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Then G = S: h where $S \in DSPACE((log n)^l)$ and $|h(x)| \le k \log_2 |x|$, for some k. Then $x \in G \Leftrightarrow \exists w \forall w' \ Win(w, w', x)$, where w and w' range over all strings of length $\le k \log_2 |x|$. Clearly space $0((log n)^l)$ suffices to deterministically enumerate all pairs (w, w') and, for each, to play out Strat(w) against Strat(w') from position x, with the help of repeated calls on a deterministic space $(\log n)^l$ recognizer for S. It follows that

$$G \in DSPACE ((log n)^l)$$
.

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5. The Self-Reducibility Method

The "hardest" problems in complexity classes defined by bounds on nondeterministic time or space often possess a structural property called *self-reducibility*. Various formal definitions of self-reducibility can be found in the literature ([12, 18, 20]). Here is one version of the idea. Let K be a subset of $\{0, 1\}^*$. A *self-reducibility structure* for K is specified by a partial ordering < of $\{0, 1\}^*$ such that

- (i) A, the set of minimal elements in <, is recursive and
- (ii) $A \cap K$ is recursive

together with a pair of computable functions G_0 and G_1 mapping $\{0, 1\}^* - A$ into $\{0, 1\}^*$, such that, for all $x \in \{0, 1\}^* - A$,

(iii)
$$G_0(x) < x$$
, $G_1(x) < x$, $|G_0(x)| = |G_1(x)| = |x|$
and $x \in K \Leftrightarrow G_0(x) \in K$ or $G_1(x) \in K$.

If K has a self-reducibility structure, then K is called self-reducible.

To illustrate the concept, we give self-reducibility structures for two important examples. The first example is the satisfiability problem for propositional formulas, encoded so that the following property holds: Let $F(t_1, t_2, ..., t_n)$ be a formula in which the variables $t_1, t_1, ..., t_n$ appear, and let $F(a, t_2, ..., t_n)$ be the same formula with the Boolean constant a substituted for t_1 . Let $\langle F(t_1, t_2, ..., t_n) \rangle$ and $\langle F(a, t_2, ..., t_n) \rangle$ denote the encodings of these two formulas as strings. Then

$$|\langle F(t_1, t_2, ..., t_n) \rangle| = |\langle F(a, t_2, ..., t_n) \rangle|.$$

Let SAT denote this version of the satisfiability problem. The set SAT has a self-reducibility structure in which A is the set of propositional formulas containing no variables,

$$G_0(\langle F(t_1, t_2, ..., t_n) \rangle) = \langle F(0, t_2, ..., t_n) \rangle$$
 and $G_1(\langle F(t_1, t_2, ..., t_n) \rangle) = \langle F(1, t_2, ..., t_n) \rangle$.

As a second example, let DAG denote the set of encodings of triples (Ψ, s, t) such that

- (i) Ψ is a directed acyclic graph in which the out-degree of each vertex is either 0 or 2; if v has out-degree 2 then its successor vertices are denoted $\sigma_0(v)$ and $\sigma_1(v)$;
- (ii) s is a vertex and t is a vertex of out-degree 0;
- (iii) there exists a directed path from s to t.

Assume that, for any directed acyclic graph G, any vertex t of out-degree 0, and any two vertices v and w, the encodings of (Ψ, v, t) and (Ψ, w, t) are of the same length. Then DAG is clearly self-reducible. Let A be the set of triples (Ψ, s, t) such that s is of out-degree 0, and let $G_0((\Psi, s, t)) = (\Psi, \sigma_0(s), t)$ and $G_1((\Psi, s, t)) = (\Psi, \sigma_1(s), t)$.

It is possible to relate the uniform complexity of a self-reducible set K to its nonuniform complexity. Suppose K has a self-reducibility structure $(<, A, G_0, G_1)$ and K = S : h. For each $w \in \{0, 1\}^*$ define $reduct_w$, a total function over $\{0, 1\}^*$, by the following recursive definition:

$$reduct_{w}(x) = \text{if } x \in A \text{ then } x \text{ else}$$

$$\text{if } w \cdot G_{0}(x) \in S \text{ then } reduct_{w}(G_{0}(x)) \text{ else}$$

$$reduct_{w}(G_{1}(x)).$$

Then, for all w, $reduct_w(x) \in A$. Also, $reduct_w(x) \in K \Rightarrow x \in K$ and $x \in K \Leftrightarrow reduct_{h(|x|)}(x) \in K$. These observations imply the following lemma.

