Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON BOOLEAN ALGEBRAS WITH DISTINGUISHED SUBALGEBRAS

Autor: Koppelberg, Sabine

**Kapitel:** 2. Relative automorphisms of finite extensions

**DOI:** https://doi.org/10.5169/seals-52239

## Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

## **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

## Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF: 22.11.2025** 

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

- 1.3. Remark. a) Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.
- b) Suppose A is relatively complete in B and B is complete. Then A is complete.

## 2. Relative automorphisms of finite extensions

We first give an internal description of a finite extension (B, A) where  $B = A(u_1 \dots u_n)$  and  $n \in \omega$ . We shall always assume that  $u_1, \dots, u_n$  are the atoms of the subalgebra of B generated by  $u_1, \dots, u_n$ ; i.e. that they are non-zero, pairwise disjoint and  $u_1 + \dots + u_n = 1$ . Let  $I_r = \{a \in A \mid a \cdot u_r = 0\}$  for  $1 \le r \le n$ . Clearly, each  $I_r$  is a proper ideal of A and  $I_1 \cap \dots \cap I_n = \{0\}$ . The family  $(I_r \mid 1 \le r \le n)$  completely characterizes the extension (B, A):

2.1. Remark. Suppose  $C = A(v_1 \dots v_n)$  is a finite extension of A where  $v_1, \dots, v_n$  are pairwise disjoint and  $1 = v_1 + \dots + v_n$ . Let  $B = A(u_1 \dots u_n)$  be as above. There is an isomorphism g from B onto C satisfying g(a) = a for  $a \in A$  and  $g(u_r) = v_r$  iff, for each r,  $\{a \in A \mid a \cdot v_r = 0\} = I_r$ .

Proof. By Theorem 12.4 in [7].

2.2. Remark. A is relatively complete in  $B = A(u_1 \dots u_n)$  iff, for each r,  $I_r$  is a principal ideal.

*Proof.* The only—if part follows by the definition of relative completeness. Now suppose  $\alpha_r \in A$  generates  $I_r$ ; let  $b \in B$  and  $I = \{a \in A \mid a \cdot b = 0\}$ . There are  $a_1, ..., a_n \in A$  such that  $b = a_1 \cdot u_1 + ... + a_n \cdot u_n$ . It follows that I is the principal ideal generated by  $\alpha = (-a_1 + \alpha_1) \cdot ... \cdot (-a_n + \alpha_n)$ .

Conversely, given any family  $(I_r \mid 1 \le r \le n)$  of proper ideals in A satisfying  $I_1 \cap ... \cap I_n = \{0\}$ , there is an extension  $A(u_1 ... u_n)$  of A such that  $I_r = \{a \in A \mid a \cdot u_r = 0\}$ : let  $D = A(x_1 ... x_n)$  be the free product of A and a finite BA with atoms  $x_1, ..., x_n$ . Let

$$K = \{i_1 \cdot x_1 + ... + i_n \cdot x_n \mid i_1 \in I_1, ..., i_n \in I_n\}.$$

K is an ideal of D; the canonical epimorphism  $\pi$  from D onto B = D/K is one- one on A, and for  $a \in A$ ,  $\pi(a) \cdot u_r = 0$  iff  $a \in I_r$  where  $u_r = \pi(x_r)$ . Now identify A with the subalgebra  $\pi(A)$  of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf  $\mathcal{S} = (S, \pi, X, \mu)$  of Boolean algebras over a

Boolean space X; we use the abbreviations of section 1. For  $p \in X$ ,  $B_p = \{b(p) \mid b \in B\}$ . Since  $b(p) \in \{0, 1\}$  for  $b \in A$  and  $B = A(u_1 \dots u_n)$ ,  $B_p$  is a finite BA with atoms  $\{u_r(p) \mid 1 \leqslant r \leqslant n\} \setminus \{0\}$ .

Let  $G = \operatorname{Aut}_A B$  be the group of those automorphisms of B leaving A pointwise fixed, i.e. G is the Galois group of B over A. Suppose  $g \in G$  and  $p \in X$ . Since g(a) = a for  $a \in A$ , g induces an automorphism of  $B_p$  which, in turn, is induced by a permutation of the (at most n) atoms of  $B_p$ . This gives rise to the following definitions  $(S_n$  is the group of permutations of  $\{1, ..., n\}$ ).

