**Zeitschrift:** L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

**Band:** 28 (1982)

**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TURING MACHINES THAT TAKE ADVICE

Autor: Karp, Richard M. / Lipton, Richard J.

**Kapitel:** 2. NONUNIFORM COMPLEXITY MEASURES

**DOI:** https://doi.org/10.5169/seals-52237

### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF: 23.11.2025** 

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

always": clearly there are sets with small circuits that are not even recursive. The very trivial nature of such "counter-examples" suggests, however, that a more careful investigation may still yield insight. Indeed, as we will show, if one considers not arbitrary sets but rather "well behaved ones" it is possible to achieve our goal. For example, we will show that if SAT has small circuits, then the Meyer-Stockmeyer [19] hierarchy collapses.

Thus, here is an example of a nonuniform upper bound that has uniform consequences. The proof, of course, will depend on the fact that SAT is not a "pathological" set, but is rather well behaved.

Our results also serve to rule out some plausible speculations about the complexity of problems in NP. For example, one might imagine that  $P \neq NP$ , but SAT is tractable in the following sense: for every l there is a very short program that runs in time  $n^2$  and correctly treats all instances of length l. Theorem 5.2 shows that, if "very short" means "of length c log 1", then this speculation is false.

Finally, we mention that the proof techniques presented here were put to use by S. Mahaney in his proof that  $P \neq NP$  implies the nonexistence of sparse NP-complete problems [11], and by S. A. Cook in his proof that

$$P \subseteq HARDWARE (log n) \Rightarrow P \subseteq DSPACE (log n log log n)$$
[5].

# 2. Nonuniform Complexity Measures

In this section we will define our basic notion of nonuniform complexity and relate it to circuit complexity.

Let S be a subset of  $\{0, 1\}^*$ . Let  $h: N \to \{0, 1\}^*$  where N is the set of natural numbers. Define  $S: h = \{x \mid h(|x|) \cdot x \in S\}$ . Next, let V be any collection of subsets of  $\{0, 1\}^*$  and let F be any collection of functions from N to N. The key definition is

$$V/F = \{S: h \mid S \in V \text{ and } h \in F\}$$

Intuitively, V/F is the collection of subsets of  $\{0, 1\}^*$  that can be accepted by V with an amount of advice bounded by F. The idea behind this definition is foreshadowed in papers by Pippenger [14] and Plaisted [15].

We are mainly interested in poly, the collection of all polynomially-bounded functions, and log, the collection of all functions that are 0 (log n). Indeed, many of our results will concern the classes P/poly and P/log.

If f is a function, V/f is synonymous with  $V/\{f\}$ . Some preliminary facts are:

- (1) for all V, V/0 = V;
- (2) any subset of  $\{0, 1\}^*$  is in  $P/2^n$ ;
- (3) if f is infinitely often nonzero, then P/f contains nonrecursive sets;
- (4) if  $g(n) < f(n) \le 2^n$  (i.o.) then  $P/f \subseteq P/g$ .

The class P/poly can be characterized in terms of classic circuit complexity. An n-input m-gate Boolean circuit C is a function

$$C: \{n+1, ..., n+m\} \to \{0, 1\}^4 \times \{1, ..., n+m\}^2$$

satisfying: if  $C(i) = \langle B, j, k \rangle$  then j < i and k < i. The interpretation of C is that gate i uses the truth table B on inputs j and k to produce its output. If  $1 \leqslant j \leqslant n$  then input j is simply the input variable  $x_j$ ; otherwise, input j is the output of gate j. In the usual way we define what it means for a circuit C to realize the Boolean function f. Then let L(f) denote the minimum number of gates in a Boolean circuit realizing the Boolean function f. Next, as in the introduction, if S is a subset of  $\{0, 1\}^*$ , then  $S_n: \{0, 1\}^n \to \{0, 1\}$  is defined by

$$S_n(x_1, x_2, ..., x_n) = \begin{cases} 1, & \text{if } x_1 x_2 ... x_n \in S \\ 0, & \text{otherwise} \end{cases}$$

Finally, recall that a set S has *small circuits* if  $L(S_n)$  is bounded by a polynomial in n.

The following simple theorem, which is given in [14], characterizes P/poly.

THEOREM 2.1. Let S be a subset of  $\{0, 1\}^*$ . Then the following are equivalent.

- (1) S has small circuits.
- (2) S is in P/poly.

Another way we can gain insight into our classes V/F is to use them to restate other known results. For example, the result in [2] that there are short universal traversal sequences for undirected graphs can be restated as

Here UGAP is the undirected maze problem. As another example, we have Adleman's [1] result that R (the set of languages accepted in polynomial time by randomizing Turing machines) has small circuits, which can be restated as

$$R$$
 is a subset of  $P/poly$ .

It may be interesting that both these results use the probabilistic method of Erdös to prove the existence of the required advice bits.

## 3. Summary of Main Results

We will discuss a variety of complexity classes. These include the basic time and space classes DTIME(T(n)), DSPACE(S(n)) and NSPACE(S(n)) and the classes:

P = the set of languages accepted in deterministic polynomial time,

R = the set of languages accepted in polynomial time by randomizing Turing machines [1],

NP = the set of languages accepted in nondeterministic polynomial time,

PSPACE = the set of languages accepted in polynomial space,

$$EXPTIME = \bigcup_{i>0} DTIME (2^{ni})$$
.

Also important is the polynomial-time hierarchy of Meyer and Stockmeyer [19]. For  $i \ge 1$  we let  $\sum_{i=1}^{p} p$  (respectively  $\prod_{i=1}^{p} p$ ) denote those languages accepted in polynomial time by Turing machines that make i alternations starting from an existential (respectively universal) state. Note that  $NP = \sum_{i=1}^{p} p$  and co- $NP = \prod_{i=1}^{p} p$ . Finally, note that P, PSPACE and EXPTIME can be viewed as complexity classes associated with alternating Turing machines; specifically, P = ASPACE (log n), PSPACE = AP and EXPTIME = APSPACE [3, 10].

Many of the following theorems take the form

$$L \subseteq S/F \Rightarrow L \subseteq S'$$

where L and S' are uniform complexity classes and V/F is a nonuniform complexity class. The proof usually consists of showing that