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**Autor:** Koornwinder, Tom H.  
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3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of  $\mathbf{R}^2$  and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of  $SL(2, \mathbf{R})$  and [41, p. 560, Cor. 2] for the spherical principal series of  $F_{4(-20)}$ .

3.3.6. The method of this section does not show in an *a priori* way that a  $K$ -multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

#### 4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

##### 4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of  $K$ -multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let  $G$  be an lcsc. group.

*Definition 4.1.* Let  $\sigma$  and  $\tau$  be Hilbert representations of  $G$ . The representation  $\sigma$  is called *Naimark related* to  $\tau$  if there is a closed (possibly unbounded) injective linear operator  $A$  from  $\mathcal{H}(\sigma)$  to  $\mathcal{H}(\tau)$  with domain  $\mathcal{D}(A)$  dense in  $\mathcal{H}(\sigma)$  and range  $\mathcal{R}(A)$  dense in  $\mathcal{H}(\tau)$  such that  $\mathcal{D}(A)$  is  $\sigma$ -invariant and  $A\sigma(g)v = \tau(g)Av$  for all  $v \in \mathcal{D}(A)$ ,  $g \in G$ . Then we use the notation  $\sigma \stackrel{A}{\simeq} \tau$  or  $\sigma \simeq \tau$ .

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of  $K$ -multiplicity free representations,  $K$  abelian.

Two unitary representations  $\sigma$  and  $\tau$  of  $G$  are called *unitarily equivalent* if there is an isometry  $A$  from  $\mathcal{H}(\sigma)$  onto  $\mathcal{H}(\tau)$  such that  $A\sigma(g)v = \tau(g)Av$  for all  $v \in \mathcal{H}(\sigma)$ ,  $g \in G$ . Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group  $G$  are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let  $K$  be a compact abelian subgroup of  $G$ . Let  $\sigma$  and  $\tau$  be  $K$ -multiplicity free representations of  $G$ . Let  $\{\phi_\delta\}$  and  $\{\psi_\delta\}$  be  $K$ -bases for  $\mathcal{H}(\sigma)$  and  $\mathcal{H}(\tau)$ , respectively.

LEMMA 4.3. *If  $\sigma \stackrel{A}{\simeq} \tau$  then  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ ,  $\phi_\delta \in \mathcal{D}(A)$  and  $\psi_\delta \in \mathcal{R}(A)$  ( $\delta \in \mathcal{M}(\sigma)$ ), and there are nonzero complex numbers  $c_\delta$  ( $\delta \in \mathcal{M}(\sigma)$ ) such that*

$$(4.1) \quad (Av, \psi_\delta) = c_\delta(v, \phi_\delta), \quad v \in \mathcal{D}(A).$$

*In particular*

$$(4.2) \quad A\phi_\delta = c_\delta\psi_\delta.$$

*Proof.* Let  $\delta \in \mathcal{M}(\sigma)$ . Let  $v \in \mathcal{D}(A)$ . We have, by the intertwining property of  $A$ ,

$$\begin{aligned} \int_K \delta(k^{-1})\sigma(k)vdk &= (v, \phi_\delta)\phi_\delta, \\ \int_K \delta(k^{-1})A\sigma(k)vdk &= \int_K \delta(k^{-1})\sigma(k)Avdk \\ &= \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases} \end{aligned}$$

Since  $A$  is closed, we conclude that  $(v, \phi_\delta)\phi_\delta \in \mathcal{D}(A)$  and

$$A((v, \phi_\delta)\phi_\delta) = \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases}$$

Since  $A$  is injective with dense domain, the left hand side is nonzero for certain  $v \in \mathcal{D}(A)$ . Hence  $\delta \in \mathcal{M}(\tau)$ ,  $\phi_\delta \in \mathcal{D}(A)$  and (4.2) and (4.1) hold for certain nonzero  $c_\delta$ . Finally, since  $A$  is closed with dense range,  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ .  $\square$

LEMMA 4.4. *Let  $A$  be a possibly unbounded, not necessarily closed, injective linear operator from  $\mathcal{H}(\sigma)$  to  $\mathcal{H}(\tau)$  which satisfies all other properties of Definition 4.1. Suppose that  $\phi_\delta \in \mathcal{D}(A)$  for all  $\delta \in \mathcal{M}(\sigma)$ ,  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$  and,*

for each  $\delta \in \mathcal{M}(\sigma)$ , there is a complex number  $c_\delta$  such that  $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$  for all  $v \in \mathcal{D}(A)$ . Then the closure  $\bar{A}$  of  $A$  is one-valued and injective,  $\bar{A}$  satisfies all properties of Definition 4.1 and

$$(4.3) \quad \mathcal{D}(\bar{A}) = \left\{ v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty \right\}.$$

