [\beta] \| | \beta \in B \}.$$

By our inductive assumption, u is an open subset of X. Choose, by Zorn's lemma, a maximal family  $F = \{(b_i, c_i) \mid i \in I\}$  such that  $b_i \in B$ ,  $c_i$  is a clopen subset of u,  $c_i \subseteq \|\psi[b_i]\|$ ,  $i \neq j$  implies  $c_i \cap c_j = \phi$ . It follows that c, the closure of  $\bigcup_{i \in I} c_i$ , includes u (by maximality of F). A is a cBA,  $i \in I$ 

hence X is extremally disconnected and c is clopen. By completeness of B, there is some  $b \in B$  such that  $b \cdot e(c_i) = b_i$  for  $i \in I$ . Thus, for  $i \in I$ ,  $c_i \subseteq \|\psi[b]\|$ . So, for  $\beta \in B$ ,  $\|\psi[\beta]\| \subseteq u \subseteq c \subseteq \|\psi[b]\| = \|\exists x \psi(x)\|$ .

Finally we show that  $B_p$  is separated for each  $p \in X$ . Let  $\alpha(x)$  be the  $\mathcal{L}_{BA}$ -formula stating that x is an atom and let  $\beta(x)$ ,  $\gamma(x)$  be the  $\mathcal{L}_{BA}$ -formulas  $\alpha(x) \vee x = 0$  resp.  $\forall y (\alpha(y) \to y \leqslant x)$ . Put  $M = \{f \in B \mid \|\beta[f]\| = 1 \|$  and let b be the supremum of M in B. We show that b(p) is, for each  $p \in X$ , the supremum of the atoms of  $B_p$ .

First suppose  $s \in B_p$  is an atom of  $B_p$ . There is some  $f \in M$  such that f(p) = s (note that  $\| \alpha [f] \|$  is clopen for each  $f \in B$ ). So  $f \leqslant b$  and  $s = f(p) \leqslant b(p)$ . — On the other hand, suppose  $t \in B_p$  and  $s \leqslant t$  for every atom s of  $B_p$ . Choose  $g \in B$  such that g(p) = t. Then  $p \in c = \| \gamma [g] \|$ . For  $f \in M$ ,  $e(c) \cdot f \leqslant g$ , since  $g \in C$  implies that f(g) is zero or an atom of  $B_q$  and thus  $f(g) \leqslant g(g)$ . By the definition of  $g \in C$ , this implies (by  $g \in C$ )  $g \in C$  and  $g \in C$  implies that  $g(g) \in C$  i

## 4. Decidability and completions of Th(K)

Call  $T_{sBA} = T_{BA} \cup \{\sigma\}$  the theory of separated BAs, where  $T_{BA}$  is the theory of BAs and  $\sigma$  was defined in section 3. We give a short review of the completions of  $T_{sBA}$ . Let, for  $n \in \omega$ ,  $\varphi_n$  be the  $\mathcal{L}_{BA}$ -sentence stating that there are exactly n atoms and  $\psi$  the  $\mathcal{L}_{BA}$ -sentence stating that there is a non-zero atomless element. Let  $\chi_n = \neg (\varphi_0 \vee ... \vee \varphi_{n-1})$ ; so  $\chi_n$  says that there are at least n atoms. Define, for  $n \in \omega + 1$  and  $i \in 2 = \{0, 1\}$ , an  $\mathcal{L}_{BA}$ -theory  $T_{ni}$  by

$$T_{n0} = T_{sBA} \cup \{ \varphi_n, \neg \psi \}$$

$$T_{n1} = T_{sBA} \cup \{ \varphi_n, \psi \}$$