Let  $p \in X$ . For  $1 \le r$ ,  $l \le n$ , say  $u_r \sim u_l$  at p if there is a neighbourhood u of p such that, for  $q \in u$ ,  $u_r(q) = 0$  iff  $u_l(q) = 0$ .  $\pi \in S_n$  is said to be compatible with p if  $u_r \sim u_{\pi(r)}$  at p for  $1 \le r \le n$ .  $g \in G$  is said to be induced by  $\pi$  at p if  $g(u_r)(p) = u_{\pi(r)}(p)$  for  $1 \le r \le n$ . Note that, if one of these definitions holds (for fixed  $u_r$ ,  $u_l$ ,  $\pi \in S_n$ ,  $g \in G$ ) for some  $p \in X$ , then it holds (for the same  $u_r$ ,  $u_l$ ,  $\pi \in S_n$ ,  $g \in G$ ) for every q in some neighbourhood of p. And  $u_r \sim u_l$  at p means that there is a clopen subset c of X such that  $p \in c$  and, for  $a \in A$  satisfying  $a \le e(c)$ ,  $a \in I_r$  iff  $a \in I_l$ .

2.3. Lemma. Suppose  $p \in X$  and  $\pi \in S_n$ . Then  $\pi$  is compatible with p iff there is some  $g \in G$  which is induced by  $\pi$  at p.

*Proof.* Suppose  $\pi$  induces g at p and  $1 \leqslant r \leqslant n$ . Let u be a neighbourhood of p such that  $g(u_r)(q) = u_{\pi(r)}(q)$  for  $q \in u$ . Thus, for  $q \in u$ ,  $u_{\pi(r)}(q) = 0$  iff  $g(u_r)(q) = 0$  iff  $u_r(q) = 0$  since g induces an automorphism of  $B_q$ .

Conversely, suppose  $\pi$  is compatible with p. Choose a clopen neighbourhood c of p such that  $u_r(q) = 0$  iff  $u_{\pi(r)}(q) = 0$  for  $1 \le r \le n$  and  $q \in u$ . Let a = e(c). By 2.1 and the remark preceding this lemma, there is some  $g \in G$  such that  $g(u_r) = -a \cdot u_r + a \cdot u_{\pi(r)}$  for every r. This g is induced by  $\pi$  at p, since a(p) = 1 and hence  $g(u_r)(p) = u_{\pi(r)}(p)$ .

2.4. Theorem. a) Let  $X = \bigcup \{c_{\pi} \mid \pi \in S_n\}$  be a partition of X into pairwise disjoint clopen subsets such that, for every  $p \in c_{\pi}$ ,  $\pi$  is compatible with p. Put  $a_{\pi} = e(c_{\pi})$  for  $\pi \in S_n$ . Then there is  $g \in G$  such that, for  $1 \leqslant r \leqslant n$ ,

$$g(u_r) = \sum \{a_{\pi} \cdot u_{\pi(r)} \mid \pi \in S_n\}.$$

b) Conversely, let  $g \in G$ . Then there is a partition  $X = \bigcup \{c_{\pi} \mid \pi \in S_n\}$  of X into pairwise disjoint clopen subsets such that, for  $p \in c_{\pi}$ ,  $\pi$  is compatible with p, and  $g(u_r) = \sum \{a_{\pi} \cdot u_{\pi(r)} \mid \pi \in S_n\}$ , where  $a_{\pi} = e(c_{\pi})$ .

*Proof.* First note that  $g \in G$ ,  $a_{\pi} = e(c_{\pi})$  where  $(c_{\pi} \mid \pi \in S_n)$  is a partition of X and  $g(u_r) = \sum_{n} \{a_n \cdot u_{\pi(r)} \mid \pi \in S_n\}$  imply that  $\pi$  is compatible with p for  $p \in c_{\pi}$ : by  $p \in c_{\pi}$ , we get  $a_{\pi}(p) = 1$  and  $a_{\rho}(p) = 0$  for  $\rho \in S_n$ ,  $\rho \neq \pi$ . So  $g(u_r)(p) = u_{\pi(r)}(p)$ , g is induced by  $\pi$  at p, and  $\pi$  is compatible with p.

To prove a), note that  $\{a_n \cdot u_r \mid \pi \in S_n, 1 \leqslant r \leqslant n\}$  is a set of pairwise disjoint elements of B with supremum 1 and generating B over A. The existence of g follows by 2.1 and the remark preceding 2.3.

To prove b), let  $g \in G$ . For  $\pi \in S_n$ , put

$$v_{\pi} = \{ p \in X \mid \pi \text{ induces } g \text{ at } p \}.$$

Each  $v_n$  is an open subset of X, and  $X = \bigcup \{v_n \mid \pi \in S_n\}$ : suppose  $p \in X$ . Define  $\pi \in S_n$  as follows: let  $1 \leqslant r \leqslant n$ . If  $u_r(p) = 0$ , then  $g(u_r)(p) = 0$ ; put  $\pi(r) = r$ . If  $u_r(p) \neq 0$ ,  $u_r(p)$  and hence  $g(u_r)(p)$  is an atom of  $B_p$ ; let  $\pi(r) = l$  where  $g(u_r)(p) = u_l(p)$ . Clearly,  $p \in v_n$ .

