

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 28 (1982)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE REPRESENTATION THEORY OF $SL(2, \mathbb{R})$, A NON-INFINITESIMAL APPROACH
Autor: Koornwinder, Tom H.
Kapitel: 5.6. Notes
DOI: <https://doi.org/10.5169/seals-52233>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

By the continuity and homomorphism property of α we have, for $f \in \mathcal{D}_{\text{even}}(\mathbf{R})$:

$$\alpha(f_1)\alpha(f) = \alpha(f_1 * f) = \int_{-\infty}^{\infty} \alpha(\lambda(y)f_1)f(y)dy.$$

Hence

$$\alpha(f) = \int_{-\infty}^{\infty} f(y)\beta(y)dy, \quad f \in \mathcal{D}_{\text{even}}(\mathbf{R}),$$

where

$$\beta(y) := \frac{1}{2}(\alpha(f_1))^{-1}(\alpha(\lambda(y)f_1) + \alpha(\lambda(-y)f_1)).$$

Then β is even and it is a continuous function by the continuity of α . It follows from the homomorphism property of α and from the fact that β is even, that

$$\beta(x)\beta(y) = \frac{1}{2}(\beta(x+y) + \beta(x-y)),$$

so $\beta(0) = 1$. This is d'Alembert's functional equation. By continuity, $\operatorname{Re} \beta(x) > 0$ if $0 \leq x \leq x_0$ for some $x_0 > 0$. Then $\beta(x_0) = \cosh c$ for some complex $c = a + ib$ with $a \geq 0$, $-\frac{1}{2}\pi < b < \frac{1}{2}\pi$. Now, following the proof in ACZEL [1, 2.4.1] it can be shown ¹⁾ that for all integer $n, m \geq 0$

$$\beta\left(\frac{n}{2^m}x_0\right) = \cosh\left(\frac{c}{x_0}\frac{n}{2^m}x_0\right).$$

So, by continuity and evenness of β :

$$\beta(x) = \cosh\left(\frac{c}{x_0}x\right) \text{ for all } x \in \mathbf{R}. \quad \square$$

5.6. NOTES

5.6.1. Some other examples of Gelfand pairs $(G \times K, K^*)$ are provided by $G = SO_0(n, 1)$, $K = SO(n)$ and $G = SU(n, 1)$, $K = S(U(n) \times U(1))$, cf. BOERNER [4, Ch. VII, §12; Ch. V, §6], DIXMIER [8] or KOORNWINDER [27, Theorems 5.7, 5.8].

5.6.2. The main Theorem 5.4, which was first proved in the case of unitary representations by BARGMANN [2], is a special case of the *subrepresentation*

¹⁾ I thank H. van Haeringen for this reference.

theorem for noncompact semisimple Lie groups due to Casselman (cf. WALLACH [47, Cor. 7.5]). Casselman's theorem improves HARISH-CHANDRA's [22, Theorem 4] *subquotient theorem*.

5.6.3. The generalized Abel transform $f \rightarrow F_f^\delta$ can be defined for general K -type δ . It was introduced by HARISH-CHANDRA [24, p. 595] in the spherical case, TAKAHASHI [40, §2] in the case $G = SO_0(n, 1)$ and WARNER [49, 6.2.2] in the general case. The injectivity of this transform holds generally, cf. WARNER [49]. The image of $I_{c, \delta}^\infty(G)$ under this transform is known in the spherical case (cf. GANGOLLI [16]) and if G has real rank 1 and δ is one-dimensional (cf. WALLACH [46]), but seems to be unknown in the general case (cf. WARNER [49, p. 36]).

5.6.4. In [39] TAKAHASHI also reduces the proof of Theorem 5.4 to Proposition 5.5. However, he proves Prop. 5.5 by considering eigenfunctions of the Casimir operator, since he did not know, then, how to invert the transform $f \rightarrow F_f^n$. In [42] he independently obtained a proof of Prop. 5.5 similar to ours. Earlier, in [40, §4.1] he used a similar method in the spherical case of $G = SO_0(n, 1)$. NAIMARK [34, Ch. 3, §9] proved the subquotient theorem for $SL(2, \mathbf{C})$ by methods somewhat related to ours.

5.6.5. Part of Lemma 5.8 is contained in WHITNEY [50]. See SCHWARZ [37] for a theorem on C^∞ -functions which are invariant under a more general Weyl group.

5.6.6. Theorem 5.10 more generally holds with Gegenbauer polynomials of integer or half integer order as kernels, cf. DEANS [6], [7], KOORNWINDER [27, §5.9]. Deans' proof uses the inversion formula for the Radon transform. The author's proof uses Weyl fractional integral transforms and generalized fractional integral transforms studied by SPRINKHUIZEN [38]. MATSUSHITA [30, §2.3] considers the transformation $f \rightarrow F_f^n$ for general real n in the context of the universal covering group of $SL(2, \mathbf{R})$ and he derives the inversion formula with a proof due to T. Shintani, which uses Mellin transforms.

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

6.1. A CRITERIUM FOR UNITARIZABILITY

Remember that a representation of an lcsc. group G on a Hilbert space is strongly continuous if and only if it is weakly continuous (cf. WARNER [48, Prop. 4.2.2.1]). Thus, if τ is a (strongly continuous) Hilbert representation of G then $\tilde{\tau}$ defined by