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These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where  $\phi$  is a spherical function on  $G \times K$ . If  $\phi$  is a spherical function on  $G \times K$  then there is a  $\delta \in \hat{K}$  such that for all  $x \in G$  the function  $k \rightarrow \phi(xk)$  on  $K$  belongs to  $\delta$ . Then  $\delta$  is called a *spherical function of type  $\delta$*  on  $G$  (with respect to  $K$ ), cf. GODEMENT [19]. It is funny that spherical functions of type  $\delta$  are on the one hand generalizations of ordinary spherical functions for  $(G, K)$ , on the other hand restrictions to  $G$  of ordinary spherical functions for  $(G \times K, K^*)$ .

For convenience, we take a one-dimensional  $\delta \in \hat{K}$ . Then a spherical function  $\phi$  on  $G \times K$  is of type  $\delta$  iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \quad x \in G, k \in K.$$

Let

$$\begin{aligned} & I_{c, \delta}(G) \text{ (or } I_{c, \delta}^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(xk) = f(kx) \\ & \quad = \delta(k)f(x), x \in G, k \in K\}. \end{aligned}$$

These are closed subalgebras of  $I_c(G)$  (or  $I_c^\infty(G)$ ) and their characters are precisely of the form (5.1), where  $\phi$  is a spherical function of type  $\delta$ . Finally, if  $\tau$  is a  $K$ -unitary representation of  $G$  and if  $\mathcal{H}(\tau)$  contains a unit vector  $v$  satisfying  $\tau(k)v = \delta(k)v$ , unique up to a constant factor, then  $x \rightarrow (\tau(x)v, v)$  is a spherical function of type  $\delta$ .

### 5.3. THE GENERALIZED ABEL TRANSFORM

Let  $G$  be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures  $dk, da, dn$  on  $K, A, N$ , respectively, normalize the Haar measure on  $G$  such that

$$(5.2) \quad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)} dk da dn, f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \quad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, f \in C_c(N), a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).

For  $\lambda \in \mathfrak{a}_\mathbb{C}^*$  let  $U^\lambda$  be the representation of  $G$  induced by the one-dimensional representation  $an \rightarrow e^{\lambda(\log a)}$  of the subgroup  $AN$ :

$$(5.4) \quad (U^\lambda(g)f)(k) := e^{-(\rho+\lambda)H(g^{-1}k)} f(u(g^{-1}k)), \quad f \in L^2(K), g \in G, k \in K.$$

The representation  $U^\lambda$  is easily seen to split as a direct sum of principal series representations  $\pi_{\xi, \lambda}$ .  $U^\lambda$  restricted to  $K$  is the left regular representation of  $K$ .

Let  $\delta \in \widehat{K}$ . For convenience, suppose that  $\delta$  is one-dimensional. The *generalized Abel transform*  $f \rightarrow F_f^\delta: I_{c, \delta}(G) \rightarrow C_c(A)$  is defined by

$$(5.5) \quad F_f^\delta(a) := e^{\rho(\log a)} \int_N f(an) dn, \quad a \in A.$$

If  $G = SU(1, 1)$  and  $\delta = 1$  then this transform can be rewritten as the classical Abel transform, cf. §5.4.

**PROPOSITION 5.3.** *The mapping  $f \rightarrow F_f^\delta$  is a continuous homomorphism (with respect to convolution on  $G$  and  $A$ , respectively) from  $I_{c, \delta}^\infty(G)$  to  $C_c^\infty(A)$ . Furthermore,*

$$(5.6) \quad \int_A F_f^\delta(a) e^{-\lambda(\log a)} da = \int_G f(g) (U^\lambda(g^{-1})\check{\delta}, \check{\delta}) dg, \quad f \in I_{c, \delta}^\infty(G), \lambda \in \mathfrak{a}_\mathbb{C}^*,$$

where  $(\cdot, \cdot)$  denotes the inner product on  $L^2(K)$ .

*Proof.* The continuity is immediate. The homomorphism property follows easily from (5.2) and (5.3) (cf. WARNER [49, pp. 34, 35]). For the proof of (5.6) substitute (5.4) into the right hand side of (5.6):

$$\begin{aligned} \int_G f(g) (U^\lambda(g^{-1})\check{\delta}, \check{\delta}) dg &= \int_G \int_K f(g) e^{-(\rho+\lambda)H(gk)} \delta((u(gk))^{-1}k) dk dg \\ &= \int_G f(g) e^{-(\rho+\lambda)H(g)} \delta((u(g))^{-1}) dg \\ &= \int_{K \times A \times N} f(kan) e^{(\rho-\lambda)\log a} \delta(k^{-1}) dk da dn \\ &= \int_A \int_N f(an) e^{(\rho-\lambda)\log a} dn da \\ &= \int_A F_f^\delta(a) e^{-\lambda(\log a)} da. \end{aligned}$$

□

Now let  $G = SU(1, 1)$ . Write  $F_f^n(t)$  and  $I_{c, n}^\infty(G)$  instead of  $F_f^{\delta_n}(a_t)$  and  $I_{c, \delta_n}^\infty(G)$ , respectively. If  $n \in \mathbf{Z} + \xi$  then (5.5) and (5.6) take the form

$$(5.7) \quad F_f^n(t) = e^{\frac{1}{2}t} \int_{-\infty}^{\infty} f(a_t n_z) dz$$

and

$$(5.8) \quad \int_{-\infty}^{\infty} F_f^n(t) e^{-\lambda t} dt = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \quad f \in I_{c, n}^\infty(G), \lambda \in \mathbf{C},$$

where  $dg = (2\pi)^{-1} e^t d\theta dt dz$  if  $g = u_\theta a_t n_z$ .

#### 5.4. THE MAIN THEOREM

It is the purpose of this section to prove:

**THEOREM 5.4.** *Let  $\tau$  be an irreducible  $K$ -unitary representation of  $SU(1, 1)$  which is  $K$ -finite or unitary. Then  $\tau$  is Naimark equivalent to an irreducible subrepresentation of some principal series representation  $\pi_{\xi, \lambda}$ .*

By Proposition 5.2  $\tau$  is  $K$ -multiplicity free. If  $\delta_n \in \mathcal{M}(\tau)$  then write  $\tau_{n, n}$  instead of  $\tau_{\delta_n, \delta_n}$ . In view of Theorem 4.5 and Remark 4.8 it is sufficient for the proof of Theorem 5.4 to show that for some  $\delta_n \in \mathcal{M}(\tau)$ , for some  $\lambda \in \mathbf{C}$  and for  $\xi \in \{0, \frac{1}{2}\}$  with  $n \in \mathbf{Z} + \xi$  we have

$$(5.9) \quad \tau_{n, n} = \pi_{\xi, \lambda, n, n}.$$

Both sides of (5.9) are spherical functions of type  $\delta_n$ . Then (5.9) holds if the corresponding characters on  $I_{c, n}^\infty(G)$  are equal. Hence Theorem 5.4 will follow from

**PROPOSITION 5.5.** *Let  $G = SU(1, 1)$ ,  $n \in \frac{1}{2}\mathbf{Z}$ . Let  $\alpha$  be a continuous character on  $I_{c, n}^\infty(G)$ . Then*

$$(5.10) \quad \alpha(f) = \int_G f(g) \pi_{\xi, \lambda, n, n}(g^{-1}) dg, \quad f \in I_{c, n}^\infty(G),$$

for some  $\lambda \in \mathbf{C}$  and for  $\xi \in \{0, \frac{1}{2}\}$  such that  $n \in \mathbf{Z} + \xi$ .