Since X is a Boolean space, there is a family  $(c_{\pi} \mid \pi \in S_n)$  such that  $c_{\pi}$  is a clopen subset of  $v_{\pi}$ ,  $X = \bigcup \{c_{\pi} \mid \pi \in S_n\}$  and the  $c_{\pi}$  are pairwise disjoint. Put  $a_{\pi} = e(c_{\pi})$ . Suppose  $1 \leqslant r \leqslant n$  and  $p \in X$ , e.g.  $p \in c_{\pi}$ . Then  $p \in v_{\pi}$  and

$$\left(\sum \left\{a_{\pi}\cdot u_{\pi(r)} \mid \pi\in S_{n}\right\}\right)(p) = g(u_{r})(p).$$

Theorem 2.4 says that the automorphisms of B over A are completely determined by certain partitions  $(a_n \mid \pi \in S_n)$  of A resp.  $(c_n \mid \pi \in S_n)$  of C. Unfortunately, for a given  $g \in G$ , a partition  $(c_n \mid \pi \in S_n)$  defining g is not uniquely determined, since there may be different possibilities of choosing a clopen disjoint refinement of  $(v_n \mid \pi \in S_n)$ . We conclude this section by illustrating 2.4 by several examples.

If H is any group and A a BA, let X be the Stone space of A and

$$H[A] = \{f: X \to H \mid f \text{ is continuous} \}$$

where H is given the discrete topology. H[A] is a subgroup of  $H^X$  and is usually called the bounded Boolean power of H by A. Recall that, for  $B = A(u_1 \dots u_n)$ , A and the subalgebra of B generated by  $u_1, \dots, u_n$  are independent iff  $a \cdot u_r \neq 0$  for  $a \in A \setminus \{0\}$ ,  $1 \leqslant r \leqslant n$ . A is then relatively complete in B. Conversely, suppose A is relatively complete in B. Then there is a partition  $(a_k \mid 1 \leqslant k \leqslant n)$  of A (some of the  $a_k$  may equal zero) such that, for each k, the relative algebra  $B \mid a_k = \{x \in B \mid x \leqslant a_k\}$  is generated over  $A \mid a_k$  by k disjoint elements  $v_1, \dots, v_k$  which are independent from  $A \mid a_k$ : for  $1 \leqslant r, l \leqslant n$ , the set of those  $p \in X$  such that  $u_r(p) = u_l(p)$  is clopen. Hence, for  $1 \leqslant k \leqslant n$ ,  $c_k = \{p \in X \mid B_p \text{ has exactly } k \text{ atoms}\}$  is

clopen; put  $a_k = e(c_k)$ . By a compactness argument, construct  $v_1, ..., v_k \in B \upharpoonright a_k$  by patching together some of the  $u_r$  such that for  $p \in c_k$ , the atoms of  $B_p$  are  $v_1(p), ..., v_k(p)$ .

2.5. Example. Suppose  $a \cdot u_r \neq 0$  for  $1 \leqslant r \leqslant n$  and  $a \in A \setminus \{0\}$ . Then  $\operatorname{Aut}_A B \cong S_n[A]$ .

*Proof.* Our assumption says that  $u_r(p) \neq 0$  for each r and each  $p \in X$ . Hence each  $\pi \in S_n$  is compatible with each  $p \in X$  and, for fixed  $g \in G$ , the open sets  $v_{\pi}$  in the proof of 2.4 are disjoint, hence  $c_{\pi} = v_{\pi}$ . An isomorphism  $\varphi : G \to S_n[A]$  is established by defining  $\varphi(g)(p) = \pi$  iff  $p \in v_{\pi}$ .

2.6. EXAMPLE. Suppose A is relatively complete in B. Then there is a partition  $(a_k \mid 1 \leqslant k \leqslant n)$  of A such that

$$\operatorname{Aut}_A B \cong S_1 [A \upharpoonright a_1] \times ... \times S_n [A \upharpoonright a_n].$$

*Proof.* Choose, for  $1 \le k \le n$ ,  $a_k \in A$  as indicated above and let  $G_k$  be the Galois group of  $B \upharpoonright a_k$  over  $A \upharpoonright a_k$ . Clearly,

$$\operatorname{Aut}_A B \cong G_1 \times ... \times G_n,$$

since  $a_k \in A$ . By 2.5,  $G_k \cong S_k [A \mid a_k]$ .

- 2.7. Proposition. The following conditions on (B, A) are equivalent:
- a) A is relatively complete in B;
- b) there is some  $g \in G$  such that  $g(b) \neq b$  for  $b \in B \setminus A$ ;
- c) there is some finite subgroup H of G such that, for every  $b \in B \setminus A$ , there is some  $g \in H$  satisfying  $g(b) \neq b$ .

