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Autor: Karp, Richard M. / Lipton, Richard J.
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Since DAG is logspace complete in $NSPACE(\log n)$, it suffices to show that

$$\text{DAG} \in \text{DSPACE}(\log n)/\log \Rightarrow \text{DAG} \in \text{DSPACE}(\log n) .$$

Suppose that $\text{DAG} = S : h$, where $S \in \text{DSPACE}(\log n)$ and

$$|h(n)| \leq k \log_2 n .$$

Then, guided by the self-reducibility of DAG, we can test whether $(\Psi, s, t) \in \text{DAG}$ by performing the following computation for each string w of length $\leq k \log_2 n$:

```

 $v := s;$ 
while  $v$  has out-degree 2 do
 $v :=$  if  $w \cdot (\Psi, v_0, t) \in S$  then  $v_0$  else  $v_1$ .

```

If v is ever set equal to t then accept (Ψ, s, t) ; otherwise, reject it. It is clear that this method recognizes DAG deterministically within space $O(\log n)$. ■

6. THE METHOD OF RECURSIVE DEFINITION

Let K be a subset of $\{0, 1\}^*$, and let $C_K : \{0, 1\}^* \rightarrow \{0, 1\}$ be the characteristic function of K . By a recursive definition of C_K we mean a rule that specifies C_K on a “basis set” $A \subseteq \{0, 1\}^*$, and uniquely determines C_K on the rest of $\{0, 1\}^*$ by a recurrence formula of the form

$$\begin{aligned}
 C_K(x) &= F(x, C_K(f_1(x)), C_K(f_2(x)), \dots, C_K(f_t(x))), \\
 x &\in \{0, 1\}^* - A .
 \end{aligned}$$

Example 1. Let G be a game, as defined in Section 4, and let G be the set of positions from which the player to move can force a win. Then G is uniquely determined by

- (i) if $x \in W$ then $x \in G$
- (ii) if $x \in \{0, 1\}^* - W$ then $x \in G \Leftrightarrow F_0(x) \notin G$ or $F_1(x) \notin G$.

Example 2. Let (\langle, A, G_0, G_1) be a self-reducibility structure for the set $K \subseteq \{0, 1\}^*$. Then K is determined uniquely by its intersection with A , together with the recurrence

$$\text{for } x \notin A, x \in K \Leftrightarrow G_0(x) \in K \cup G_1(x) \in K .$$

The theme of the present section is that, when C_K has a simple enough recursive definition, bounds on the nonuniform complexity of K yield bounds on its uniform complexity. The idea is as follows. Suppose $K = S : h$, and C_K is determined by its values on A , together with the recurrence formula

$$C_K(x) = F(x, C_K(f_1(x)), \dots, C_K(f_t(x))), \quad x \in \{0, 1\}^* - A,$$

where

$$|f_1(x)| = |f_t(x)| = |x|.$$

For any string w , define $K_w = \{x \mid wx \in S\}$. Then, for $x \in A$, we can make the following assertion:

$$\begin{aligned} x \in K &\Leftrightarrow \exists w [x \in K_w] \wedge \forall y [C_{K_w}(y) \\ &= F(y, C_{K_w}(f_1(y)), \dots, C_{K_w}(f_t(y))]. \end{aligned}$$

Here, w ranges over all strings of length $|h(|x|)|$, and y ranges over all strings of the same length as x . The above formula suggests a uniform algorithm to test membership in K by searching through all choices of w and y . Further, the quantifier structure of the formula allows us to conclude that K lies in \sum_2^p , provided that $|h(n)|$ is bounded by a polynomial in n , S is in P , and F is computable in polynomial time.

As an illustration of this approach, we prove that, if NP has small circuits, then $\bigcup_i \sum_i^p = \sum_2^p$, i.e., the polynomial-time hierarchy collapses.

Originally we proved this with \sum_2^p replaced by \sum_3^p . The improvement is due to M. Sipser.

THEOREM 6.1. If $NP \subseteq P/poly$ then $\sum_2^p = \bigcup_{i=1}^{\infty} \sum_i^p$.

The proof of this theorem requires the following lemma.

LEMMA 6.2. If $NP \subseteq P/poly$ then $\bigcup_{i=1}^{\infty} \sum_i^p \subseteq P/poly$.

Proof. Let E_i be the set of encodings of true sentences of the form

$$(*) \quad Q_1 \vec{x}_1 Q_2 \vec{x}_2 \dots Q_i \vec{x}_i F(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_i)$$

where $Q_1 = \exists$, the Q_j are alternately \exists and \forall , \vec{x}_j is shorthand for the triple $x_{j_1}, x_{j_2}, \dots, x_{j_{r_j}}$ of Boolean variables, and F is a propositional formula. Let A_i be defined in the same way, except that $Q_1 = \forall$. It is known that E_i is logspace complete in \sum_i^p , and A_i is logspace complete in

\prod_i^p . Also, it is clear that $A_i \in P/poly \Leftrightarrow E_i \in P/poly$. It suffices for the lemma to prove that $E_i \in P/poly$ for all i .

