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(b)  $\Rightarrow$  (a) : Assume (b). Then  $A$  is self-adjoint and positive definite. Define a new inner product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{D}(A)$  by  $\langle v, w \rangle := (Av, w)$ . Then, for  $v, w \in \mathcal{D}(A)$ ,  $g \in G$ , we have:

$$\begin{aligned} \langle \tau(g)v, \tau(g)w \rangle &= (A\tau(g)v, \tau(g)w) = (\tilde{\tau}(g^{-1})A\tau(g)v, w) \\ &= (A\tau(g^{-1})\tau(g)v, w) = (Av, w) = \langle v, w \rangle, \end{aligned}$$

i.e.  $\langle \tau(g)v, \tau(g)w \rangle = \langle v, w \rangle$ . Thus  $\tau$  is a unitary representation on  $\mathcal{D}(A)$  with respect to the new inner product. (Weak continuity of  $\tau$  is easily proved.) Let  $\sigma$  be the extension of this representation to a unitary representation in the Hilbert space completion  $\mathcal{H}(\sigma)$  of  $\mathcal{D}(A)$  with respect to  $\langle \cdot, \cdot \rangle$ . Then  $\tau \simeq \sigma$ , where  $B$  is the closure of the identity operator on  $\mathcal{D}(A)$  (cf. Lemma 4.4). Note that we have also proved the last part of the theorem.

The equivalence of (c) or (d) with (b) follows from Theorem 4.5.  $\square$

## 6.2. THE CASE $SU(1, 1)$

It follows from (2.30) that

$$(6.4) \quad \overline{c_{\xi, \lambda, n, m}} = (-1)^{m-n} c_{\xi, -\bar{\lambda}, m, n}.$$

Combination of (6.3), (2.29) and (6.4) yields

$$(6.5) \quad \tilde{\pi}_{\xi, \lambda} = \pi_{\xi, -\bar{\lambda}}.$$

In §6.1 we showed that a necessary condition for unitarizability of an irreducible subquotient representation  $\tau$  of  $\pi_{\xi, \lambda}$  is the equivalence of  $\tau$  and  $\tilde{\tau}$ . In view of (6.5) and Theorem 4.7 this is only possible if  $\bar{\lambda} = \pm\lambda$ , that is, if  $\lambda$  is real or imaginary. If  $\lambda$  is imaginary then  $\tilde{\pi}_{\xi, \lambda} = \pi_{\xi, \lambda}$ , so  $\pi_{\xi, \lambda}$  is already unitary. Let us now examine the case that  $\lambda$  is real and nonzero. Then  $\tilde{\pi}_{\xi, \lambda} = \pi_{\xi, -\lambda}$ . If  $\tau$  is an irreducible subquotient representation of  $\pi_{\xi, \lambda}$  then  $\tau \stackrel{A}{\simeq} \tilde{\tau}$  with (cf. (4.10))

$$(6.6) \quad A\phi_m = c_{\xi, \lambda, m} \phi_m, \phi_m \in \mathcal{H}(\tau),$$

where  $c_{\xi, \lambda, m}$  is given by (4.9). Now a sufficient condition for the unitarizability of  $\tau$  is that the coefficients  $c_{\xi, \lambda, m}$  are all positive or all negative for  $\phi_m \in \mathcal{H}(\tau)$ . Referring to the classification in Theorem 3.4 we will examine these coefficients. (Because of equivalence, it is not necessary to treat the cases where  $\lambda < 0$ .)

$$(a) \quad \pi_{0, \lambda}(\lambda > 0, \lambda \notin \mathbf{Z} + \frac{1}{2}).$$

$$c_{0, \lambda, m} = \frac{(-\lambda + \frac{1}{2})_{|m|}}{(\lambda + \frac{1}{2})_{|m|}}, \quad m \in \mathbf{Z}.$$

$c_{0, \lambda, m}$  has fixed sign iff  $0 < \lambda < \frac{1}{2}$ .

$$(b) \quad \pi_{\frac{1}{2}, \lambda}(\lambda > 0, \lambda \notin \mathbf{Z}).$$

$$c_{\frac{1}{2}, \lambda, m} = \frac{(-\lambda)_{m + \frac{1}{2}}}{(\lambda)_{m + \frac{1}{2}}}, \quad m + \frac{1}{2} \in \{0, 1, 2, \dots\}.$$

