

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 28 (1982)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** NON-STANDARD MODELS OF PEANO ARITHMETIC  
**Autor:** Kochen, Simon / Kripke, Saul  
**Kapitel:** V. Construction of the Model  
**DOI:** <https://doi.org/10.5169/seals-52238>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

Proposition 2 may be expressed by a  $\prod_2^0$  formula. First it is clear that we can construct a  $\sum_1^0$ -formula  $\phi_i$  that expresses the properties that

1.  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$  is a primitive recursive partition
2.  $z_1 < z_2 < \dots < z_{n_k}$
3.  $\{z_1, \dots, z_{n_k}\}$  is homogeneous for  $P_i$
4.  $k \leq n_k$
5.  $2^{2^{z_1}} \leq n_k$

Proposition 2 asserts that for every  $k$

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i .$$

### V. CONSTRUCTION OF THE MODEL

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let  $P_i$  be an effective enumeration of all primitive recursive partitions  $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ . By Proposition 2 we have that for every  $k$

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where  $\phi_i$  is the  $\sum_1^0$ -formula of Section IV expressing the conditions (1)-(5) satisfied by the partition  $P_i$ .

Following the prescription given in Section III we let  $a_{kn_k}$  be the smallest number such that  $a_{k_1}, \dots, a_{kn_k}$  is an increasing sequence satisfying the formula  $\bigwedge_{j \leq k} \phi_j$ . Now we define the functions  $h_j$  by

$$h_0(k) = n_k \quad \text{for every } k$$

and for  $j > 0$

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leq n_k \\ h_{j-1}(k)^2 & \text{for } j > n_k . \end{cases}$$

Let  $\mathcal{F} = \{f \mid f \leq h_j\}$ .

Since  $1 \leq h_0$  the function 1 is automatically in  $\mathcal{F}$ .

By Theorem 2 the sequence  $\{h_j\}$  satisfies  $\bigwedge_{j < \infty} \phi_j$  in  $\mathcal{F}/D$ . We now prove that this implies that the sequence  $\{h_j\}$  satisfies the Stability and

Closure Conditions in  $\mathcal{F}/D$ . As we saw in Section III it suffices for this purpose to show that for each  $k$

$$\begin{aligned} \mathbf{N} \vdash \exists z_1 \dots \exists z_{n_k} \bigwedge_{\substack{1 \leq i < j, j' < n_k \\ 1 \leq s < k}} [(\forall y < z_i) (\psi_s(y; z_j) \\ \leftrightarrow \psi_s(y; z_{j'}) \wedge z_{j-1}^2 < z_j)] \end{aligned} \quad (*)$$

Let  $t_i$  be the length of the sequence  $y$  in  $\psi_i(y; z)$ . Define the partitions  $T: [\mathbf{N}]^2 \rightarrow 2$ ,  $Q_i: \mathbf{N} \rightarrow t_i^2 + 1$ , and  $S_i: [\mathbf{N}]^{2e+1} \rightarrow 2$  by:

$$T(a, b) = \begin{cases} 1 & \text{if } a^2 < b \\ 0 & \text{if not} \end{cases}$$

$$Q_i(a) = \min(a, [t_i \log_2 a] + 1)$$

and for  $a \in \mathbf{N}$ ,  $c, c' \in [\mathbf{N}]^e$

$$S_i(a, c, c') = \begin{cases} 1 & \text{if } (\forall y < a) (\psi_i(y; c) \leftrightarrow \psi_i(y; c')) \\ 0 & \text{if not.} \end{cases}$$

The partitions  $T$ ,  $Q_i$ , and  $S_i$  are clearly primitive recursive since  $\psi_i(y; z)$  is a limited formula. Hence  $T, Q_1, \dots, Q_k, S_1, \dots, S_k$  occur in the sequence  $\{P_i\}$ . Thus by looking sufficiently far in the sequence we can find a set  $X_k = \{a_{k1}, \dots, a_{kn_k}\}$  which is homogeneous for  $T, Q_1, \dots, Q_k, S_1, \dots, S_k$  with  $n_k \geq k, 2^{2^{a_{k1}}}$ .

Since  $a_{kn_k} > a_{k1}^2$ ,  $T(a_{k1}, a_{kn_k}) = 1$ . Hence, by homogeneity,

$$T(a, b) = 1, \quad \text{i.e. } a^2 < b,$$

for all  $a < b$  in  $X_k$ .

Since  $\#X > 1$ , and  $X_k$  is homogeneous for  $Q_i$ ,  $a_{k1} \geq t_i \log_2 a_{k1}$ .

The number of sequences of numbers  $< a_{k1}$  of length  $t_i$  is  $< a_{k1}^{t_i}$ . The number of distinct sequences of truth values of length  $a_{k1}^{t_i}$  is  $< 2^{a_{k1}^{t_i}}$ . Now  $n_k > 2^{2^{a_{k1}}} > 2^{a_{k1}^{t_i}}$  since  $a_{k1} \geq t_i \log_2 a_{k1}$ . Thus there are distinct  $c, c' > a$  in  $X_k$  such that

$$(\forall y < a_{k1}) (\psi_i(y; c) \leftrightarrow \psi_i(y; c')),$$

i.e.  $S_i(a_{k1}, c, c') = 1$ .

By homogeneity

$$S_i(a, b, b') = 1 \quad \text{for all } a < b, b' \text{ in } X_k$$

proving (\*).

We have thereby shown that  $\mathcal{F}/D$  is a model of the Peano axioms. Since  $a_{kn_k}$  was chosen minimal, Proposition 2 is false in  $\mathcal{F}/D$ , and hence independent of the Peano axioms.

Proposition 1 is also false in  $\mathcal{F}/D$ . In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let  $P_i : [\mathbb{N}]^e \rightarrow r_i, 1 \leq i \leq n$ , be  $n$  partitions. There is a partition  $P : [\mathbb{N}]^e \rightarrow r$  such that for all subsets  $H$  of  $\mathbb{N}$  of cardinality  $> e$ ,  $H$  is homogeneous for  $P$  if and only if  $H$  is homogeneous for all the  $P_i$ .*

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers  $e, r$ , and  $k$  there exists an  $N$ , such that for all partitions  $P : [N]^e \rightarrow r$  there exists a subset  $X$  of  $N$ , with  $\# X \geq k$  and  $\# X \geq 2^{2^{\min X}}$ , which is homogeneous for  $P$ .*

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that  $\# X \geq 2^{2^{\min X}}$ , then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if  $P : [\mathbb{N}]^e \rightarrow r$  is a partition and  $k$  is a number then by considering the partition  $P \upharpoonright [N]^e$ , where  $N$  is the number provided by Proposition 3 we obtain the required homogeneous set  $X$  for  $P \upharpoonright [N]^e$  and hence for  $P$ . This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

## VI. A SIMPLER MODEL

The condition in Proposition 1 that  $\# X \geq 2^{2^{\min X}}$  can be simplified and so yield a simpler sequence  $\{h_i\}$  of functions which define the model  $\mathcal{F}/D$ . In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem which is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let  $P : [\mathbb{N}]^e \rightarrow r$  be a primitive recursive partition. For every  $k$  there exists a finite subset  $X$  of  $\mathbb{N}$ , with  $\# X \geq k$  and  $\# X \geq \min X$ , which is homogeneous for the partition  $P$ .*

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].