Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH

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Kapitel: 5.6. Notes

DOI: https://doi.org/10.5169/seals-52233

By the continuity and homomorphism property of α we have, for $f \in \mathcal{D}_{\text{even}}(\mathbf{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)dy .$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)dy, \; f \in \mathcal{D}_{\text{even}}(\mathbf{R}) ,$$

where

$$\beta(y) := \tfrac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1)) .$$

Then β is even and it is a continuous function by the continuity of α. It follows from the homomorphism property of α and from the fact that β is even, that

$$\beta(x)\beta(y) = \tfrac{1}{2}(\beta(x+y) + \beta(x-y)) ,$$

so $\beta(0) = 1$. This is d'Alembert's functional equation. By continuity, $Re\, \beta(x) > 0$ if $0 \leqslant x \leqslant x_0$ for some $x_0 > 0$. Then $\beta(x_0) = \cosh c$ for some complex $c = a + ib$ with $a \geqslant 0$, $-\frac{1}{2}\pi < b < \frac{1}{2}\pi$. Now, following the proof in ACZEL [1, 2.4.1] it can be shown [1]) that for all integer $n, m \geqslant 0$

$$\beta\left(\frac{n}{2^m} x_0\right) = \cosh\left(\frac{c}{x_0}\frac{n}{2^m} x_0\right) .$$

So, by continuity and evenness of β:

$$\beta(x) = \cosh\left(\frac{c}{x_0} x\right) \text{ for all } x \in \mathbf{R} . \qquad \square$$

5.6. NOTES

5.6.1. Some other examples of Gelfand pairs $(G \times K, K^*)$ are provided by $G = SO_0(n, 1)$, $K = SO(n)$ and $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, cf. BOERNER [4, Ch. VII, §12; Ch. V, §6], DIXMIER [8] or KOORNWINDER [27, Theorems 5.7, 5.8].

5.6.2. The main Theorem 5.4, which was first proved in the case of unitary representations by BARGMANN [2], is a special case of the *subrepresentation*

[1]) I thank H. van Haeringen for this reference.

theorem for noncompact semisimple Lie groups due to Casselman (cf. WALLACH [47, Cor. 7.5]). Casselman's theorem improves HARISH-CHANDRA'S [22, Theorem 4] *subquotient theorem.*

5.6.3. The generalized Abel transform $f \to F_f^\delta$ can be defined for general K-type δ. It was introduced by HARISH-CHANDRA [24, p. 595] in the spherical case, TAKAHASHI [40, §2] in the case $G = SO_0(n, 1)$ and WARNER [49, 6.2.2] in the general case. The injectivity of this transform holds generally, cf. WARNER [49]. The image of $I_{c,\delta}^\infty(G)$ under this transform is known in the spherical case (cf. GANGOLLI [16]) and if G has real rank 1 and δ is one-dimensional (cf. WALLACH [46]), but seems to be unknown in the general case (cf. WARNER [49, p. 36]).

5.6.4. In [39] TAKAHASHI also reduces the proof of Theorem 5.4 to Proposition 5.5. However, he proves Prop. 5.5 by considering eigenfunctions of the Casimir operator, since he did not know, then, how to invert the transform $f \to F_f^n$. In [42] he independently obtained a proof of Prop. 5.5 similar to ours. Earlier, in [40, §4.1] he used a similar method in the spherical case of $G = SO_0(n, 1)$. NAIMARK [34, Ch. 3, §9] proved the subquotient theorem for $SL(2, \mathbf{C})$ by methods somewhat related to ours.

5.6.5. Part of Lemma 5.8 is contained in WHITNEY [50]. See SCHWARZ [37] for a theorem on C^∞-functions which are invariant under a more general Weyl group.

5.6.6. Theorem 5.10 more generally holds with Gegenbauer polynomials of integer of half integer order as kernels, cf. DEANS [6], [7], KOORNWINDER [27, §5.9]. Deans' proof uses the inversion formula for the Radon transform. The author's proof uses Weyl fractional integral transforms and generalized fractional integral transforms studied by SPRINKHUIZEN [38]. MATSUSHITA [30, §2.3] considers the transformation $f \to F_f^n$ for general real n in the context of the universal covering group of $SL(2, \mathbf{R})$ and he derives the inversion formula with a proof due to T. Shintani, which uses Mellin transforms.

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

6.1. A CRITERIUM FOR UNITARIZABILITY

Remember that a representation of an lcsc. group G on a Hilbert space is strongly continuous if and only if it is weakly continuous (cf. WARNER [48, Prop. 4.2.2.1]). Thus, if τ is a (strongly continuous) Hilbert representation of G then $\tilde{\tau}$ defined by