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We have thereby shown that $\mathscr{F}/D$ is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in $\mathscr{F}/D$, and hence independent of the Peano axioms.

Proposition 1 is also false in $\mathscr{F}/D$. In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let* $P_i : [\mathbf{N}]^{e_i} \to r_i$, $1 \leqslant i \leqslant n$, *be* n *partitions. There is a partition* $P : [\mathbf{N}]^e \to r$ *such that for all subsets* H *of* $\mathbf{N}$ *of cardinality* $> e$, H *is homogeneous for* P *if and only if* H *is homogeneous for all the* P_i.

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers* e, r, *and* k *there exists an* N, *such that for all partitions* $P : [N]^e \to r$ *there exists a subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant 2^{2^{\min X}}$, *which is homogeneous for* P.

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geqslant 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathbf{N}]^e \to r$ is a partition and k is a number then by considering the partition $P \mid [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \mid [N]^e$ and hence for P. This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geqslant 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{ h_i \}$ of functions which define the model $\mathscr{F}/D$. In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem wich is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let* $P : [\mathbf{N}]^e \to r$ *be a primitive recursive partition. For every* k *there exists a finite subset* X *of* $\mathbf{N}$, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is homogeneous for the partition* P.

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].

LEMMA 3. *Let* $P : [\mathbf{N}]^e \to r\ (e \geqslant 2)$ *be a partition. There is a partition* $P^* : [\mathbf{N}]^e \to r^*$ *(where* r^* *depends only on* m, e, *and* r*) such that if* X^* *is a finite subset of* $\mathbf{N}$, *homogeneous for* P^* *with* $\# X^* \geqslant e+1$ *and* $\# X \geqslant \min X$, *then the set* $X = [\log_2 \log_2](X^*)$ *is homogeneous for* P, *and*

$$\# X \geqslant e + 1 \quad \text{and} \quad \# X \geqslant 2^{2^{\min X}} .$$

Moreover, if P *is a primitive recursive partition, then* P^* *can be chosen to be primitive recursive.*[1])

Since this proof that Proposition 4 implies Proposition 1 may be carried out in Peano arithmetic, it follows that Proposition 4 is also false in our model $\mathscr{F}/D$. However, our aim here is not merely to give a simple independent statement but to construct a simpler model for Peano arithmetic. Once again we actually use a version of the combinatorial principle which applies to several partitions. The following result is implied by Proposition 4 in Peano arithmetic.

PROPOSIIION 5. *Let* $P_i : [\mathbf{N}]^{e_i} \to r_i$, $1 \leqslant i \leqslant n$, *be a set of primitive recursive partitions. For every* k *there exists a finite subset of* $\mathbf{N}$, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is simultaneously homogeneous for all the partitions* $P_1, \ldots, P_n$.

We now construct a non-standard model via Proposition 5. Let $\{ P_i \}$ again be an effective enumeration of all the primitive recursive partitions $P_i : [\mathbf{N}]^{e_i} \to r_i$. Let $c_{k1}, \ldots, c_{kn_k}$ be an increasing sequence with c_{kn_k} the least number such that $c_{k1}, \ldots, c_{kn_k}$ is homogeneous for all $P_1, \ldots, P_k$, with $c_{k1} \leqslant n_k$ and $k \leqslant n_k$. Define the functions g_j by

$$g_0(k) = n_k \qquad \text{for every } k$$

and for $j > 0$

$$g_j(k) = \begin{cases} c_{kj} & \text{for } j \leqslant n_k \\ g_{j-1}(k)^2 & \text{for } j > n_k . \end{cases}$$

Let $\mathscr{F} = \{ f \mid \exists j f \leqslant g_j \}$.

We shall show that $\mathscr{F}/D$ is a model of Peano arithmetic by proving that there is an increasing sequence $\{ h_j \}$ which lies in and is cofinal with $\mathscr{F}$ and which satisfies the Stability and Closure Conditions. We set

$$h_j = [\log_2 \log_2 g_j] .$$

[1]) Here, as is customary, $[x]$ is the greatest integer $\leqslant x$.