*Proof.* Let  $\{v_n\}$  be a sequence in  $\mathcal{D}(A)$  such that  $v_n \rightarrow v$  in  $\mathcal{H}(\sigma)$  and  $Av_n \rightarrow w$  in  $\mathcal{H}(\tau)$ . Then, for each  $\delta \in \mathcal{M}(\sigma)$ ,

$$(w, \psi_\delta) = \lim_{n \rightarrow \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \rightarrow \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence  $v = 0$  iff  $w = 0$ , so  $\bar{A}$  is one-valued and injective.

To prove the domain invariance and intertwining property of  $\bar{A}$ , let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \rightarrow v, Av_n \rightarrow \bar{A}v$$

for some sequence  $\{v_n\}$  in  $\mathcal{D}(A)$ . If  $g \in G$  then

$$\sigma(g)v_n \rightarrow \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \rightarrow \tau(g)\bar{A}v,$$

so  $\sigma(g)v \in \mathcal{D}(\bar{A})$  and  $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$ .

Finally, to prove (4.3), first suppose that  $v \in \mathcal{H}(\sigma)$  and

$$\sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$\begin{aligned} v &= \sum (v, \phi_\delta)\phi_\delta, w := \sum c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta \\ &= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}). \end{aligned}$$

Conversely, let  $v \in \mathcal{D}(\bar{A})$ . Then  $\bar{A}v = \sum (\bar{A}v, \psi_\delta)\psi_\delta = \sum c_\delta(v, \phi_\delta)\psi_\delta$  (note  $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$  by (4.1)). Hence  $\sum |c_\delta(v, \phi_\delta)|^2 < \infty$ .  $\square$

Next we will prove a criterium for Naimark relatedness of  $K$ -multiplicity free representations  $\sigma$  and  $\tau$  in terms of the canonical matrix elements.

**THEOREM 4.5.** *Let  $G$  be an lcsc. group with compact abelian subgroup  $K$ . Let  $\sigma$  and  $\tau$  be  $K$ -multiplicity free representations of  $G$ . Let  $\{\phi_\delta\}$  and  $\{\psi_\delta\}$  be  $K$ -bases of  $\mathcal{H}(\sigma)$  and  $\mathcal{H}(\tau)$ , respectively. For each  $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$  let  $0 \neq c_\delta \in \mathbb{C}$ . Then the following two statements are equivalent:*

(a)  $\sigma \stackrel{A}{\simeq} \tau$  and  $A\phi_\delta = c_\delta\psi_\delta$ ,  $\delta \in \mathcal{M}(\sigma)$ .

(b)  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$  and, for all  $\gamma, \delta \in \mathcal{M}(\sigma)$ ,

$$(4.4) \quad \tau_{\gamma, \delta} = C_{\gamma, \delta} \sigma_{\gamma, \delta}$$

with  $C_{\gamma, \delta} = c_\gamma/c_\delta$ .

If, moreover,  $\sigma$  and  $\tau$  are irreducible then (a) and (b) are also equivalent to :

(c) For some  $\gamma, \delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$  (4.4) holds for some nonzero complex  $C_{\gamma, \delta}$ .

*Proof.*

(a)  $\Rightarrow$  (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma). \end{aligned}$$

(b)  $\Rightarrow$  (a): Define  $A$  on the domain  $\{v \in \mathcal{H}(\sigma) \mid \sum |c_\delta(v, \phi_\delta)|^2 < \infty\}$  by  $Av := \sum c_\delta(v, \phi_\delta)\psi_\delta$ . Then  $A$  is injective with dense domain and range and  $A$  satisfies (4.1). We will prove that  $\mathcal{D}(A)$  is  $G$ -invariant and that  $A$  is an intertwining operator. Let  $v \in \mathcal{D}(A)$ ,  $g \in G$ . Then, by (4.4) and the definition of  $Av$ :

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma \sum_\delta (v, \phi_\delta) \sigma_{\gamma, \delta}(g) \\ &= \sum_\delta c_\delta (v, \phi_\delta) \tau_{\gamma, \delta}(g) = (\tau(g)Av, \psi_\gamma). \end{aligned}$$

Hence

$$\sum_\gamma |c_\gamma(\sigma(g)v, \phi_\gamma)|^2 = \|\tau(g)Av\|^2 < \infty.$$

So  $\sigma(g)v \in \mathcal{D}(A)$  and  $A\sigma(g)v = \tau(g)Av$ . Now apply Lemma 4.4.

