

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 28 (1982)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: NON-STANDARD MODELS OF PEANO ARITHMETIC
Autor: Kochen, Simon / Kripke, Saul
Kapitel: VI. A Simpler Model
DOI: <https://doi.org/10.5169/seals-52238>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

We have thereby shown that \mathcal{F}/D is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in \mathcal{F}/D , and hence independent of the Peano axioms.

Proposition 1 is also false in \mathcal{F}/D . In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let $P_i : [\mathbb{N}]^e \rightarrow r_i, 1 \leq i \leq n$, be n partitions. There is a partition $P : [\mathbb{N}]^e \rightarrow r$ such that for all subsets H of \mathbb{N} of cardinality $> e$, H is homogeneous for P if and only if H is homogeneous for all the P_i .*

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers e, r , and k there exists an N , such that for all partitions $P : [N]^e \rightarrow r$ there exists a subset X of N , with $\# X \geq k$ and $\# X \geq 2^{2^{\min X}}$, which is homogeneous for P .*

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geq 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathbb{N}]^e \rightarrow r$ is a partition and k is a number then by considering the partition $P \upharpoonright [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \upharpoonright [N]^e$ and hence for P . This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geq 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{h_i\}$ of functions which define the model \mathcal{F}/D . In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem which is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let $P : [\mathbb{N}]^e \rightarrow r$ be a primitive recursive partition. For every k there exists a finite subset X of \mathbb{N} , with $\# X \geq k$ and $\# X \geq \min X$, which is homogeneous for the partition P .*

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].

LEMMA 3. Let $P : [\mathbf{N}]^e \rightarrow r$ ($e \geq 2$) be a partition. There is a partition $P^* : [\mathbf{N}]^e \rightarrow r^*$ (where r^* depends only on $m, e,$ and r) such that if X^* is a finite subset of \mathbf{N} , homogeneous for P^* with $\# X^* \geq e+1$ and $\# X \geq \min X$, then the set $X = [\log_2 \log_2](X^*)$ is homogeneous for P , and

$$\# X \geq e + 1 \quad \text{and} \quad \# X \geq 2^{2^{\min X}}.$$

Moreover, if P is a primitive recursive partition, then P^* can be chosen to be primitive recursive.¹⁾

Since this proof that Proposition 4 implies Proposition 1 may be carried out in Peano arithmetic, it follows that Proposition 4 is also false in our model \mathcal{F}/D . However, our aim here is not merely to give a simple independent statement but to construct a simpler model for Peano arithmetic. Once again we actually use a version of the combinatorial principle which applies to several partitions. The following result is implied by Proposition 4 in Peano arithmetic.

PROPOSITION 5. Let $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$, $1 \leq i \leq n$, be a set of primitive recursive partitions. For every k there exists a finite subset of \mathbf{N} , with $\# X \geq k$ and $\# X \geq \min X$, which is simultaneously homogeneous for all the partitions P_1, \dots, P_n .

We now construct a non-standard model via Proposition 5. Let $\{P_i\}$ again be an effective enumeration of all the primitive recursive partitions $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$. Let c_{k1}, \dots, c_{kn_k} be an increasing sequence with c_{kn_k} the least number such that c_{k1}, \dots, c_{kn_k} is homogeneous for all P_1, \dots, P_k , with $c_{k1} \leq n_k$ and $k \leq n_k$. Define the functions g_j by

$$g_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$g_j(k) = \begin{cases} c_{kj} & \text{for } j \leq n_k \\ g_{j-1}(k)^2 & \text{for } j > n_k. \end{cases}$$

Let $\mathcal{F} = \{f \mid \exists j f \leq g_j\}$.

We shall show that \mathcal{F}/D is a model of Peano arithmetic by proving that there is an increasing sequence $\{h_j\}$ which lies in and is cofinal with \mathcal{F} and which satisfies the Stability and Closure Conditions. We set

$$h_j = [\log_2 \log_2 g_j].$$

¹⁾ Here, as is customary, $[x]$ is the greatest integer $\leq x$.

Since $h_j < g_j$, $h_j \in \mathcal{F}$. It follows from Lemma 2.13 of [3] that there is a primitive recursive partition R such that if X is homogeneous for R , with $\# X \geq \min X$ and $\# X \geq 3$, then for every $x, y, \in X$, $x < y$ implies $2^{2^x} < y$. Since this partition appears in the enumeration $\{P_i\}$ at some point k , it follows that, for all $i \geq k$ and $j < n_i$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. Thus, if for a given j we choose an $m \geq k$ such that $n_m \geq j$, then, for all $i \geq m$, $2^{2^{g_j(i)}} < g_{j+1}(i)$. For every $i < m$ choose an s_i with $2^{2^{g_j(i)}} < g_{s_i}(i)$. Let

$$s = \max (s_1, \dots, s_{m-1}, j + 1)$$

Then

$$2^{2^{g_i}} < g_s .$$

Thus $h_s = [\log_2 \log_2 g_s] > g_j$, proving that $\{h_i\}$ is cofinal in \mathcal{F} .

