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1. The sheaf representation of Boolean algebra extensions

Let \mathscr{L} be any language for first-order predicate logic. Suppose X is a non-empty set and for every $p \in X$ we have an \mathscr{L} -structure $\mathscr{B}_p = (B_p, ...)$; put $S = \bigcup_{p \in X} B_p$. Suppose $\varphi(x_1 ... x_n)$ is an \mathscr{L} -formula, $u \subseteq X$ and $f_1, ..., f_n : u \to S$ are such that $f_i(p) \in B_p$ for $1 \le i \le n$ and $p \in u$. Then let

$$\| \varphi [f_1 ... f_n] \| = \{ p \in u \mid \mathcal{B}_p \mid = \varphi [f_1 (p) ... f_n (p)] \}.$$

We may think of $\| \varphi [f_1 ... f_n] \| \subseteq X$ as being a (Boolean) truth value of $\varphi [f_1 ... f_n]$ in the power set of X.

A sheaf of \mathcal{L} -structures is a sequence

$$\mathcal{S} = (S, \pi, X, \mu)$$

such that a) S and X are topological spaces and $\pi: S \to X$ is a continuous open local homeomorphism from S onto X, b) μ is a function assigning to each $p \in X$ an \mathcal{L} -structure $\mathcal{B}_p = (B_p, ...)$ such that the B_p are pairwise disjoint, $S = \bigcup_{p \in X} B_p$ and $\pi(s) = p$ iff $s \in B_p$, c) for every open subset u of X and continuous $f_1, ..., f_n: u \to S$ satisfying $f_i(p) \in B_p$ for $p \in u$ and every atomic \mathcal{L} -formula $\varphi(x_1 ... x_n)$, $\|\varphi[f_1 ... f_n]\|$ is an open subset of u.

The \mathcal{L} -structure \mathcal{B}_p is called the stalk of \mathcal{L} over p. — Let, if \mathcal{L} is a sheaf of \mathcal{L} -structures, $\Gamma(\mathcal{L})$ be the set of all continuous functions $f: X \to S$ satisfying $f(p) \in \mathcal{B}_p$ for $p \in X$ (the set of "global sections" of \mathcal{L}). So $\Gamma(\mathcal{L})$ is, if non-empty, (the underlying set of) a substructure of the product structure $\prod_{p \in Y} \mathcal{B}_p$, hence an \mathcal{L} -structure.

For the rest of the paper, let $\mathcal{L} = \{+, \cdot, -, 0, 1, U\}$ where U is a unary predicate. We indicate how, for a given BA extension (B, A), B may be represented by Γ (\mathcal{S}) where \mathcal{S} is a sheaf of \mathcal{L} -structures over a Boolean space. We omit most of the proofs which are easy and entirely analoguous to well-known representation theorems for lattices over Boolean spaces. Let X be the Stone space of A, i.e. the set of all ultrafilters of A with the usual topology. For $p \in X$, let be the filter of B generated by p. Let $\pi_p: B \to B/ = B_p$ be the canonical epimorphism. So B_p is a BA with at least two elements. For $p, q \in X$ and $p \neq q$, B_p and B_q are disjoint. Let $S = \bigcup_{p \in X} B_p$ and $\pi: S \to X$ be defined as stated in b) above. Let, for $p \in X$, $\mu(p)$ be the \mathcal{L} -structure $(B_p, ..., \{0, 1\})$. For $u \subseteq X$ open and $b \in B$, let $M_{ub} = \{\pi_p(b) \mid p \in u\}$. The set of these M_{ub} constitutes a base

for a topology of S, and this makes $\mathscr{S} = (S, \pi, X, \mu)$ a sheaf of \mathscr{L} -structures. Furthermore, for $b \in B$, $\sigma_b : X \to S$ defined by $\sigma_b(p) = \pi_p(b)$ is a global section of \mathscr{S} and

$$\begin{array}{c}
\sigma: B \to \Gamma\left(\mathcal{S}\right) \\
b \mapsto \sigma_{b}
\end{array}$$

is an isomorphism from B onto Γ (\mathcal{S}). We shall now identify B with Γ (\mathcal{S}); so every $b \in B$ is a function from X to S. This identifies A with those $b \in B$ such that for every $p \in X$ b (p) = 0 or b (p) = 1, i.e. with those $b \in B$ satisfying ||U(b)|| = X. Let C be the BA of clopen subsets of X and e (c) the characteristic function of c for $c \in C$. Thus e is an isomorphism from C onto $A \subseteq B$.

