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3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of $\mathbf{R}^2$ and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of $SL(2, \mathbf{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.

3.3.6. The method of this section does not show in an *a priori* way that a K-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lcsc. group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called *Naimark related* to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ-invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \underset{A}{\simeq} \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathscr{H}(\sigma)$ onto $\mathscr{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathscr{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

PROPOSITION 4.2. *Two unitary representations of an lcsc. group G are Naimark related if and only if they are unitarily equivalent.*

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G. Let σ and τ be K-multiplicity free representations of G. Let $\{\phi_\delta\}$ and $\{\psi_\delta\}$ be K-bases for $\mathscr{H}(\sigma)$ and $\mathscr{H}(\tau)$, respectively.

LEMMA 4.3. *If* $\sigma \overset{A}{\simeq} \tau$ *then* $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ *and* $\psi_\delta \in \mathscr{R}(A)$ $(\delta \in \mathscr{M}(\sigma))$, *and there are nonzero complex numbers* $c_\delta(\delta \in \mathscr{M}(\sigma))$ *such that*

$$(Av, \psi_\delta) = c_\delta(v, \phi_\delta), \quad v \in \mathscr{D}(A). \tag{4.1}$$

In particular

$$A\phi_\delta = c_\delta \psi_\delta. \tag{4.2}$$

Proof. Let $\delta \in \mathscr{M}(\sigma)$. Let $v \in \mathscr{D}(A)$. We have, by the intertwining property of A,

$$\int_K \delta(k^{-1})\sigma(k)v\,dk = (v, \phi_\delta)\phi_\delta,$$

$$\int_K \delta(k^{-1})A\sigma(k)v\,dk = \int_K \delta(k^{-1})\sigma(k)Av\,dk$$

$$= \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is closed, we conclude that $(v, \phi_\delta)\phi_\delta \in \mathscr{D}(A)$ and

$$A((v, \phi_\delta)\phi_\delta) = \begin{cases} (Av, \psi_\delta)\psi_\delta & \text{if } \delta \in \mathscr{M}(\tau), \\ 0 & \text{if } \delta \notin \mathscr{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathscr{D}(A)$. Hence $\delta \in \mathscr{M}(\tau)$, $\phi_\delta \in \mathscr{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_δ. Finally, since A is closed with dense range, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$. □

LEMMA 4.4. *Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that* $\phi_\delta \in \mathscr{D}(A)$ *for all* $\delta \in \mathscr{M}(\sigma)$, $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ *and,* *for each* $\delta \in \mathcal{M}(\sigma)$, *there is a complex number* c_δ *such that* $(Av, \psi_\delta) = c_\delta(v, \phi_\delta)$ *for all* $v \in \mathcal{D}(A)$. *Then the closure* $\bar{A}$ *of* A *is one-valued and injective,* $\bar{A}$ *satisfies all properties of Definition* 4.1 *and*

$$\mathcal{D}(\bar{A}) = \{v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty\}. \tag{4.3}$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \to v$ in $\mathcal{H}(\sigma)$ and $Av_n \to w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_\delta) = \lim_{n \to \infty} (Av_n, \psi_\delta) = c_\delta \lim_{n \to \infty} (v_n, \phi_\delta) = c_\delta(v, \phi_\delta).$$

Hence $v = 0$ iff $w = 0$, so $\bar{A}$ is one-valued and injective.

To prove the domain invariance and intertwining property of $\bar{A}$, let

$$v \in \mathcal{D}(\bar{A}), \text{ so } v_n \to v,\ Av_n \to \bar{A}v$$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \to \sigma(g)v \text{ and } A\sigma(g)v_n = \tau(g)Av_n \to \tau(g)\bar{A}v,$$

so $\sigma(g)v \in \mathcal{D}(\bar{A})$ and $\bar{A}\sigma(g)v = \tau(g)\bar{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\Sigma_{\delta \in \mathcal{M}(\sigma)} |c_\delta(v, \phi_\delta)|^2 < \infty.$$

Then

$$v = \Sigma(v, \phi_\delta)\phi_\delta,\ w := \Sigma c_\delta(v, \phi_\delta)\psi_\delta \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_\delta$$
$$= c_\delta\psi_\delta, \text{ so, } w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A}).$$

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \Sigma(\bar{A}v, \psi_\delta)\psi_\delta = \Sigma c_\delta(v, \phi_\delta)\psi_\delta$ (note $(\bar{A}v, \psi_\delta) = c_\delta(v, \phi_\delta)$ by (4.1)). Hence $\Sigma\,|c_\delta(v, \phi_\delta)|^2 < \infty$. □

Next we will prove a criterium for Naimark relatedness of K-multiplicity free representations σ and τ in terms of the canonical matrix elements.