For each partition P_k in the sequence $\{P_i\}$ there exists another partition $P_t (= P_k^*)$ satisfying the conditions of Lemma 3. By the definition of the functions g_j , the set $\{g_1(t), \dots, g_{n_t}(t)\}$ is homogeneous for P_t and $n_t \geq t$, $n_t \geq g_1(t)$. Hence, by Lemma 3, the set

$$\{h_1(t), \dots, h_{n_t}(t)\} = \{[\log_2 \log_2 g_1(t)], \dots, [\log_2 \log_2 g_{n_t}(t)]\}$$

is homogeneous for P_k and $n_t \geq 2^{2^{h_1(t)}}$. Thus, as in the previous section, the sequence $\{h_j\}$ fulfills the conditions which ensure the satisfaction of the Stability and Closure Conditions. This proves that \mathcal{F}/D is a model of the Peano axioms. Once again, since c_{kn_k} was chosen as minimal, it follows that Proposition 5, and hence Proposition 4, is false in \mathcal{F}/D , and therefore independent of Peano arithmetic.

As before we may formulate a finite version of this combinatorial principle.

PROPOSITION 6. *For every e, r , and k there exists an N such that for every partition $P : [N]^e \rightarrow r$ there exists a subset X of N , with $\# X \geq k$ and $\# X \geq \min X$, which is homogeneous for P .*

Again it is provable in Peano arithmetic that Proposition 6 implies Proposition 4, so that Proposition 6 is false in our model. Proposition 6 was first proved independent of Peano arithmetic in [3] by showing that it implies the consistency of Peano arithmetic and then applying Gödel's Theorem.

Let $C_k = \{i \mid i \leq c_{kn_k}\}$. The model \mathcal{F}/D is an initial segment not only of the ultrapower \mathbf{N}^I/D but also of the smaller ultraproduct $\prod_{k \in \mathbf{N}} C_k/D$.

This indicates that the function C given by $C(k) = c_{knk}$ is a very rapidly growing function. In fact the function C majorizes every recursive function which is a provably total function in Peano arithmetic.

THEOREM 5. *Let f be a recursive function. Let ψ be an elementary statement expressing the condition that f is a total function. If ψ is provable in Peano arithmetic, $f(k) < C(k)$ for all sufficiently large k .*

Proof. Suppose $t = \{k \mid f(k) \geq C(k)\}$ is infinite. Let D be a non-principal ultrafilter such that $t \in D$. Then $f^* \geq C^*$. On the other hand, $f^* = f(\mathbf{1}^*) \in \mathcal{F}/D$, so that $f^* < C^*$, a contradiction.

It follows a fortiori that if N is the smallest integer to satisfy Theorem 5 then this function N also majorizes every provably total recursive function (c.f. Theorem 3.2 in [3]).

We mentioned in the introduction that a by-product of our construction is a new proof of Specker's theorem that there exists a recursive partition with no recursively enumerable infinite homogeneous set. In fact we may obtain the stronger theorem that for each $e \geq 2$, there exists a primitive recursive partition: $P : [\mathbb{N}]^e \rightarrow 2$ such that P has no infinite homogeneous set in \sum_e^0 (c.f. Jockusch [10], Theorem 5.1). We outline the proof of this result. Let $\phi(y)$ be any formula. As in Section III, the limited associate $\hat{\phi}(y; z)$ of $\phi(y)$ defines a partition $P : [\mathbb{N}]^e \rightarrow 2$ such that every sequence $\{b_i\}$ of natural numbers homogeneous for P satisfies the Stability Condition for $\hat{\phi}(y; z)$ in \mathbb{N} . Hence, for any vector a in \mathbb{N} $\phi(a)$ holds in \mathbb{N} if and only if $\hat{\phi}(a; b)$ does. It follows that the set $\{a \mid \mathbb{N} \models \phi(a)\}$ is recursive in the set $\{b_i\}$. Thus the set $\{b_i\}$ is not in \sum_e^0 .

VII. VARIATIONS

We conclude with a series of remarks on various modifications of our construction.

(a) It is easily proved that if \mathcal{F} is closed under $<$ and contains $\mathbf{1}$, then \mathcal{F}/D is non-denumerable, for every non-principal ultrafilter D . Thus, this construction leads only to non-denumerable models. However, a slight variation of the basic construction yields denumerable models. Note that in the proof of Theorem 1 the function g is primitive recursive in f . It follows that we may define $\mathcal{F} = \{f \mid \exists j f \leq h_j \text{ and } f \text{ is primitive recursive}$