*Proof.* Assume a). There is a finite partition T of C such that, for  $1 \le r$   $\le n$ ,  $t \in T$  and  $p, q \in t$ ,  $u_r(p) = 0$  iff  $u_r(q) = 0$ . For  $t \in T$ , let  $\pi_t \in S_n$  such that, for  $p \in t$ ,  $\pi_t(r) = r$  if  $u_r(p) = 0$  and  $u_r(p) \mapsto u_{\pi_t(r)}(p)$  is a cyclic permutation of the atoms of  $B_p$  which moves all these atoms.  $\pi_t$  is compatible with each  $p \in t$ ; hence there is some  $g \in G$  such that g is induced by  $\pi_t$  for  $p \in t$ ,  $t \in T$ . Now let  $b \in B \setminus A$ . Choose  $p \in X$ , e.g.  $p \in t$  where  $t \in T$ , such that  $b(p) \notin \{0, 1\}$ ; put b' = g(b). Let  $At(B_p)$  be the set of atoms of  $B_p$ ,  $M = \{\alpha \in At(B_p) \mid \alpha \le b(p)\}$ ,  $g_p$  the automorphism of  $B_p$  induced by g,  $M' = \{g_p(\alpha) \mid \alpha \in M\}$ . By the choice of  $\pi_t$  and g,

$$b'(p) = g_p(b(p)) = \sum M' \neq \sum M = b(p)$$

which proves  $b' \neq b - \text{since}$ , if  $\pi$  is a cyclic permutation of a finite set Y moving every element of Y and  $M \subseteq Y$  satisfies  $M = \{\pi(m) \mid m \in M\}$ , then  $M = \phi$  or M = Y.

To prove that b) implies c) it is sufficient to know that every finitely generated subgroup of G is finite. We indicate a construction for finite subgroups of G. Let  $T \subseteq C$  be a finite partition of C. A function  $\varphi: T \to S_n$  is said to be compatible if, for every  $t \in T$  and  $p \in t$ ,  $\varphi(t)$  is compatible with p. For each compatible  $\varphi: T \to S_n$  let  $g_{\varphi}$  be the element of G mapping  $u_r$  to  $\sum \{e(t) \cdot u_{\varphi(t)}(r) \mid t \in T\}$ . It is easily seen that

$$G_T = \{ g_{\varphi} \mid \varphi : T \to S_n \text{ compatible} \}$$

is a finite subgroup of G and that every finite subset of G is contained in some  $G_T$ .

Now suppose c), i.e. there is some finite subgroup H of G moving every  $b \in B \setminus A$ . We may assume that  $H = G_T$  for some finite partition T of C. Assume that A is not relatively complete in B. By 2.2 there is some r such that  $I_r$  is not a principal ideal; w.l.o.g., r = 1. Let  $\sigma = \{p \in X \mid u_1(p) = 0\}$ .  $\sigma$  is a subset of X which is open but not closed; choose  $p \in X$  which lies in the closure of  $\sigma$  but not in  $\sigma$ . W.l.o.g., for some k satisfying  $1 \leq k \gg n$ ,

$$\{r \mid 1 \leqslant r \leqslant n \text{ and } u_r \sim u_1 \text{ at } p\} = \{1, ..., k\}.$$

Let c be a clopen neighbourhood of p such that, for  $1 \le r \le k$  and  $q \in c$ ,  $u_r(q) = 0$  iff  $u_1(q) = 0$ . W.l.o.g.,  $c \in T$ . There is some l such that  $k < l \le n$  and  $u_l(p) \ne 0$ ; otherwise, let  $c' \subseteq c$  a neighbourhood of p such that  $u_l(q) = 0$  for  $q \in c'$  and  $k < l \le n$ . Choose  $q \in c' \cap \sigma$  (since p lies in the closure of  $\sigma$ ). In  $B_q$ , which has at least two elements,  $1 = u_1(q) + \dots + u_n(q) = 0 + \dots + 0 = 0$ , a contradiction. — Put a = e(c) and  $b = a \cdot u_1 + \dots + a \cdot u_k$ .  $b \in B \setminus A$ , since  $0 < b(p) = u_1(p) + \dots + u_k(p) < 1$  by our preceding claim. We prove that, for  $g \in H = G_T$ , g(b) = b, thus arriving at a final contradiction: there is some compatible  $\varphi: T \to S_n$  such that  $g = g_{\varphi}$ . Consider  $k \le n$ ,  $c \in T$  and  $p \in c$  as constructed above. Since  $\varphi$  is compatible,  $\pi = \varphi(c)$  is compatible with p; hence  $\pi$  maps the set  $\{1, \dots, k\}$  into itself,  $g_{\varphi}(a \cdot u_r) = a \cdot u_{\pi(r)}$  for  $1 \le r \le k$  (where a = e(c)) and g(b) = b.