THEOREM 4.5. *Let* G *be an lcsc. group with compact abelian subgroup* K. *Let* σ *and* τ *be* K*-multiplicity free representations of* G. *Let* $\{\phi_\delta\}$ *and* $\{\psi_\delta\}$ *be* K*-bases of* $\mathcal{H}(\sigma)$ *and* $\mathcal{H}(\tau)$, *respectively. For each* $\delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ *let* $0 \neq c_\delta \in \mathbf{C}$. *Then the following two statements are equivalent:*

(a) $\sigma \overset{A}{\simeq} \tau$ *and* $A\phi_\delta = c_\delta \psi_\delta, \delta \in \mathcal{M}(\sigma)$.

(b) $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ *and, for all* $\gamma, \delta \in \mathcal{M}(\sigma)$,

$$\tau_{\gamma,\delta} = C_{\gamma,\delta}\sigma_{\gamma,\delta} \tag{4.4}$$

with $C_{\gamma,\delta} = c_\gamma/c_\delta$.

If, moreover, σ *and* τ *are irreducible then* (a) *and* (b) *are also equivalent to:*

(c) *For some* $\gamma, \delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ (4.4) *holds for some nonzero complex* $C_{\gamma,\delta}$.

Proof.

(a) $\Rightarrow$ (b): Apply Lemma 4.3. By using (4.1) we have

$$\begin{aligned} c_\gamma(\sigma(g)\phi_\delta, \phi_\gamma) &= (A\sigma(g)\phi_\delta, \psi_\gamma) = (\tau(g)A\phi_\delta, \psi_\gamma) \\ &= c_\delta(\tau(g)\psi_\delta, \psi_\gamma). \end{aligned}$$

(b) $\Rightarrow$ (a): Define A on the domain $\{v \in \mathcal{H}(\sigma) \mid \Sigma \mid c_\delta(v, \phi_\delta)\mid^2 < \infty\}$ by $Av := \Sigma c_\delta(v, \phi_\delta)\psi_\delta$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathcal{D}(A)$ is G-invariant and that A is an intertwining operator. Let $v \in \mathcal{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av:

$$\begin{aligned} c_\gamma(\sigma(g)v, \phi_\gamma) &= c_\gamma \Sigma_\delta (v, \phi_\delta)\sigma_{\gamma,\delta}(g) \\ &= \Sigma_\delta c_\delta (v, \phi_\delta)\tau_{\gamma,\delta}(g) = (\tau(g)Av, \psi_\gamma). \end{aligned}$$

Hence

$$\Sigma_\gamma \mid c_\gamma(\sigma(g)v, \phi_\gamma)\mid^2 = \| \tau(g)Av \|^2 < \infty .$$

So $\sigma(g)v \in \mathcal{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

(c) $\Rightarrow$ (b): (σ, τ irreducible): We will first show that $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for each $\beta \in \mathcal{M}(\sigma)$, $\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta}$ and $\tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta}$ for some nonzero complex $C_{\gamma,\beta}$ and $C_{\beta,\delta}$. It follows from (4.4) evaluated for $g = g_1 k g_2$ that

$$\begin{aligned} &\sum_{\beta \in \mathcal{M}(\tau)} \beta(k)\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) \\ &= C_{\delta,\gamma} \sum_{\beta \in \mathcal{M}(\sigma)} \beta(k)\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2), \quad g_1, g_2 \in G, k \in K. \end{aligned}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ, for each $\beta \in \mathcal{M}(\tau)$

respectively $\beta \in \mathscr{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2. Hence $\mathscr{M}(\sigma) = \mathscr{M}(\tau)$ and

$$\tau_{\gamma,\beta}(g_1)\tau_{\beta,\delta}(g_2) = C_{\gamma,\delta}\sigma_{\gamma,\beta}(g_1)\sigma_{\beta,\delta}(g_2).$$

This implies

$$\tau_{\gamma,\beta} = C_{\gamma,\beta}\sigma_{\gamma,\beta} \text{ and } \tau_{\beta,\delta} = C_{\beta,\delta}\sigma_{\beta,\delta} \text{ with } C_{\gamma,\beta}C_{\beta,\delta} = C_{\gamma,\delta}.$$

