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1.3. REMARK. a) Let  $A$  and the inclusion map from  $A$  to  $B$  be complete. Then  $A$  is relatively complete in  $B$ .

b) Suppose  $A$  is relatively complete in  $B$  and  $B$  is complete. Then  $A$  is complete.

## 2. RELATIVE AUTOMORPHISMS OF FINITE EXTENSIONS

We first give an internal description of a finite extension  $(B, A)$  where  $B = A(u_1 \dots u_n)$  and  $n \in \omega$ . We shall always assume that  $u_1, \dots, u_n$  are the atoms of the subalgebra of  $B$  generated by  $u_1, \dots, u_n$ ; i.e. that they are non-zero, pairwise disjoint and  $u_1 + \dots + u_n = 1$ . Let  $I_r = \{a \in A \mid a \cdot u_r = 0\}$  for  $1 \leq r \leq n$ . Clearly, each  $I_r$  is a proper ideal of  $A$  and  $I_1 \cap \dots \cap I_n = \{0\}$ . The family  $(I_r \mid 1 \leq r \leq n)$  completely characterizes the extension  $(B, A)$ :

2.1. REMARK. Suppose  $C = A(v_1 \dots v_n)$  is a finite extension of  $A$  where  $v_1, \dots, v_n$  are pairwise disjoint and  $1 = v_1 + \dots + v_n$ . Let  $B = A(u_1 \dots u_n)$  be as above. There is an isomorphism  $g$  from  $B$  onto  $C$  satisfying  $g(a) = a$  for  $a \in A$  and  $g(u_r) = v_r$  iff, for each  $r$ ,  $\{a \in A \mid a \cdot v_r = 0\} = I_r$ .

*Proof.* By Theorem 12.4 in [7].

2.2. REMARK.  $A$  is relatively complete in  $B = A(u_1 \dots u_n)$  iff, for each  $r$ ,  $I_r$  is a principal ideal.

*Proof.* The only-if part follows by the definition of relative completeness. Now suppose  $\alpha_r \in A$  generates  $I_r$ ; let  $b \in B$  and  $I = \{a \in A \mid a \cdot b = 0\}$ . There are  $a_1, \dots, a_n \in A$  such that  $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$ . It follows that  $I$  is the principal ideal generated by  $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$ .

Conversely, given any family  $(I_r \mid 1 \leq r \leq n)$  of proper ideals in  $A$  satisfying  $I_1 \cap \dots \cap I_n = \{0\}$ , there is an extension  $A(u_1 \dots u_n)$  of  $A$  such that  $I_r = \{a \in A \mid a \cdot u_r = 0\}$ : let  $D = A(x_1 \dots x_n)$  be the free product of  $A$  and a finite BA with atoms  $x_1, \dots, x_n$ . Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\}.$$

$K$  is an ideal of  $D$ ; the canonical epimorphism  $\pi$  from  $D$  onto  $B = D/K$  is one-one on  $A$ , and for  $a \in A$ ,  $\pi(a) \cdot u_r = 0$  iff  $a \in I_r$  where  $u_r = \pi(x_r)$ . Now identify  $A$  with the subalgebra  $\pi(A)$  of  $B$ .

For the rest of this section we think, as in section 1, of  $B$  as being the set of global sections of a sheaf  $\mathcal{S} = (S, \pi, X, \mu)$  of Boolean algebras over a

Boolean space  $X$ ; we use the abbreviations of section 1. For  $p \in X$ ,  $B_p = \{b(p) \mid b \in B\}$ . Since  $b(p) \in \{0, 1\}$  for  $b \in A$  and  $B = A(u_1 \dots u_n)$ ,  $B_p$  is a finite BA with atoms  $\{u_r(p) \mid 1 \leq r \leq n\} \setminus \{0\}$ .