By hypothesis, $E_1 \in P/poly$. We proceed by induction on i . Assume $E_{i-1} \in P/poly$; then $A_{i-1} \in P/poly$. Thus there exists a set $S \in P$, a constant k and a function $h: N \rightarrow \{0, 1\}^*$ such that $|h(n)| \leq k + n^k$ and $x \in A_{i-1} \Leftrightarrow h(|x|) \cdot x \in S$.

If y is the encoding of a sentence of the form (*), and \vec{a} is a t_1 -tuple of boolean variables, let $y_{\vec{a}}$ denote the encoding of the sentence that results from y by deleting the quantifier Q_1 and substituting \vec{a} for \vec{x}_i in $F(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_i)$. We choose our encoding conventions and method of substitution so that the length of $y_{\vec{a}}$ is equal to the length of y .

Since $S \in P$, the following set T is in NP :

$$T = \{wy \mid \text{for some } \vec{a}, w \cdot y_{\vec{a}} \in S\}.$$

By hypothesis $T \in P/poly$, so there exist $S' \in P$, $k' \in N$ and $h': N \rightarrow \{0, 1\}^*$ so that $|h'(n)| \leq k' + n^{k'}$ and $x \in T \Leftrightarrow h'(|x|) \cdot x \in S$. Then $y \in A_i \Leftrightarrow$ for some \vec{a} , $y_{\vec{a}} \in E_{i-1} \Leftrightarrow$ for some a ,

$$h(|y_{\vec{a}}|) \cdot y_{\vec{a}} \in S \Leftrightarrow h(|y_{\vec{a}}|) \cdot y \in T \Leftrightarrow h'(|h(|y_{\vec{a}}|) \cdot y|) \cdot h(|y_{\vec{a}}|) \cdot y \in S'.$$

But the prefix $h'(|h(|y_{\vec{a}}|) \cdot y|) \cdot h(|y_{\vec{a}}|)$ is a polynomial-bounded function of $|y|$; also $S' \in P$. These two facts together establish that $A_i \in P/poly$. ■

Proof of Theorem 6.1. It suffices to prove that $NP \subseteq P/poly \Rightarrow \prod_3^p \subseteq \sum_2^p$; for this it is sufficient to prove that the set A_3 is in \sum_2^p . Our proof is based on the fact that A_3 has an easy-to-evaluate recursive definition of the form $C_{A_3}(y) = R(y, C_{A_3}(y'), C_{A_3}(y''))$. Consider a sentence y of the form

$$Q_1 x_1 Q_2 x_2 \dots Q_n x_n F(x_1, x_2, \dots, x_n)$$

where the string of quantifiers $Q_1 Q_2 \dots Q_n$ is contained in $\forall^* \exists^* \forall^*$.

Let

$$y' = Q_2 x_2 \dots Q_n x_n F(0, x_2, \dots, x_n)$$

and

$$y'' = Q_2 x_2 \dots Q_n x_n F(1, x_2, \dots, x_n).$$

Then

$C_{A_3}(y) = (\text{if } Q_1 = \forall \text{ then } C_{A_3}(y') \wedge C_{A_3}(y'') \text{ else } C_{A_3}(y') \cup C_{A_3}(y''))$. C_{A_3} is uniquely determined by this recursive definition which is of the form $C_{A_3}(y) = R((y, C_{A_3}(y'), C_{A_3}(y''))$, coupled with its values on the “basis set” consisting of sentences without quantifiers.

By Lemma 6.2, $A_3 \in P/poly$. Thus $A_3 = S:h$ where $S \in P$ and $|h(n)| \leq k + n^k$. For each $w \in \{0, 1\}^*$ define $f_w: \{0, 1\}^* \rightarrow \{0, 1\}$ by $f_w(x) = 1 \Leftrightarrow wx \in S$. Then membership of y in A_3 , in the case where y contains at least one quantifier, is expressed by the following formula:

$$(**) \quad \exists w \forall z [f_w(y) = 1 \wedge f_w(z) = R(z, f_w(z'), f_w(z''))].$$

Here w ranges over all strings of length $\leq k + |y|^k$, and z ranges over all strings of length $|y|$. Also, with the help of a polynomial-time algorithm to test membership in S , the property $f_w(y) = 1$ and

$$f_w(z) = R(z, f_w(z'), f_w(z''))$$

can be tested in polynomial time. Thus the $\exists \forall$ form of $(**)$ establishes that $A_3 \in \sum_2^p$. ■

Theorem 6.1 has a number of corollaries.

