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Proposition 2 may be expressed by a $\prod_2^0$ formula. First it is clear that we can construct a $\sum_1^0$-formula ϕ_i that expresses the properties that

1. $P_i : [\mathrm{N}]^{e\, i} \to r_i$ is a primitive recursive partition
2. $z_1 < z_2 < \dots < z_{n_k}$
3. $\{ z_1, \dots, z_{n_k} \}$ is homogeneous for P_i
4. $k \leqslant n_k$
5. $2^{2^{z_1}} \leqslant n_k$

Proposition 2 asserts that for every k

$$\mathbf{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i \, .$$

V. Construction of the Model

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let P_i be an effective enumeration of all primitive recursive partitions $P_i : [\mathrm{N}]^{e\, i} \to r_i$. By Proposition 2 we have that for every k

$$\mathbf{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where ϕ_i is the $\sum_1^0$-formula of Section IV expressing the conditions (1)-(5) satisfied by the partition P_i.

Following the prescription given in Section III we let a_{kn_k} be the smallest number such that $a_{k1}, \dots, a_{kn_k}$ is an increasing sequence satisfying the formula $\bigwedge_{j \leq k} \phi_j$. Now we define the functions h_j by

$$h_0 (k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$h_j (k) = \begin{cases} a_{kj} & \text{for } j \leqslant n_k \\ h_{j-1} (k)^2 & \text{for } j > n_k \, . \end{cases}$$

Let $\mathscr{F} = \{ f \mid f \leqslant h_j \}$.

Since $\mathbf{1} \leqslant h_o$ the function $\mathbf{1}$ is automatically in $\mathscr{F}$.

By Theorem 2 the sequence $\{ h_j \}$ satisfies $\bigwedge_{j < \infty} \phi_j$ in $\mathscr{F}/D$. We now prove that this implies that the sequence $\{ h_j \}$ satisfies the Stability and

Closure Conditions in $\mathscr{F}/D$. As we saw in Section III it suffices for this purpose to show that for each k

$$\mathbf{N} \vdash \exists z_1 \ldots \exists z_{n_k} \bigwedge_{\substack{1 \leqq i < j, j' < n_k \\ 1 \leqq s < k}} [(\forall y < z_i)(\psi_s(y; z_j) \leftrightarrow \psi_s(y; z_{j'}) \wedge z_{j-1}^2 < z_j] \qquad (*)$$

Let t_i be the length of the sequence y in $\psi_i(y; z)$. Define the partitions $T : [\mathbf{N}]^2 \to 2$, $Q_i : \mathbf{N} \to t_i^2 + 1$, and $S_i : [\mathbf{N}]^{2e+1} \to 2$ by :

$$T(a, b) = \begin{cases} 1 & \text{if } a^2 < b \\ 0 & \text{if not} \end{cases}$$

$$Q_i(a) = \min(a, [t_i \log_2 a] + 1)$$

and for $a \in \mathbf{N}$, $c, c' \in [\mathbf{N}]^e$

$$S_i(a, c, c') = \begin{cases} 1 & \text{if } (\forall y < a)(\psi_i(y; c) \leftrightarrow \psi_i(y; c')) \\ 0 & \text{if not.} \end{cases}$$

The partitions T, Q_i, and S_i are clearly primitive recursive since $\psi_i(y; z)$ is a limited formula. Hence T, $Q_1, \ldots, Q_k$, $S_1, \ldots, S_k$ occur in the sequence $\{P_i\}$. Thus by looking sufficiently far in the sequence we can find a set $X_k = \{a_{k1}, \ldots, a_{kn_k}\}$ which is homogeneous for T, $Q_1, \ldots, Q_k$, $S_1, \ldots, S_k$ with $n_k \geqslant k, 2^{2^{a_{k1}}}$.

Since $a_{kn_k} > a_{k1}^2$, $T(a_{k1}, a_{kn_k}) = 1$. Hence, by homogeneity,

$$T(a, b) = 1, \quad \text{i.e.} \quad a^2 < b,$$

for all $a < b$ in X_k.

Since $\# X > 1$, and X_k is homogeneous for Q_i, $a_{k1} \geqslant t_i \log_2 a_{k1}$.

The number of sequences of numbers $< a_{k1}$ of length t_i is $< a_{k1}^{t_i}$. The number of distinct sequences of truth values of length $a_{k1}^{t_i}$ is $< 2^{a_{k1}^{t_i}}$. Now $n_k > 2^{2^{a_{k1}}} > 2^{a_{k1}^{t_i}}$ since $a_{k1} \geqslant t_i \log_2 a_{k1}$. Thus there are distinct $c, c' > a$ in X_k such that

$$(\forall y < a_{k1}) \ (\psi_i(y; c) \leftrightarrow \psi_i(y; c')) ,$$

i.e. $S_i(a_{k1}, c, c') = 1$.

By homogeneity

$$S_i(a, b, b') = 1 \qquad \text{for } all \qquad a < b, b' \text{ in } X_k$$

proving (*).

We have thereby shown that $\mathscr{F}/D$ is a model of the Peano axioms. Since a_{kn_k} was chosen minimal, Proposition 2 is false in $\mathscr{F}/D$, and hence independent of the Peano axioms.

Proposition 1 is also false in $\mathscr{F}/D$. In fact it is provable in Peano arithmetic that Proposition 1 implies Proposition 2. This is a consequence of the following lemma, provable in Peano arithmetic (c.f. Lemma 2.9 in [3]).

LEMMA 2. *Let* $P_i : [\mathrm{N}]^{e_i} \to r_i$, $1 \leqslant i \leqslant n$, *be* n *partitions. There is a partition* $P : [\mathrm{N}]^e \to r$ *such that for all subsets* H *of* N *of cardinality* $> e$, H *is homogeneous for* P *if and only if* H *is homogeneous for all the* P_i.

We may also obtain a purely finitary combinatorial principle which is false in our model.

PROPOSITION 3. *For all natural numbers* e, r, *and* k *there exists an* N, *such that for all partitions* $P : [N]^e \to r$ *there exists a subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant 2^{2^{\min X}}$, *which is homogeneous for* P.

This result follows immediately from the infinite Ramsey Theorem by an application of König's Lemma. If we drop the condition that $\# X \geqslant 2^{2^{\min X}}$, then we obtain the usual finite Ramsey Theorem. Ramsey [11] gave a proof of the latter theorem which is formalizable in Peano arithmetic. Proposition 3 directly yields Proposition 1, for if $P : [\mathrm{N}]^e \to r$ is a partition and k is a number then by considering the partition $P \mid [N]^e$, where N is the number provided by Proposition 3 we obtain the required homogeneous set X for $P \mid [N]^e$ and hence for P. This proof may be carried out in Peano arithmetic. Thus, Proposition 3 is false in our model and independent of the Peano axioms.

VI. A SIMPLER MODEL

The condition in Proposition 1 that $\# X \geqslant 2^{2^{\min X}}$ can be simplified and so yield a simpler sequence $\{h_i\}$ of functions which define the model $\mathscr{F}/D$. In this section we describe such a model by using a combinatorial consequence of Ramsey's Theorem wich is closer to the proposition proved independent in [3].

PROPOSITION 4. *Let* $P : [\mathrm{N}]^e \to r$ *be a primitive recursive partition. For every* k *there exists a finite subset* X *of* N, *with* $\# X \geqslant k$ *and* $\# X \geqslant \min X$, *which is homogeneous for the partition* P.

Proposition 4 implies Proposition 1 via the following result, the proof of which is the same as the proof of Lemma 2.14 of [3].