No fixed sign.

$$(c) \quad \pi_{\xi, \lambda}^+ \text{ and } \pi_{\xi, \lambda}^-(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0).$$

$$c_{\xi, \lambda, m} = \frac{(|m| - (\lambda + \frac{1}{2}))!}{(2\lambda + 1)_{|m| - (\lambda + \frac{1}{2})}}, \quad m \in \mathbf{Z} + \xi, |m| \geq \lambda + \frac{1}{2}.$$

Fixed sign.

$$(d) \quad \pi_{\xi, \lambda}^0(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0).$$

$$c_{\xi, \lambda, m} = \frac{(-1)^{m - \xi}}{(\lambda - \frac{1}{2} + m)!(\lambda + \frac{1}{2} - m)!}, \quad m \in \left\{ -\lambda + \frac{1}{2}, -\lambda + \frac{3}{2}, \dots, \lambda - \frac{1}{2} \right\}.$$

No fixed sign except if  $\lambda = \frac{1}{2}, \xi = 0$ .

Combining these results with Theorems 3.4, 4.7 and 5.4 and Prop. 4.2 we reobtain BARGMANN'S [2] classification of all irreducible unitary representations of  $SU(1, 1)$ :

**THEOREM 6.2.** *Any irreducible unitary representation of  $SU(1, 1)$  is unitarily equivalent to one and only one of the following representations:*

$$1) \quad \pi_{\xi, i\nu}(\xi = 0, \frac{1}{2}, \nu > 0), \pi_{0, 0}, \pi_{\frac{1}{2}, 0}^+, \pi_{\frac{1}{2}, 0}^- \text{ (unitary principal series).}$$

$$2) \quad \pi_{0, \lambda}(0 < \lambda < \frac{1}{2}) \text{ on } Cl \text{ Span}\{\dots, \phi_{-1}, \phi_0, \phi_1, \dots\}$$

with respect to the inner product

$$\langle \phi_m \phi_n \rangle := \frac{(-\lambda + \frac{1}{2})_{|m|}}{(\lambda + \frac{1}{2})_{|m|}} \delta_{m, n} \text{ (complementary series).}$$

$$3) \quad \pi_{\xi, \lambda}^+ \text{ and } \pi_{\xi, \lambda}^- \left( \xi = 0 \text{ or } \frac{1}{2}, \lambda = \xi + \frac{1}{2}, \xi + \frac{3}{2}, \dots \right)$$

on

$$Cl \text{ Span}\{\phi_{\lambda + \frac{1}{2}}, \phi_{\lambda + 3/2}, \dots\}$$

and

$$Cl \text{ Span}\{\dots, \phi_{-\lambda - 3/2}, \phi_{-\lambda - \frac{1}{2}}\},$$

respectively, with respect to the inner product

$$\langle \phi_m, \phi_n \rangle := \frac{(|m| - (\lambda + \frac{1}{2}))!}{(2\lambda + 1)_{|m| - (\lambda + \frac{1}{2})}} \delta_{m,n} \text{ (discrete series).}$$

4)  $\pi_{0, \frac{1}{2}}^0$  (identity representation).

### 6.3. NOTES

6.3.1. Following BARGMANN [2], most authors prove Theorem 6.2 by infinitesimal methods. VILENKIN [43, Ch. VI] uses the method of the present paper. TAKAHASHI [39, §6] decides about unitarizability by considering whether  $\pi_{\xi, \lambda, n, n}$  is a positive definite function on  $G$ .

6.3.2. A method related to this section was used in FLENSTED-JENSEN & KOORNWINDER [15] in order to find all irreducible unitary spherical representations of non-compact semisimple Lie groups  $G$  of rank one. They examined the nonnegativity of the coefficients in the addition formula for the spherical functions on  $G$ . See also [27, §6.4].

6.3.3. A generalization of Theorem 6.1 can be formulated for not necessarily abelian  $K$  and, partly, for  $K$ -finite  $\tau$ , cf. [27, Theorems 6.4, 6.5].