COROLLARY 6.3. If $R = NP$ then $\bigcup_i \sum_i^p = \sum_2^p$.

This follows immediately from the observation [1] that every set in R has small circuits.

The next corollary concerns sparse sets. A set S is *sparse* [6, 7] if

$$\exists c \forall n \geq 2, |S \cap \{0, 1\}^n| \leq n^c.$$

COROLLARY 6.4. If there is a sparse set S that is complete in NP with respect to polynomial time Turing reducibility (cf. Cook [4]), then

$$\bigcup_i \sum_i^p = \sum_2^p.$$

This corollary follows immediately from Theorem 6.1 once it is noted that the existence of such an S implies that every set in NP has small circuits. Corollary 6.4 should be compared with results of Mahaney [11] and Fortune [6] which show that, if there exists a sparse or co-sparse set which is complete in NP with respect to many-one polynomial-time reducibility (Karp [8]) then $P = NP$. Note that Corollary 6.4 has a weaker conclusion than the results of Mahaney and Fortune, but also a weaker hypothesis.

Let $ZERO_S$ denote the following decision problem: given a prime q and a set $\{p_1(x), p_2(x), \dots, p_n(x)\}$ of sparse polynomials with integer coefficients, to determine whether there exists an integer x such that, for $i = 1, 2, \dots, n, p_i(x) \equiv 0 \pmod{q}$.

COROLLARY 6.5. If $ZERO\text{S} \in P/\text{poly}$, then $\cup \sum_2^p = \sum_2^p$.

This is based on Plaisted's result [15] that every problem in NP can be solved in polynomial time with the help of an oracle for $ZERO\text{S}$ together with a polynomial-bounded number of advice bits. Thus $NP \subseteq P/\text{poly}$ if $ZERO\text{S} \in P/\text{poly}$.

THEOREM 6.6. (Meyer) $EXPTIME \subseteq P/\text{poly} \Leftrightarrow EXPTIME = \sum_2^p$.

Proof. Let G be the set of strings representing positions from which the first player can win in the $EXPTIME$ -complete game mentioned in FACT 1. It suffices to prove that

$$G \in P/\text{poly} \Rightarrow G \in \sum_2^p.$$

Suppose $G = S : h$ where $S \in P$ and h is polynomial-bounded. Then

$$\begin{aligned} x \in G \Leftrightarrow \exists w \forall z [x \in W \cup z \in W \cup (wz \in S \Leftrightarrow wF_0(z) \\ \notin S \cup wF_1(z) \notin S)] \end{aligned}$$

Here w ranges over all strings of length $|h(|x|)$ and z ranges over all strings of the same length as x . Since membership in S or membership in W can be tested in polynomial time, it follows that $G \in \sum_2^p$. ■

COROLLARY 6.7. $EXPTIME \subseteq P/\text{poly} \Rightarrow P \neq NP$.

Proof. Assume for contradiction that $EXPTIME \subseteq P/\text{poly}$ and $P = NP$. The first hypothesis implies that $EXPTIME = \sum_2^p$, and the second implies that $P = \sum_2^p$. Hence $P = EXPTIME$. But this contradicts the result that $P \subsetneq EXPTIME$, which is easily proved by diagonalization. ■

Figure 1. MAIN RESULTS

$$PSPACE \subseteq P/\text{poly} \Rightarrow PSPACE = \sum_2^p \cap \sum_2^p$$

$$PSPACE \subseteq P/\log \Leftrightarrow PSPACE = P$$

$$EXPTIME \subseteq PSPACE/\text{poly} \Leftrightarrow EXPTIME = PSPACE$$

$$P \subseteq DSPACE((\log n)^l)/\log \Leftrightarrow P \subseteq DSPACE((\log n)^l)$$

$$NSPACE(\log n) \subseteq DSPACE(\log n)/\log$$

$$\Leftrightarrow NSPACE(\log n) = DSPACE(\log n)$$

$$NP \subseteq P/\log \Leftrightarrow P = NP \quad (1)$$

$$NP \subseteq P/\text{poly} \Rightarrow \cup \sum_i^p = \sum_2^p \quad (2)$$

$$EXPTIME \subseteq P/\text{poly} \Rightarrow EXPTIME = \sum_2^p \Rightarrow P \neq NP \quad (3)$$

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(¹) Obtained jointly with Ravindran Kannan.

(²) An improvement by Michael Sipser of an early result of ours.

(³) Due to Albert Meyer.