(c)  $\Rightarrow$  (b): ( $\sigma, \tau$  irreducible): We will first show that  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$  and, for each  $\beta \in \mathcal{M}(\sigma)$ ,  $\tau_{\gamma, \beta} = C_{\gamma, \beta} \sigma_{\gamma, \beta}$  and  $\tau_{\beta, \delta} = C_{\beta, \delta} \sigma_{\beta, \delta}$  for some nonzero complex  $C_{\gamma, \beta}$  and  $C_{\beta, \delta}$ . It follows from (4.4) evaluated for  $g = g_1 k g_2$  that

$$\begin{aligned} &\sum_{\beta \in \mathcal{M}(\tau)} \beta(k) \tau_{\gamma, \beta}(g_1) \tau_{\beta, \delta}(g_2) \\ &= C_{\delta, \gamma} \sum_{\beta \in \mathcal{M}(\sigma)} \beta(k) \sigma_{\gamma, \beta}(g_1) \sigma_{\beta, \delta}(g_2), \quad g_1, g_2 \in G, k \in K. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in  $k \in K$ . Because of Theorem 3.2 and the irreducibility of  $\sigma$  and  $\tau$ , for each  $\beta \in \mathcal{M}(\tau)$

respectively  $\beta \in \mathcal{M}(\sigma)$  the Fourier coefficient at the left respectively right hand side does not vanish identically in  $g_1, g_2$ . Hence  $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$  and

$$\tau_{\gamma, \beta}(g_1)\tau_{\beta, \delta}(g_2) = C_{\gamma, \delta}\sigma_{\gamma, \beta}(g_1)\sigma_{\beta, \delta}(g_2).$$

This implies

$$\tau_{\gamma, \beta} = C_{\gamma, \beta}\sigma_{\gamma, \beta} \text{ and } \tau_{\beta, \delta} = C_{\beta, \delta}\sigma_{\beta, \delta} \text{ with } C_{\gamma, \beta}C_{\beta, \delta} = C_{\gamma, \delta}.$$

By repeating this argument we prove that  $\tau_{\alpha, \beta} = C_{\alpha, \beta}\sigma_{\alpha, \beta}$  for all  $\alpha, \beta \in \mathcal{M}(\sigma)$  and that  $C_{\alpha, \beta}C_{\beta, \delta} = C_{\alpha, \delta}$ , i.e.  $C_{\alpha, \beta} = C_{\alpha, \delta}/C_{\beta, \delta}$ .  $\square$

**COROLLARY 4.6.** *Let  $G$  be an lcsc. group with compact abelian subgroup  $K$ . Then Naimark relatedness is an equivalence relation in the class of  $K$ -multiplicity free representations of  $G$ .*

#### 4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of  $\pi_{\xi, \lambda}$  as classified in Theorem 3.4. By comparing  $K$ -contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi, \lambda} \simeq \pi_{\xi, \mu}(\lambda + \xi, \mu + \xi \notin \mathbf{Z} + \frac{1}{2}, \lambda \neq \mu)$$

and

$$\begin{aligned} \pi_{\xi, \lambda}^+ &\simeq \pi_{\xi, -\lambda}^+, & \pi_{\xi, \lambda}^0 &\simeq \pi_{\xi, -\lambda}^0, & \pi_{\xi, \lambda}^- &\simeq \pi_{\xi, -\lambda}^- \\ & & & & & (\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0). \end{aligned}$$

Suppose that  $\sigma$  and  $\tau$  are irreducible subquotient representations of  $\pi_{\xi, \lambda}$  and  $\pi_{\xi, \mu}$ , respectively, and that  $\phi_m \in \mathcal{H}(\sigma) \cap \mathcal{H}(\tau)$  for some  $m \in \mathbf{Z} + \xi$ . It follows from Theorem 4.5 that  $\sigma \simeq \tau$  iff  $\tau_{\xi, \lambda, m, m} = \pi_{\xi, \mu, m, m}$ . This last identity already holds if it is valid for the restrictions to  $A$ . In view of (2.29) and (2.30) we have:  $\sigma \simeq \tau$  iff

$$(4.5) \quad \phi_{2i\lambda}^{(0, 2m)}(t) = \phi_{2i\mu}^{(0, 2m)}(t), \quad t \in \mathbf{R}.$$

Formula (4.5) holds if  $\lambda = \pm\mu$  (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in  $-(sh t)^2$  by using (2.23) and (2.20). The coefficients of  $-(sh t)^2$  yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence  $\lambda = \pm\mu$ . We have proved: