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Then $G = S : h$ where $S \in \text{DSPACE}((\log n)^l)$ and $|h(x)| \leq k \log_2 |x|$, for some k . Then $x \in G \Leftrightarrow \exists w \forall w' \text{Win}(w, w', x)$, where w and w' range over all strings of length $\leq k \log_2 |x|$. Clearly space $O((\log n)^l)$ suffices to deterministically enumerate all pairs (w, w') and, for each, to play out $\text{Strat}(w)$ against $\text{Strat}(w')$ from position x , with the help of repeated calls on a deterministic space $(\log n)^l$ recognizer for S . It follows that

$$G \in \text{DSPACE}((\log n)^l). \quad \blacksquare$$

5. THE SELF-REDUCIBILITY METHOD

The “hardest” problems in complexity classes defined by bounds on nondeterministic time or space often possess a structural property called *self-reducibility*. Various formal definitions of self-reducibility can be found in the literature ([12, 18, 20]). Here is one version of the idea. Let K be a subset of $\{0, 1\}^*$. A *self-reducibility structure* for K is specified by a partial ordering $<$ of $\{0, 1\}^*$ such that

- (i) A , the set of minimal elements in $<$, is recursive and
- (ii) $A \cap K$ is recursive

together with a pair of computable functions G_0 and G_1 mapping $\{0, 1\}^* - A$ into $\{0, 1\}^*$, such that, for all $x \in \{0, 1\}^* - A$,

- (iii) $G_0(x) < x, G_1(x) < x, |G_0(x)| = |G_1(x)| = |x|$
and $x \in K \Leftrightarrow G_0(x) \in K \text{ or } G_1(x) \in K$.

If K has a self-reducibility structure, then K is called *self-reducible*.

To illustrate the concept, we give self-reducibility structures for two important examples. The first example is the satisfiability problem for propositional formulas, encoded so that the following property holds: Let $F(t_1, t_2, \dots, t_n)$ be a formula in which the variables t_1, t_1, \dots, t_n appear, and let $F(a, t_2, \dots, t_n)$ be the same formula with the Boolean constant a substituted for t_1 . Let $\langle F(t_1, t_2, \dots, t_n) \rangle$ and $\langle F(a, t_2, \dots, t_n) \rangle$ denote the encodings of these two formulas as strings. Then

$$|\langle F(t_1, t_2, \dots, t_n) \rangle| = |\langle F(a, t_2, \dots, t_n) \rangle|.$$

Let SAT denote this version of the satisfiability problem. The set SAT has a self-reducibility structure in which A is the set of propositional formulas containing no variables,

$$G_0(\langle F(t_1, t_2, \dots, t_n) \rangle) = \langle F(0, t_2, \dots, t_n) \rangle \text{ and}$$

$$G_1(\langle F(t_1, t_2, \dots, t_n) \rangle) = \langle F(1, t_2, \dots, t_n) \rangle.$$

As a second example, let DAG denote the set of encodings of triples (Ψ, s, t) such that

- (i) Ψ is a directed acyclic graph in which the out-degree of each vertex is either 0 or 2; if v has out-degree 2 then its successor vertices are denoted $\sigma_0(v)$ and $\sigma_1(v)$;
- (ii) s is a vertex and t is a vertex of out-degree 0;
- (iii) there exists a directed path from s to t .

Assume that, for any directed acyclic graph G , any vertex t of out-degree 0, and any two vertices v and w , the encodings of (Ψ, v, t) and (Ψ, w, t) are of the same length. Then DAG is clearly self-reducible. Let A be the set of triples (Ψ, s, t) such that s is of out-degree 0, and let $G_0((\Psi, s, t)) = (\Psi, \sigma_0(s), t)$ and $G_1((\Psi, s, t)) = (\Psi, \sigma_1(s), t)$.

It is possible to relate the uniform complexity of a self-reducible set K to its nonuniform complexity. Suppose K has a self-reducibility structure $(\langle, \text{A}, G_0, G_1 \rangle)$ and $\text{K} = \text{S} : h$. For each $w \in \{0, 1\}^*$ define reduct_w , a total function over $\{0, 1\}^*$, by the following recursive definition:

$$\begin{aligned} \text{reduct}_w(x) = & \text{ if } x \in \text{A} \text{ then } x \text{ else} \\ & \text{ if } w \cdot G_0(x) \in \text{S} \text{ then } \text{reduct}_w(G_0(x)) \text{ else} \\ & \text{reduct}_w(G_1(x)). \end{aligned}$$

Then, for all w , $\text{reduct}_w(x) \in \text{A}$. Also, $\text{reduct}_w(x) \in \text{K} \Rightarrow x \in \text{K}$ and $x \in \text{K} \Leftrightarrow \text{reduct}_{h(|x|)}(x) \in \text{K}$. These observations imply the following lemma.