Let  $G = \text{Aut}_A B$  be the group of those automorphisms of  $B$  leaving  $A$  pointwise fixed, i.e.  $G$  is the Galois group of  $B$  over  $A$ . Suppose  $g \in G$  and  $p \in X$ . Since  $g(a) = a$  for  $a \in A$ ,  $g$  induces an automorphism of  $B_p$  which, in turn, is induced by a permutation of the (at most  $n$ ) atoms of  $B_p$ . This gives rise to the following definitions ( $S_n$  is the group of permutations of  $\{1, \dots, n\}$ ).

Let  $p \in X$ . For  $1 \leq r, l \leq n$ , say  $u_r \sim u_l$  at  $p$  if there is a neighbourhood  $u$  of  $p$  such that, for  $q \in u$ ,  $u_r(q) = 0$  iff  $u_l(q) = 0$ .  $\pi \in S_n$  is said to be compatible with  $p$  if  $u_r \sim u_{\pi(r)}$  at  $p$  for  $1 \leq r \leq n$ .  $g \in G$  is said to be induced by  $\pi$  at  $p$  if  $g(u_r)(p) = u_{\pi(r)}(p)$  for  $1 \leq r \leq n$ . Note that, if one of these definitions holds (for fixed  $u_r, u_l, \pi \in S_n, g \in G$ ) for some  $p \in X$ , then it holds (for the same  $u_r, u_l, \pi \in S_n, g \in G$ ) for every  $q$  in some neighbourhood of  $p$ . And  $u_r \sim u_l$  at  $p$  means that there is a clopen subset  $c$  of  $X$  such that  $p \in c$  and, for  $a \in A$  satisfying  $a \leq e(c)$ ,  $a \in I_r$  iff  $a \in I_l$ .

2.3. LEMMA. Suppose  $p \in X$  and  $\pi \in S_n$ . Then  $\pi$  is compatible with  $p$  iff there is some  $g \in G$  which is induced by  $\pi$  at  $p$ .

*Proof.* Suppose  $\pi$  induces  $g$  at  $p$  and  $1 \leq r \leq n$ . Let  $u$  be a neighbourhood of  $p$  such that  $g(u_r)(q) = u_{\pi(r)}(q)$  for  $q \in u$ . Thus, for  $q \in u$ ,  $u_{\pi(r)}(q) = 0$  iff  $g(u_r)(q) = 0$  iff  $u_r(q) = 0$  since  $g$  induces an automorphism of  $B_q$ .

Conversely, suppose  $\pi$  is compatible with  $p$ . Choose a clopen neighbourhood  $c$  of  $p$  such that  $u_r(q) = 0$  iff  $u_{\pi(r)}(q) = 0$  for  $1 \leq r \leq n$  and  $q \in c$ . Let  $a = e(c)$ . By 2.1 and the remark preceding this lemma, there is some  $g \in G$  such that  $g(u_r) = -a \cdot u_r + a \cdot u_{\pi(r)}$  for every  $r$ . This  $g$  is induced by  $\pi$  at  $p$ , since  $a(p) = 1$  and hence  $g(u_r)(p) = u_{\pi(r)}(p)$ .

2.4. THEOREM. a) Let  $X = \cup \{c_\pi \mid \pi \in S_n\}$  be a partition of  $X$  into pairwise disjoint clopen subsets such that, for every  $p \in c_\pi$ ,  $\pi$  is compatible with  $p$ . Put  $a_\pi = e(c_\pi)$  for  $\pi \in S_n$ . Then there is  $g \in G$  such that, for  $1 \leq r \leq n$ ,

$$g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}.$$

b) Conversely, let  $g \in G$ . Then there is a partition  $X = \cup \{c_\pi \mid \pi \in S_n\}$  of  $X$  into pairwise disjoint clopen subsets such that, for  $p \in c_\pi$ ,  $\pi$  is compatible with  $p$ , and  $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$ , where  $a_\pi = e(c_\pi)$ .

*Proof.* First note that  $g \in G$ ,  $a_\pi = e(c_\pi)$  where  $(c_\pi \mid \pi \in S_n)$  is a partition of  $X$  and  $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$  imply that  $\pi$  is compatible with  $p$  for  $p \in c_\pi$ : by  $p \in c_\pi$ , we get  $a_\pi(p) = 1$  and  $a_\rho(p) = 0$  for  $\rho \in S_n$ ,  $\rho \neq \pi$ . So  $g(u_r)(p) = u_{\pi(r)}(p)$ ,  $g$  is induced by  $\pi$  at  $p$ , and  $\pi$  is compatible with  $p$ .

To prove a), note that  $\{a_\pi \cdot u_r \mid \pi \in S_n, 1 \leq r \leq n\}$  is a set of pairwise disjoint elements of  $B$  with supremum 1 and generating  $B$  over  $A$ . The existence of  $g$  follows by 2.1 and the remark preceding 2.3.

To prove b), let  $g \in G$ . For  $\pi \in S_n$ , put

$$v_\pi = \{p \in X \mid \pi \text{ induces } g \text{ at } p\}.$$

Each  $v_\pi$  is an open subset of  $X$ , and  $X = \cup \{v_\pi \mid \pi \in S_n\}$ : suppose  $p \in X$ . Define  $\pi \in S_n$  as follows: let  $1 \leq r \leq n$ . If  $u_r(p) = 0$ , then  $g(u_r)(p) = 0$ ; put  $\pi(r) = r$ . If  $u_r(p) \neq 0$ ,  $u_r(p)$  and hence  $g(u_r)(p)$  is an atom of  $B_p$ ; let  $\pi(r) = l$  where  $g(u_r)(p) = u_l(p)$ . Clearly,  $p \in v_\pi$ .

Since  $X$  is a Boolean space, there is a family  $(c_\pi \mid \pi \in S_n)$  such that  $c_\pi$  is a clopen subset of  $v_\pi$ ,  $X = \cup \{c_\pi \mid \pi \in S_n\}$  and the  $c_\pi$  are pairwise disjoint. Put  $a_\pi = e(c_\pi)$ . Suppose  $1 \leq r \leq n$  and  $p \in X$ , e.g.  $p \in c_\pi$ . Then  $p \in v_\pi$  and

$$\left(\sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}\right)(p) = g(u_r)(p).$$

Theorem 2.4 says that the automorphisms of  $B$  over  $A$  are completely determined by certain partitions  $(a_\pi \mid \pi \in S_n)$  of  $A$  resp.  $(c_\pi \mid \pi \in S_n)$  of  $C$ . Unfortunately, for a given  $g \in G$ , a partition  $(c_\pi \mid \pi \in S_n)$  defining  $g$  is not uniquely determined, since there may be different possibilities of choosing a clopen disjoint refinement of  $(v_\pi \mid \pi \in S_n)$ . We conclude this section by illustrating 2.4 by several examples.

If  $H$  is any group and  $A$  a BA, let  $X$  be the Stone space of  $A$  and

$$H[A] = \{f : X \rightarrow H \mid f \text{ is continuous}\}$$

where  $H$  is given the discrete topology.  $H[A]$  is a subgroup of  $H^X$  and is usually called the bounded Boolean power of  $H$  by  $A$ . Recall that, for  $B = A(u_1 \dots u_n)$ ,  $A$  and the subalgebra of  $B$  generated by  $u_1, \dots, u_n$  are independent iff  $a \cdot u_r \neq 0$  for  $a \in A \setminus \{0\}$ ,  $1 \leq r \leq n$ .  $A$  is then relatively complete in  $B$ . Conversely, suppose  $A$  is relatively complete in  $B$ . Then there is a partition  $(a_k \mid 1 \leq k \leq n)$  of  $A$  (some of the  $a_k$  may equal zero) such that, for each  $k$ , the relative algebra  $B \upharpoonright a_k = \{x \in B \mid x \leq a_k\}$  is generated over  $A \upharpoonright a_k$  by  $k$  disjoint elements  $v_1, \dots, v_k$  which are independent from  $A \upharpoonright a_k$ : for  $1 \leq r, l \leq n$ , the set of those  $p \in X$  such that  $u_r(p) = u_l(p)$  is clopen. Hence, for  $1 \leq k \leq n$ ,  $c_k = \{p \in X \mid B_p \text{ has exactly } k \text{ atoms}\}$  is