By repeating this argument we prove that $\tau_{\alpha,\beta} = C_{\alpha,\beta}\sigma_{\alpha,\beta}$ for all $\alpha, \beta \in \mathscr{M}(\sigma)$ and that $C_{\alpha,\beta}C_{\beta,\delta} = C_{\alpha,\delta}$, i.e. $C_{\alpha,\beta} = C_{\alpha,\delta}/C_{\beta,\delta}$. □

COROLLARY 4.6. *Let G be an lcsc. group with compact abelian subgroup K. Then Naimark relatedness is an equivalence relation in the class of K-multiplicity free representations of G.*

4.2. THE CASE $SU(1, 1)$

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\xi,\lambda} \simeq \pi_{\xi,\mu}(\lambda+\xi,\ \mu+\xi \notin \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq \mu)$$

and

$$\pi^{+}_{\xi,\lambda} \simeq \pi^{+}_{\xi,-\lambda},\quad \pi^{0}_{\xi,\lambda} \simeq \pi^{0}_{\xi,-\lambda},\quad \pi^{-}_{\xi,\lambda} \simeq \pi^{-}_{\xi,-\lambda}$$
$$(\lambda+\xi \in \mathbf{Z}+\tfrac{1}{2},\ \lambda \neq 0).$$

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi,\lambda}$ and $\pi_{\xi,\mu}$, respectively, and that $\phi_m \in \mathscr{H}(\sigma) \cap \mathscr{H}(\tau)$ for some $m \in \mathbf{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi,\lambda,m,m} = \pi_{\xi,\mu,m,m}$. This last identity already holds if it is valid for the restrictions to A. In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

$$\phi^{(0,\,2m)}_{2i\lambda}(t) = \phi^{(0,\,2m)}_{2i\mu}(t), \qquad t \in \mathbf{R}. \tag{4.5}$$

Formula (4.5) holds if $\lambda = \pm\mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh\ t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh\ t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm\mu$. We have proved:

THEOREM 4.7. *Let* σ *and* $\tau(\sigma\neq\tau)$ *be irreducible subquotient representations of the principal series. Then* σ *is Naimark equivalent to* τ *in precisely the following situations (cf. the notation of Theorem 3.4):*

$$\text{(a)}\qquad \pi_{\xi,\lambda}\simeq\pi_{\xi,-\lambda}(\lambda+\xi\notin\mathbf{Z}+\tfrac{1}{2},\ \lambda\neq 0)$$

$$\text{(b)}\qquad \pi^{+}_{\xi,\lambda}\simeq\pi^{+}_{\xi,-\lambda},\ \pi^{0}_{\xi,\lambda}\simeq\pi^{0}_{\xi,-\lambda},\ \pi^{-}_{\xi,\lambda}\simeq\pi^{-}_{\xi,-\lambda}\ (\lambda+\xi\in\mathbf{Z}+\tfrac{1}{2},\ \lambda\neq 0).$$

Remark 4.8. It follows from Theorem 3.4 and Theorem 4.7 that each irreducible subquotient representation of some $\pi_{\xi,\lambda}$ is Naimark equivalent to some irreducible subrepresentation of some $\pi_{\xi,\lambda}$.

It follows from Theorems 4.7 and 4.5 that for each $\xi\in\{0,\frac{1}{2}\}$ and $\lambda\in\mathbf{C}\backslash\{0\}$ we have identities

$$\pi_{\xi,-\lambda,m,n}=C_{\xi,\lambda,m,n}\,\pi_{\xi,\lambda,m,n}\tag{4.6}$$

for certain nonzero complex constants $C_{\xi,\lambda,m,n}$, where $m,n\in\mathbf{Z}+\xi$ and, if $\lambda+\xi\in\mathbf{Z}+\frac{1}{2}$, we have the further restriction that $m,n\in(-\infty,-|\lambda|-\frac{1}{2}]$ or $m,n\in[-|\lambda|+\frac{1}{2},|\lambda|-\frac{1}{2}]$ or $m,n\in[|\lambda|+\frac{1}{2},\infty)$. Indeed, it follows from (2.29) and (2.26) that (4.6) holds with

$$C_{\xi,\lambda,m,n}=\frac{c_{\xi,-\lambda,m,n}}{c_{\xi,\lambda,m,n}}.\tag{4.7}$$