LEMMA 5.1. Let w range over some set which includes h(|x|). Then $x \in K \Leftrightarrow \exists w [reduct_w(x) \in K]$.

Lemma 5.1 suggests a uniform way of testing membership in K: for each w in a suitable set, compute $reduct_w(x)$ and test whether

$$reduct_{w}(x) \in A \cap K$$
.

The complexity of this algorithm will depend on the time and space needed to test membership in A, and in $A \cap K$, on the lengths of chains in the

partial ordering <, and on the number of strings w that need to be considered.

Now we are ready to give some applications of self-reducibility.

Theorem 5.2. $P = NP \Leftrightarrow NP \subseteq P/log$.

Proof. The implication $P = NP \Rightarrow NP \subseteq P/log$ is immediate. Since SAT is NP-complete, the reverse implication will follow once we prove that

$$SAT \in P/log \Rightarrow SAT \in P$$
.

Assume that SAT $\in P/log$. Then SAT = S: h, where S $\in P$ and, for some k, $|h(n)| \le k \log_2 n$.

Using the self-reducibility structure for SAT given above, coupled with the method of lemma 5.1, we can test whether string x is in SAT. It is necessary to compute $reduct_w(x)$ for each of the polynomially-many strings w of length $\leq k \log_2 n$ and, for each, to test whether

$$reduct_{w}(x) \in A \cap K$$
.

Each such computation can be done in polynomial time. Hence we conclude that $SAT \in P$.

By similar methods we can relate the nonuniform and uniform complexities of other self-reducible problems. For example, we can state the following result.

THEOREM 5.3. Let *Factor* denote the set of triples of integers $\langle x, y, z \rangle$ such that x has a factor between y and z. Then

$$Factor \in P \mid log \Leftrightarrow Factor \in P$$
.

As another application of the self-reducibility method, we give the following theorem.

THEOREM 5.4.

$$NSPACE (log n) / log \subseteq DSPACE (log n) / log$$

 $\Leftrightarrow NSPACE (log n) = DSPACE (log n).$

Proof. It is sufficient to prove

$$NSPACE (log n)/log \subseteq DSPACE (log n)/log$$

 $\Rightarrow NSPACE (log n) = DSPACE (log n)$.

Since DAG is logspace complete in NSPACE (log n), it suffices to show that

$$DAG \in DSPACE (log n)/log \Rightarrow DAG \in DSPACE (log n)$$
.

Suppose that DAG = S: h, where $S \in DSPACE (log n)$ and

$$|h(n)| \leq k \log_2 n$$
.

Then, guided by the self-reducibility of DAG, we can test whether $(\Psi, s, t) \in DAG$ by performing the following computation for each string w of length $\leq k \log_2 n$:

$$v:=s;$$

while v has out-degree 2 do

$$v:=$$
 if $w\cdot (\Psi, v_0, t)\in S$ then v_0 else v_1 .

If v is ever set equal to t then accept (Ψ, s, t) ; otherwise, reject it. It is clear that this method recognizes DAG deterministically within space 0 (log n).

6. The Method of Recursive Definition

Let K be a subset of $\{0, 1\}^*$, and let $C_K : \{0, 1\}^* \to \{0, 1\}$ be the characteristic function of K. By a recursive definition of C_K we mean a rule that specifies C_K on a "basis set" $A \subseteq \{0, 1\}^*$, and uniquely determines C_K on the rest of $\{0, 1\}^*$ by a recurrence formula of the form

$$C_K(x) = F(x, C_K(f_1(x)), C_K(f_2(x)), ..., C_K(f_t(x))),$$

$$x \in \{0, 1\}^* - A.$$

Example 1. Let G be a game, as defined in Section 4, and let G be the set of positions from which the player to move can force a win. Then G is uniquely determined by

- (i) if $x \in W$ then $x \in G$
- (ii) if $x \in \{0, 1\}^*$ W then $x \in G \Leftrightarrow F_0(x) \notin G$ or $F_1(x) \notin G$.

Example 2. Let $(<, A, G_0, G_1)$ be a self-reducibility structure for the set $K \subseteq \{0, 1\}^*$. Then K is determined uniquely by its intersection with A, together with the recurrence

for
$$x \notin A$$
, $x \in K \Leftrightarrow G_0(x) \in K \cup G_1(x) \in K$.