LEMMA 5.1. Let w range over some set which includes $h(|x|)$. Then

$$x \in \text{K} \Leftrightarrow \exists w [\text{reduct}_w(x) \in \text{K}].$$

Lemma 5.1 suggests a uniform way of testing membership in K : for each w in a suitable set, compute $\text{reduct}_w(x)$ and test whether

$$\text{reduct}_w(x) \in \text{A} \cap \text{K}.$$

The complexity of this algorithm will depend on the time and space needed to test membership in A , and in $\text{A} \cap \text{K}$, on the lengths of chains in the

partial ordering $<$, and on the number of strings w that need to be considered.

Now we are ready to give some applications of self-reducibility.

THEOREM 5.2. $P = NP \Leftrightarrow NP \subseteq P/\log$.

Proof. The implication $P = NP \Rightarrow NP \subseteq P/\log$ is immediate. Since SAT is NP -complete, the reverse implication will follow once we prove that

$$\text{SAT} \in P/\log \Rightarrow \text{SAT} \in P.$$

Assume that $\text{SAT} \in P/\log$. Then $\text{SAT} = S : h$, where $S \in P$ and, for some k , $|h(n)| \leq k \log_2 n$.

Using the self-reducibility structure for SAT given above, coupled with the method of lemma 5.1, we can test whether string x is in SAT. It is necessary to compute $\text{reduct}_w(x)$ for each of the polynomially-many strings w of length $\leq k \log_2 n$ and, for each, to test whether

$$\text{reduct}_w(x) \in A \cap K.$$

Each such computation can be done in polynomial time. Hence we conclude that $\text{SAT} \in P$. ■

By similar methods we can relate the nonuniform and uniform complexities of other self-reducible problems. For example, we can state the following result.

THEOREM 5.3. Let *Factor* denote the set of triples of integers $\langle x, y, z \rangle$ such that x has a factor between y and z . Then

$$\text{Factor} \in P/\log \Leftrightarrow \text{Factor} \in P.$$

As another application of the self-reducibility method, we give the following theorem.

THEOREM 5.4.

$$\begin{aligned} \text{NSPACE}(\log n)/\log &\subseteq \text{DSPACE}(\log n)/\log \\ \Leftrightarrow \text{NSPACE}(\log n) &= \text{DSPACE}(\log n). \end{aligned}$$

Proof. It is sufficient to prove

$$\begin{aligned} \text{NSPACE}(\log n)/\log &\subseteq \text{DSPACE}(\log n)/\log \\ \Rightarrow \text{NSPACE}(\log n) &= \text{DSPACE}(\log n). \end{aligned}$$

Since DAG is logspace complete in $NSPACE(\log n)$, it suffices to show that

$$\text{DAG} \in \text{DSPACE}(\log n)/\log \Rightarrow \text{DAG} \in \text{DSPACE}(\log n) .$$

Suppose that $\text{DAG} = S : h$, where $S \in \text{DSPACE}(\log n)$ and

$$|h(n)| \leq k \log_2 n .$$

Then, guided by the self-reducibility of DAG, we can test whether $(\Psi, s, t) \in \text{DAG}$ by performing the following computation for each string w of length $\leq k \log_2 n$:

```

 $v := s;$ 
while  $v$  has out-degree 2 do
 $v :=$  if  $w \cdot (\Psi, v_0, t) \in S$  then  $v_0$  else  $v_1$ .

```

If v is ever set equal to t then accept (Ψ, s, t) ; otherwise, reject it. It is clear that this method recognizes DAG deterministically within space $O(\log n)$. ■

6. THE METHOD OF RECURSIVE DEFINITION

Let K be a subset of $\{0, 1\}^*$, and let $C_K : \{0, 1\}^* \rightarrow \{0, 1\}$ be the characteristic function of K . By a recursive definition of C_K we mean a rule that specifies C_K on a “basis set” $A \subseteq \{0, 1\}^*$, and uniquely determines C_K on the rest of $\{0, 1\}^*$ by a recurrence formula of the form

$$\begin{aligned}
 C_K(x) &= F(x, C_K(f_1(x)), C_K(f_2(x)), \dots, C_K(f_t(x))), \\
 x &\in \{0, 1\}^* - A .
 \end{aligned}$$

Example 1. Let G be a game, as defined in Section 4, and let G be the set of positions from which the player to move can force a win. Then G is uniquely determined by

- (i) if $x \in W$ then $x \in G$
- (ii) if $x \in \{0, 1\}^* - W$ then $x \in G \Leftrightarrow F_0(x) \notin G$ or $F_1(x) \notin G$.

Example 2. Let (\langle, A, G_0, G_1) be a self-reducibility structure for the set $K \subseteq \{0, 1\}^*$. Then K is determined uniquely by its intersection with A , together with the recurrence

$$\text{for } x \notin A, x \in K \Leftrightarrow G_0(x) \in K \cup G_1(x) \in K .$$