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- 3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of \mathbb{R}^2 and for the harmonic oscillator group, Takahashi [39, §3.4] for the discrete series of $SL(2, \mathbb{R})$ and [41, p. 560, Cor. 2] for the spherical principal series of $F_{4(-20)}$.
- 3.3.6. The method of this section does not show in an *a priori* way that a *K*-multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. Wallach [45, Theorem 8.13.3].

4. Equivalences between irreducible subquotient representations of the principal series

4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of K-multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let G be an lese, group.

Definition 4.1. Let σ and τ be Hilbert representations of G. The representation σ is called Naimark related to τ if there is a closed (possibly) unbounded) injective linear operator A from $\mathscr{H}(\sigma)$ to $\mathscr{H}(\tau)$ with domain $\mathscr{D}(A)$ dense in $\mathscr{H}(\sigma)$ and range $\mathscr{R}(A)$ dense in $\mathscr{H}(\tau)$ such that $\mathscr{D}(A)$ is σ -invariant and $A\sigma(g)v = \tau(G)Av$ for all $v \in \mathscr{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\sigma \simeq \tau$.

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of K-multiplicity free representations, K abelian.

Two unitary representations σ and τ of G are called *unitarily equivalent* if there is an isometry A from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$ such that $A\sigma(g)v = \tau(g)Av$ for all $v \in \mathcal{H}(\sigma)$, $g \in G$. Clearly unitary equivalence is an equivalence relation.

Proposition 4.2. Two unitary representations of an lesse, group G are Naimark related if and only if they are unitarily equivalent.

See WARNER [48, Prop. 4.3.1.4] for the proof.

Let K be a compact abelian subgroup of G. Let σ and τ be K-multiplicity free representations of G. Let $\{\phi_{\delta}\}$ and $\{\psi_{\delta}\}$ be K-bases for $\mathscr{H}(\sigma)$ and $\mathscr{H}(\tau)$, respectively.

LEMMA 4.3. If $\sigma \stackrel{A}{\simeq} \tau$ then $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$, $\phi_{\delta} \in \mathcal{D}(A)$ and $\psi_{\delta} \in \mathcal{R}(A)$ $(\delta \in \mathcal{M}(\sigma))$, and there are nonzero complex numbers $c_{\delta}(\delta \in \mathcal{M}(\sigma))$ such that

$$(4.1) (Av, \psi_{\delta}) = c_{\delta}(v, \phi_{\delta}), v \in \mathcal{D}(A).$$

In particular

$$(4.2) A \phi_{\delta} = c_{\delta} \psi_{\delta}.$$

Proof. Let $\delta \in \mathcal{M}(\sigma)$. Let $v \in \mathcal{D}(A)$. We have, by the intertwining property of A,

$$\int_{K} \delta(k^{-1}) \sigma(k) v dk = (v, \phi_{\delta}) \phi_{\delta},$$

$$\int_{K} \delta(k^{-1}) A \sigma(k) v dk = \int_{K} \delta(k^{-1}) \sigma(k) A v dk$$

$$= \begin{cases} (Av, \psi_{\delta}) \psi_{\delta} & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases}$$

Since A is closed, we conclude that $(v, \phi_{\delta})\phi_{\delta} \in \mathcal{D}(A)$ and

$$A((v, \phi_{\delta})\phi_{\delta}) = \begin{cases} (Av, \psi_{\delta})\psi_{\delta} & \text{if } \delta \in \mathcal{M}(\tau), \\ 0 & \text{if } \delta \notin \mathcal{M}(\tau). \end{cases}$$

Since A is injective with dense domain, the left hand side is nonzero for certain $v \in \mathcal{D}(A)$. Hence $\delta \in \mathcal{M}(\tau)$, $\phi_{\delta} \in \mathcal{D}(A)$ and (4.2) and (4.1) hold for certain nonzero c_{δ} . Finally, since A is closed with dense range, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$.

Lemma 4.4. Let A be a possibly unbounded, not necessarily closed, injective linear operator from $\mathcal{H}(\sigma)$ to $\mathcal{H}(\tau)$ which satisfies all other properties of Definition 4.1. Suppose that $\phi_{\delta} \in \mathcal{D}(A)$ for all $\delta \in \mathcal{M}(\sigma)$, $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and,

for each $\delta \in \mathcal{M}(\sigma)$, there is a complex number c_{δ} such that $(Av, \psi_{\delta}) = c_{\delta}(v, \phi_{\delta})$ for all $v \in \mathcal{D}(A)$. Then the closure \bar{A} of A is one-valued and injective, \bar{A} satisfies all properties of Definition 4.1 and

$$\mathcal{D}(\bar{A}) = \left\{ v \in \mathcal{H}(\sigma) \mid \sum_{\delta \in \mathcal{M}(\sigma)} |c_{\delta}(v, \phi_{\delta})|^{2} < \infty \right\}.$$

Proof. Let $\{v_n\}$ be a sequence in $\mathcal{D}(A)$ such that $v_n \to v$ in $\mathcal{H}(\sigma)$ and $Av_n \to w$ in $\mathcal{H}(\tau)$. Then, for each $\delta \in \mathcal{M}(\sigma)$,

$$(w, \psi_{\delta}) = \lim_{n \to \infty} (Av_n, \psi_{\delta}) = c_{\delta} \lim_{n \to \infty} (v_n, \phi_{\delta}) = c_{\delta}(v, \phi_{\delta}).$$

Hence v = 0 iff w = 0, so \bar{A} is one-valued and injective.