In the rest of this section, we show that the property of being a Hausdorff sheaf for $\mathcal S$ is equivalent to a property of the extension (B,A) which reflects, in a way which is first-order expressible in $\mathcal S$, completeness of the embedding of A into B. Recall that, for a sheaf $\mathcal S$ over a Boolean space X, S is a T_2 -space iff, for any $f,g\in\Gamma(\mathcal S)$, $\|f=g\|$ is a clopen subset of X; $\mathcal S$ is then said to be a Hausdorff sheaf. Call A relatively complete in B if, for every $b\in B$, there is a largest element $a\in A$ such that $a\leqslant b$, equivalently: for $b\in B$, there is a largest $a\in A$ such that $a\cdot b=0$ or: for $b\in B$, there is a smallest $a\in A$ such that $b\leqslant a$.

1.1. Proposition. \mathcal{G} is a Hausdorff sheaf iff A is relatively complete in B.

Proof. Suppose \mathcal{S} is Hausdorff and $b \in B$. Let $d \in B$ such that d(p) = 0 for every $p \in X$, let $c = \|b = d\|$ and a = e(c). Then a is the largest element of A satisfying $a \cdot b = 0$.

Conversely, let A be relatively complete in B and suppose $f, g \in B$. Let a be the largest element of A such that $a \leqslant f \cdot g + -f \cdot -g$. Let $c \in C$ such that a = e(c). Then ||f = g|| = c is a clopen subset of X.

1.2. Remark. Let A be relatively complete in B. Then the inclusion map from A to B is a complete homomorphism.

Proof. Suppose M is a subset of A having a supremum a in A. We show that a is the supremum of M in B. Clearly, a is an upper bound for M in B. Suppose that b is another upper bound for M in B. Let $\alpha \in A$ be the largest element of A such that $\alpha \leqslant b$. For every $m \in M \subseteq A$, we have $m \leqslant b$, hence $m \leqslant \alpha$ and $a \leqslant \alpha \leqslant b$.

The following facts are trivial:

- 1.3. Remark. a) Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.
- b) Suppose A is relatively complete in B and B is complete. Then A is complete.

2. Relative automorphisms of finite extensions

We first give an internal description of a finite extension (B, A) where $B = A(u_1 \dots u_n)$ and $n \in \omega$. We shall always assume that u_1, \dots, u_n are the atoms of the subalgebra of B generated by u_1, \dots, u_n ; i.e. that they are non-zero, pairwise disjoint and $u_1 + \dots + u_n = 1$. Let $I_r = \{a \in A \mid a \cdot u_r = 0\}$ for $1 \le r \le n$. Clearly, each I_r is a proper ideal of A and $I_1 \cap \dots \cap I_n = \{0\}$. The family $(I_r \mid 1 \le r \le n)$ completely characterizes the extension (B, A):

2.1. Remark. Suppose $C = A(v_1 \dots v_n)$ is a finite extension of A where v_1, \dots, v_n are pairwise disjoint and $1 = v_1 + \dots + v_n$. Let $B = A(u_1 \dots u_n)$ be as above. There is an isomorphism g from B onto C satisfying g(a) = a for $a \in A$ and $g(u_r) = v_r$ iff, for each r, $\{a \in A \mid a \cdot v_r = 0\} = I_r$.

Proof. By Theorem 12.4 in [7].

2.2. Remark. A is relatively complete in $B = A(u_1 \dots u_n)$ iff, for each r, I_r is a principal ideal.

Proof. The only—if part follows by the definition of relative completeness. Now suppose $\alpha_r \in A$ generates I_r ; let $b \in B$ and $I = \{a \in A \mid a \cdot b = 0\}$. There are $a_1, ..., a_n \in A$ such that $b = a_1 \cdot u_1 + ... + a_n \cdot u_n$. It follows that I is the principal ideal generated by $\alpha = (-a_1 + \alpha_1) \cdot ... \cdot (-a_n + \alpha_n)$.

Conversely, given any family $(I_r \mid 1 \le r \le n)$ of proper ideals in A satisfying $I_1 \cap ... \cap I_n = \{0\}$, there is an extension $A(u_1 ... u_n)$ of A such that $I_r = \{a \in A \mid a \cdot u_r = 0\}$: let $D = A(x_1 ... x_n)$ be the free product of A and a finite BA with atoms $x_1, ..., x_n$. Let

$$K = \{i_1 \cdot x_1 + ... + i_n \cdot x_n \mid i_1 \in I_1, ..., i_n \in I_n\}.$$

K is an ideal of D; the canonical epimorphism π from D onto B = D/K is one- one on A, and for $a \in A$, $\pi(a) \cdot u_r = 0$ iff $a \in I_r$ where $u_r = \pi(x_r)$. Now identify A with the subalgebra $\pi(A)$ of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf $\mathcal{S} = (S, \pi, X, \mu)$ of Boolean algebras over a