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5.2. SPHERICAL FUNCTIONS OF TYPE  $\delta$ 

Let  $G$  be a unimodular lcsc. group with compact subgroup  $K$ . Let

$$K^* := \{(k, k) \in G \times K \mid k \in K\}.$$

Let  $\delta \in \hat{K}$  and let  $\tau$  be a  $K$ -unitary representation of  $G$ . Then  $\tau \otimes \check{\delta}$  ( $\check{\delta}$  the contragredient representation to  $\delta$ ) is a  $K^*$ -unitary representation of  $G \times K$  on  $\mathcal{H}(\tau) \otimes \mathcal{H}(\delta)$ .

**LEMMA 5.1.** *The multiplicity of  $\delta$  in  $\tau|_K$  is equal to the multiplicity of the representation 1 of  $K^*$  in  $\tau \otimes \check{\delta}|_{K^*}$ .  $\tau$  is irreducible iff  $\tau \otimes \check{\delta}$  is irreducible.  $\tau$  is unitary iff  $\tau \otimes \check{\delta}$  is unitary.*

This can be proved immediately. By using the results summarized in §5.1 we conclude that  $(G \times K, K^*)$  is a Gelfand pair if there exists a continuous involutive homomorphism  $\alpha$  on  $G$  such that for each  $(g, k) \in G \times K$  we have  $\alpha(g) = k_1 g^{-1} k_2$ ,  $\alpha(k) = k_1 k^{-1} k_2$  for certain  $k_1, k_2 \in K$ . Furthermore, if  $(G \times K, K^*)$  is a Gelfand pair and if the irreducible representation  $\tau$  of  $G$  is unitary or  $K$ -finite then  $\tau$  is  $K$ -multiplicity free. In particular, this applies to  $SU(1, 1)$ :

**PROPOSITION 5.2.** *If  $G = SU(1, 1)$  then  $(G \times K, K^*)$  is a Gelfand pair.*

*Proof.* For  $g \in SU(1, 1)$  define  $\alpha(g) := {}^t(g^{-1})$ . Then  $\alpha$  is a continuous involutive automorphism on  $G$  and  $\alpha(a_t) = a_{-t}$  on  $A$ ,  $\alpha(u_\theta) = u_{-\theta}$  on  $K$ . Since  $G = KAK$ ,  $\alpha$  has the required properties.  $\square$

Let  $(G \times K, K^*)$  be a Gelfand pair. Identify  $G \times \{e\}$  with  $G$ . A spherical function on  $G \times K$  is completely determined by its restriction to  $G$ . By using the results mentioned in §5.1 we obtain the following properties. First, a continuous function  $\phi$  on  $G$  is the restriction to  $G$  of a spherical function on  $G \times K$  iff  $\phi \neq 0$  and

$$\phi(x)\phi(y) = \int_K \phi(xkyk^{-1})dk, \quad x, y \in G.$$

Next, let

$$\begin{aligned} & I_c(G) \text{ (or } I_c^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(kgk^{-1}) = f(g), \\ & \quad g \in G, k \in K\}. \end{aligned}$$

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where  $\phi$  is a spherical function on  $G \times K$ . If  $\phi$  is a spherical function on  $G \times K$  then there is a  $\delta \in \hat{K}$  such that for all  $x \in G$  the function  $k \rightarrow \phi(xk)$  on  $K$  belongs to  $\delta$ . Then  $\delta$  is called a *spherical function of type  $\delta$*  on  $G$  (with respect to  $K$ ), cf. GODEMENT [19]. It is funny that spherical functions of type  $\delta$  are on the one hand generalizations of ordinary spherical functions for  $(G, K)$ , on the other hand restrictions to  $G$  of ordinary spherical functions for  $(G \times K, K^*)$ .

For convenience, we take a one-dimensional  $\delta \in \hat{K}$ . Then a spherical function  $\phi$  on  $G \times K$  is of type  $\delta$  iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \quad x \in G, k \in K.$$

Let

$$\begin{aligned} & I_{c, \delta}(G) \text{ (or } I_{c, \delta}^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(xk) = f(kx) \\ & \quad = \delta(k)f(x), x \in G, k \in K\}. \end{aligned}$$

These are closed subalgebras of  $I_c(G)$  (or  $I_c^\infty(G)$ ) and their characters are precisely of the form (5.1), where  $\phi$  is a spherical function of type  $\delta$ . Finally, if  $\tau$  is a  $K$ -unitary representation of  $G$  and if  $\mathcal{H}(\tau)$  contains a unit vector  $v$  satisfying  $\tau(k)v = \delta(k)v$ , unique up to a constant factor, then  $x \rightarrow (\tau(x)v, v)$  is a spherical function of type  $\delta$ .

### 5.3. THE GENERALIZED ABEL TRANSFORM

Let  $G$  be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures  $dk, da, dn$  on  $K, A, N$ , respectively, normalize the Haar measure on  $G$  such that

$$(5.2) \quad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)} dk da dn, \quad f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \quad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, \quad f \in C_c(N), a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).