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5.2. SPHERICAL FUNCTIONS OF TYPE δ

Let G be a unimodular lsc. group with compact subgroup K . Let

$$K^* := \{(k, k) \in G \times K \mid k \in K\}.$$

Let $\delta \in \hat{K}$ and let τ be a K -unitary representation of G . Then $\tau \otimes \check{\delta}$ ($\check{\delta}$ the contragredient representation to δ) is a K^* -unitary representation of $G \times K$ on $\mathcal{H}(\tau) \otimes \mathcal{H}(\delta)$.

LEMMA 5.1. *The multiplicity of δ in $\tau|_K$ is equal to the multiplicity of the representation 1 of K^* in $\tau \otimes \check{\delta}|_{K^*}$. τ is irreducible iff $\tau \otimes \check{\delta}$ is irreducible. τ is unitary iff $\tau \otimes \check{\delta}$ is unitary.*

This can be proved immediately. By using the results summarized in §5.1 we conclude that $(G \times K, K^*)$ is a Gelfand pair if there exists a continuous involutive homomorphism α on G such that for each $(g, k) \in G \times K$ we have $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$ for certain $k_1, k_2 \in K$. Furthermore, if $(G \times K, K^*)$ is a Gelfand pair and if the irreducible representation τ of G is unitary or K -finite then τ is K -multiplicity free. In particular, this applies to $SU(1, 1)$:

PROPOSITION 5.2. *If $G = SU(1, 1)$ then $(G \times K, K^*)$ is a Gelfand pair.*

Proof. For $g \in SU(1, 1)$ define $\alpha(g) := {}^t(g^{-1})$. Then α is a continuous involutive automorphism on G and $\alpha(a_t) = a_{-t}$ on A , $\alpha(u_\theta) = u_{-\theta}$ on K . Since $G = KAK$, α has the required properties. \square

Let $(G \times K, K^*)$ be a Gelfand pair. Identify $G \times \{e\}$ with G . A spherical function on $G \times K$ is completely determined by its restriction to G . By using the results mentioned in §5.1 we obtain the following properties. First, a continuous function ϕ on G is the restriction to G of a spherical function on $G \times K$ iff $\phi \neq 0$ and

$$\phi(x)\phi(y) = \int_K \phi(xkyk^{-1})dk, \quad x, y \in G.$$

Next, let

$$\begin{aligned} & I_c(G) \text{ (or } I_c^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(kgk^{-1}) = f(g), \\ & \quad g \in G, k \in K\}. \end{aligned}$$

These are commutative topological algebras under convolution and their characters are precisely of the form (5.1), where ϕ is a spherical function on $G \times K$. If ϕ is a spherical function on $G \times K$ then there is a $\delta \in \hat{K}$ such that for all $x \in G$ the function $k \rightarrow \phi(xk)$ on K belongs to δ . Then δ is called a *spherical function of type δ* on G (with respect to K), cf. GODEMENT [19]. It is funny that spherical functions of type δ are on the one hand generalizations of ordinary spherical functions for (G, K) , on the other hand restrictions to G of ordinary spherical functions for $(G \times K, K^*)$.

For convenience, we take a one-dimensional $\delta \in \hat{K}$. Then a spherical function ϕ on $G \times K$ is of type δ iff

$$\phi(xk) = \phi(kx) = \delta(k)\phi(x), \quad x \in G, k \in K.$$

Let

$$\begin{aligned} & I_{c, \delta}(G) \text{ (or } I_{c, \delta}^\infty(G)) \\ & := \{f \in C_c(G) \text{ (or } C_c^\infty(G)) \mid f(xk) = f(kx) \\ & \quad = \delta(k)f(x), x \in G, k \in K\}. \end{aligned}$$

These are closed subalgebras of $I_c(G)$ (or $I_c^\infty(G)$) and their characters are precisely of the form (5.1), where ϕ is a spherical function of type δ . Finally, if τ is a K -unitary representation of G and if $\mathcal{H}(\tau)$ contains a unit vector v satisfying $\tau(k)v = \delta(k)v$, unique up to a constant factor, then $x \rightarrow (\tau(x)v, v)$ is a spherical function of type δ .

5.3. THE GENERALIZED ABEL TRANSFORM

Let G be a connected noncompact real semisimple Lie group with finite center. Use the notation of §2.2. For given Haar measures dk, da, dn on K, A, N , respectively, normalize the Haar measure on G such that

$$(5.2) \quad \int_G f(g)dg = \int_{K \times A \times N} f(kan)e^{2\rho(\log a)} dk da dn, f \in C_c(G)$$

(cf. HELGASON [25, Ch. X, Prop. 1.11]). Note the property

$$(5.3) \quad \int_N f(n)dn = e^{2\rho(\log a)} \int_N f(ana^{-1})dn, f \in C_c(N), a \in A$$

(cf. [25, Ch. X, proof of Prop. 1.11]).