Since $h_j < g_j$, $h_j \in \mathscr{F}$. It follows from Lemma 2.13 of [3] that there is a primitive recursive partition R such that if X is homogeneous for R, with $\# X \geqslant \min X$ and $\# X \geqslant 3$, then for every $x, y, \in X$, $x < y$ implies $2^{2^x} < y$. Since this partition appears in the enumeration $\{ P_i \}$ at some point k, it follows that, for all $i \geqslant k$ and $j < n_i$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. Thus, if for a given j we choose an $m \geqslant k$ such that $n_m \geqslant j$, then, for all $i \geqslant m$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. For every $i < m$ choose an s_i with $2^{2^{g_j(i)}} < g_{s_i}(i)$. Let

$$s = \max (s_1, \ldots, s_{m-1}, j+1)$$

Then

$$2^{2^{g_i}} < g_s .$$

Thus $h_s = [\log_2 \log_2 g_s] > g_j$, proving that $\{ h_i \}$ is cofinal in $\mathscr{F}$.

For each partition P_k in the sequence $\{ P_i \}$ there exists another partition $P_t \, (= P_k^*)$ satisfying the conditions of Lemma 3. By the definition of the functions g_j, the set $\{ g_1(t), \ldots, g_{n_t}(t) \}$ is homogeneous for P_t and $n_t \geqslant t$, $n_t \geqslant g_1(t)$. Hence, by Lemma 3, the set

$$\{ h_1(t), \ldots, h_{n_t}(t) \} = \{ [\log_2 \log_2 g_1(t)], \ldots, [\log_2 \log_2 g_{n_t}(t)] \}$$

is homogeneous for P_k and $n_t \geqslant 2^{2^{h_1(t)}}$. Thus, as in the previous section, the sequence $\{ h_j \}$ fulfills the conditions which ensure the satisfaction of the Stability and Closure Conditions. This proves that $\mathscr{F}/D$ is a model of the Peano axioms. Once again, since c_{kn_k} was chosen as minimal, it follows that Proposition 5, and hence Proposition 4, is false in $\mathscr{F}/D$, and therefore independent of Peano arithmetic.

As before we may formulate a finite version of this combinatorial principle.

PROPOSITION 6. *For every e, r, and k there exists an N such that for every partition $P : [N]^e \to r$ there exists a subset X of N, with $\# X \geqslant k$ and $\# X \geqslant \min X$, which is homogeneous for P.*

Again it is provable in Peano arithmetic that Proposition 6 implies Proposition 4, so that Proposition 6 is false in our model. Proposition 6 was first proved independent of Peano arithmetic in [3] by showing that it implies the consistency of Peano arithmetic and then applying Gödel's Theorem.

Let $C_k = \{ i \mid i \leqslant c_{kn_k} \}$. The model $\mathscr{F}/D$ is an initial segment not only of the ultrapower $\mathbf{N}^I/D$ but also of the smaller ultraproduct $\prod_{k \in \mathbf{N}} C_k/D$.

This indicates that the function C given by $C(k) = c_{kn_k}$ is a very rapidly growing function. In fact the function C majorizes every recursive function which is a provably total function in Peano arithmetic.

THEOREM 5. *Let f be a recursive function. Let ψ be an elementary statement expressing the condition that f is a total function. If ψ is provable in Peano arithmetic, $f(k) < C(k)$ for all sufficiently large k.*

Proof. Suppose $t = \{ k \mid f(k) \geqslant C(k) \}$ is infinite. Let D be a non-principal ultrafilter such that $t \in D$. Then $f^* \geqslant C^*$. On the other hand, $f^* = f(\mathbf{1}^*) \in \mathscr{F}/D$, so that $f^* < C^*$, a contradiction.

It follows a fortiori that if N is the smallest integer to satisfy Theorem 5 then this function N also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each $e \geqslant 2$, there exists a primitive recursive partition: $P : [\mathrm{N}]^e \to 2$ such that P has no infinite homogeneous set in $\sum_e^0$ (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let $\phi(y)$ be any formula. As in Section III, the limited associate $\hat{\phi}(y; z)$ of $\phi(y)$ defines a partition $P : [\mathrm{N}]^e \to 2$ such that every sequence $\{ b_i \}$ of natural numbers homogeneous for P satisfies the Stability Condition for $\hat{\phi}(y; z)$ in N. Hence, for any vector a in N $\phi(a)$ holds in N if and only if $\hat{\phi}(a; b)$ does. It follows that the set $\{ a \mid \mathrm{N} \models \phi(a) \}$ is recursive in the set $\{ b_i \}$. Thus the set $\{ b_i \}$ is not in $\sum_e^0$.

VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if $\mathscr{F}$ is closed under $<$ and contains $\mathbf{1}$, then $\mathscr{F}/D$ is non-denumerable, for every non-principal ultrafilter D. Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function g is primitive recursive in f. It follows that we may define $\mathscr{F} = \{ f \mid \exists j f \leqslant h_j$ and f is primitive recursive