A calculation using (4.7) and (2.30) shows that

$$C_{\xi,\lambda,m,n}=c_{\xi,\lambda,m}/c_{\xi,\lambda,n}\tag{4.8}$$

with

$$\begin{aligned}c_{\xi,\lambda,m}&=\text{const.}\,\frac{\Gamma(-\lambda+m+\frac{1}{2})}{\Gamma(\lambda+m+\frac{1}{2})}=\text{const.}\,\frac{\Gamma(-\lambda-m+\frac{1}{2})}{\Gamma(\lambda-m+\frac{1}{2})}\\&=\text{const.}\,(-1)^{m-\xi}\,\Gamma(-\lambda+m+\tfrac{1}{2})\,\Gamma(-\lambda-m+\tfrac{1}{2})\\&=\text{const.}\,\frac{(-1)^{m-\xi}}{\Gamma(\lambda+m+\frac{1}{2})\,\Gamma(\lambda-m+\frac{1}{2})}.\end{aligned}\tag{4.9}$$

If $\lambda+\xi\notin\mathbf{Z}+\frac{1}{2}$ then we can use all alternatives for $c_{\xi,\lambda,m}$, but if $\lambda+\xi\in\mathbf{Z}+\frac{1}{2}$ then we can use precisely one alternative. Now, by Theorem 4.5, we obtain:

PROPOSITION 4.9. *Let* $\sigma\overset{A}{\simeq}\tau$ *be one of the equivalences of Theorem 4.7 with* σ *being a subquotient representation of* $\pi_{\xi,\lambda}$. *Then*

(4.10) $$A\phi_m = c_{\xi,\lambda,m}\,\phi_m\,,$$

where $m \in \mathbf{Z} + \xi$ *such that* $\delta_m \in \mathcal{M}(\sigma)$ *and* $c_{\xi,\lambda,m}$ *is given by (4.9).*

4.3. Notes

4.3.1. Definition 4.1 of Naimark relatedness goes back to Naimark [33]. He introduced this concept in the context of representations of the Lorentz group on a reflexive Banach space. Next he gave a much more involved definition in his book [34, Ch. 3, §9, No. 3]. Afterwards, many different versions of this definition appeared in literature, which all refer to [34]. We mention Zelobenko & Naimark [51, Def. 2] ("weak equivalence" for representations on locally convex spaces), Fell [13, §6] (Naimark relatedness for "linear system representations") and Warner [48, p. 232 and p. 242]. Warner starts with the definition of Naimark relatedness for Banach representations of an associative algebra over $\mathbf{C}$ (this definition is similar to our Definition 4.1) and next he defines Naimark relatedness for Banach representations of an lcsc. group G in terms of Naimark relatedness for the corresponding representations of $M_c(G)$ or (equivalently) $C_c(G)$. Warner's definition seems to be standard now. Poulsen [35, Def. 33] gives Naimark's original definition [33] and he calls it weak equivalence. Fell [13] (see also Warner [48, Theorem 4.5.5.2]) proved that, for K-finite Banach representations of a connected unimodular Lie group, two representations are Naimark related iff they are infinitesimally equivalent.

4.3.2. Our implication (*c*) $\Rightarrow$ (*a*) in Theorem 4.5 is related to Wallach [44, Cor. 2.1]. Theorem 4.7 can be formulated for general semisimple Lie groups G. If $\pi_{\xi,\lambda}$ is an irreducible principal series representation and if $s \in W$ then $\pi_{\xi,\lambda} \simeq \pi_{\xi^s,\, s\cdot\lambda}$ (cf. Wallach [44, Theorem 3.1]). This yields part (a). Regarding part (b) see Lepowsky's [29, Theorem 9.8] result that $\pi_{\xi,\lambda}$ and $\pi_{\xi^s,\, s\cdot\lambda}$ have equivalent composition series.

4.3.3. Theorem 4.7 was first proved in the unitarizable cases by Bargmann [2]. He used infinitesimal methods. Takahashi [39] proved Theorem 4.7 (again in the unitarizable cases) by calculating the diagonal matrix elements $\pi_{\xi,\lambda,m,n}(a_t)$ and by observing that they are even in λ. Gelfand, Graev & Vilenkin [17, Ch. VII, §4] obtained Theorem 4.7 by working in the noncompact realization of the principal series and by explicitly constructing all possible intertwining operators.

4.3.4. Analogues of the results in §4.1 hold for nonabelian K and (in Lemmas 4.3, 4.4 and Corollary 4.6) for K-finite representations, cf. [27, §4].