clopen; put  $a_k = e(c_k)$ . By a compactness argument, construct  $v_1, \dots, v_k \in B \upharpoonright a_k$  by patching together some of the  $u_r$  such that for  $p \in c_k$ , the atoms of  $B_p$  are  $v_1(p), \dots, v_k(p)$ .

2.5. EXAMPLE. Suppose  $a \cdot u_r \neq 0$  for  $1 \leq r \leq n$  and  $a \in A \setminus \{0\}$ . Then  $\text{Aut}_A B \cong S_n[A]$ .

*Proof.* Our assumption says that  $u_r(p) \neq 0$  for each  $r$  and each  $p \in X$ . Hence each  $\pi \in S_n$  is compatible with each  $p \in X$  and, for fixed  $g \in G$ , the open sets  $v_\pi$  in the proof of 2.4 are disjoint, hence  $c_\pi = v_\pi$ . An isomorphism  $\varphi : G \rightarrow S_n[A]$  is established by defining  $\varphi(g)(p) = \pi$  iff  $p \in v_\pi$ .

2.6. EXAMPLE. Suppose  $A$  is relatively complete in  $B$ . Then there is a partition  $(a_k \mid 1 \leq k \leq n)$  of  $A$  such that

$$\text{Aut}_A B \cong S_1[A \upharpoonright a_1] \times \dots \times S_n[A \upharpoonright a_n].$$

*Proof.* Choose, for  $1 \leq k \leq n$ ,  $a_k \in A$  as indicated above and let  $G_k$  be the Galois group of  $B \upharpoonright a_k$  over  $A \upharpoonright a_k$ . Clearly,

$$\text{Aut}_A B \cong G_1 \times \dots \times G_n,$$

since  $a_k \in A$ . By 2.5,  $G_k \cong S_k[A \upharpoonright a_k]$ .

2.7. PROPOSITION. *The following conditions on  $(B, A)$  are equivalent:*

- a)  $A$  is relatively complete in  $B$ ;
- b) there is some  $g \in G$  such that  $g(b) \neq b$  for  $b \in B \setminus A$ ;
- c) there is some finite subgroup  $H$  of  $G$  such that, for every  $b \in B \setminus A$ , there is some  $g \in H$  satisfying  $g(b) \neq b$ .

*Proof.* Assume a). There is a finite partition  $T$  of  $C$  such that, for  $1 \leq r \leq n$ ,  $t \in T$  and  $p, q \in t$ ,  $u_r(p) = 0$  iff  $u_r(q) = 0$ . For  $t \in T$ , let  $\pi_t \in S_n$  such that, for  $p \in t$ ,  $\pi_t(r) = r$  if  $u_r(p) = 0$  and  $u_r(p) \mapsto u_{\pi_t(r)}(p)$  is a cyclic permutation of the atoms of  $B_p$  which moves all these atoms.  $\pi_t$  is compatible with each  $p \in t$ ; hence there is some  $g \in G$  such that  $g$  is induced by  $\pi_t$  for  $p \in t$ ,  $t \in T$ . Now let  $b \in B \setminus A$ . Choose  $p \in X$ , e.g.  $p \in t$  where  $t \in T$ , such that  $b(p) \notin \{0, 1\}$ ; put  $b' = g(b)$ . Let  $At(B_p)$  be the set of atoms of  $B_p$ ,  $M = \{\alpha \in At(B_p) \mid \alpha \leq b(p)\}$ ,  $g_p$  the automorphism of  $B_p$  induced by  $g$ ,  $M' = \{g_p(\alpha) \mid \alpha \in M\}$ . By the choice of  $\pi_t$  and  $g$ ,

$$b'(p) = g_p(b(p)) = \sum M' \neq \sum M = b(p)$$

which proves  $b' \neq b$  — since, if  $\pi$  is a cyclic permutation of a finite set  $Y$  moving every element of  $Y$  and  $M \subseteq Y$  satisfies  $M = \{\pi(m) \mid m \in M\}$ , then  $M = \emptyset$  or  $M = Y$ .

To prove that b) implies c) it is sufficient to know that every finitely generated subgroup of  $G$  is finite. We indicate a construction for finite subgroups of  $G$ . Let  $T \subseteq C$  be a finite partition of  $C$ . A function  $\varphi : T \rightarrow S_n$  is said to be compatible if, for every  $t \in T$  and  $p \in t$ ,  $\varphi(t)$  is compatible with  $p$ . For each compatible  $\varphi : T \rightarrow S_n$  let  $g_\varphi$  be the element of  $G$  mapping  $u_r$  to  $\sum \{e(t) \cdot u_{\varphi(t)(r)} \mid t \in T\}$ . It is easily seen that

$$G_T = \{g_\varphi \mid \varphi : T \rightarrow S_n \text{ compatible}\}$$

is a finite subgroup of  $G$  and that every finite subset of  $G$  is contained in some  $G_T$ .

Now suppose c), i.e. there is some finite subgroup  $H$  of  $G$  moving every  $b \in B \setminus A$ . We may assume that  $H = G_T$  for some finite partition  $T$  of  $C$ . Assume that  $A$  is not relatively complete in  $B$ . By 2.2 there is some  $r$  such that  $I_r$  is not a principal ideal; w.l.o.g.,  $r = 1$ . Let  $\sigma = \{p \in X \mid u_1(p) = 0\}$ .  $\sigma$  is a subset of  $X$  which is open but not closed; choose  $p \in X$  which lies in the closure of  $\sigma$  but not in  $\sigma$ . W.l.o.g., for some  $k$  satisfying  $1 \leq k \leq n$ ,

$$\{r \mid 1 \leq r \leq n \text{ and } u_r \sim u_1 \text{ at } p\} = \{1, \dots, k\}.$$

Let  $c$  be a clopen neighbourhood of  $p$  such that, for  $1 \leq r \leq k$  and  $q \in c$ ,  $u_r(q) = 0$  iff  $u_1(q) = 0$ . W.l.o.g.,  $c \in T$ . There is some  $l$  such that  $k < l \leq n$  and  $u_l(p) \neq 0$ ; otherwise, let  $c' \subseteq c$  a neighbourhood of  $p$  such that  $u_l(q) = 0$  for  $q \in c'$  and  $k < l \leq n$ . Choose  $q \in c' \cap \sigma$  (since  $p$  lies in the closure of  $\sigma$ ). In  $B_q$ , which has at least two elements,  $1 = u_1(q) + \dots + u_n(q) = 0 + \dots + 0 = 0$ , a contradiction. — Put  $a = e(c)$  and  $b = a \cdot u_1 + \dots + a \cdot u_k$ .  $b \in B \setminus A$ , since  $0 < b(p) = u_1(p) + \dots + u_k(p) < 1$  by our preceding claim. We prove that, for  $g \in H = G_T$ ,  $g(b) = b$ , thus arriving at a final contradiction: there is some compatible  $\varphi : T \rightarrow S_n$  such that  $g = g_\varphi$ . Consider  $k \leq n$ ,  $c \in T$  and  $p \in c$  as constructed above. Since  $\varphi$  is compatible,  $\pi = \varphi(c)$  is compatible with  $p$ ; hence  $\pi$  maps the set  $\{1, \dots, k\}$  into itself,  $g_\varphi(a \cdot u_r) = a \cdot u_{\pi(r)}$  for  $1 \leq r \leq k$  (where  $a = e(c)$ ) and  $g(b) = b$ .