To prove the domain invariance and intertwining property of \bar{A} , let

$$v \in \mathcal{D}(\bar{A})$$
, so $v_n \to v$, $Av_n \to \bar{A}v$

for some sequence $\{v_n\}$ in $\mathcal{D}(A)$. If $g \in G$ then

$$\sigma(g)v_n \to \sigma(g)v$$
 and $A\sigma(g)v_n = \tau(g)Av_n \to \tau(g)\bar{A}v$,

so $\sigma(g)v \in \mathcal{D}(\overline{A})$ and $\overline{A}\sigma(g)v = \tau(g)\overline{A}v$.

Finally, to prove (4.3), first suppose that $v \in \mathcal{H}(\sigma)$ and

$$\sum_{\delta \in \mathcal{M}(\sigma)} |c_{\delta}(v, \phi_{\delta})|^2 < \infty$$
.

Then

$$v = \Sigma(v, \phi_{\delta})\phi_{\delta}, w := \Sigma c_{\delta}(v, \phi_{\delta})\psi_{\delta} \in \mathcal{H}(\tau) \text{ and } \bar{A}\phi_{\delta}$$

= $c_{\delta}\psi_{\delta}$, so, $w = \bar{A}v \text{ and } v \in \mathcal{D}(\bar{A})$.

Conversely, let $v \in \mathcal{D}(\bar{A})$. Then $\bar{A}v = \Sigma(\bar{A}v, \psi_{\delta})\psi_{\delta} = \Sigma c_{\delta}(v, \phi_{\delta})\psi_{\delta}$ (note $(\bar{A}v, \psi_{\delta}) = c_{\delta}(v, \phi_{\delta})$ by (4.1)). Hence $\Sigma |c_{\delta}(v, \phi_{\delta})|^2 < \infty$.

Next we will prove a criterium for Naimark relatedness of K-multiplicity free representations σ and τ in terms of the canonical matrix elements.

Theorem 4.5. Let G be an lcsc. group with compact abelian subgroup K. Let σ and τ be K-multiplicity free representations of G. Let $\{\varphi_{\delta}\}$ and $\{\psi_{\delta}\}$ be K-bases of $\mathscr{H}(\sigma)$ and $\mathscr{H}(\tau)$, respectively. For each $\delta \in \mathscr{M}(\sigma) \cap \mathscr{M}(\tau)$ let $0 \neq c_{\delta} \in \mathbb{C}$. Then the following two statements are equivalent:

(a)
$$\sigma \stackrel{A}{\simeq} \tau$$
 and $A\phi_{\delta} = c_{\delta}\psi_{\delta}, \delta \in \mathcal{M}(\sigma)$.

(b) $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for all $\gamma, \delta \in \mathcal{M}(\sigma)$,

$$\tau_{\gamma, \delta} = C_{\gamma, \delta} \sigma_{\gamma, \delta}$$

with $C_{\gamma,\delta} = c_{\gamma}/c_{\delta}$.

If, moreover, σ and τ are irreducible then (a) and (b) are also equivalent to:

- (c) For some $\gamma, \delta \in \mathcal{M}(\sigma) \cap \mathcal{M}(\tau)$ (4.4) holds for some nonzero complex $C_{\gamma, \delta}$.

 Proof.
 - (a) \Rightarrow (b): Apply Lemma 4.3. By using (4.1) we have

$$c_{\gamma}(\sigma(g)\phi_{\delta}, \phi_{\gamma}) = (A\sigma(g)\phi_{\delta}, \psi_{\gamma}) = (\tau(g)A\phi_{\delta}, \psi_{\gamma})$$

= $c_{\delta}(\tau(g)\psi_{\delta}, \psi_{\gamma})$.

(b) \Rightarrow (a): Define A on the domain $\{v \in \mathcal{H}(\sigma) \mid \Sigma \mid c_{\delta}(v, \phi_{\delta}) \mid^{2} < \infty\}$ by $Av := \Sigma c_{\delta}(v, \phi_{\delta})\psi_{\delta}$. Then A is injective with dense domain and range and A satisfies (4.1). We will prove that $\mathcal{D}(A)$ is G-invariant and that A is an intertwining operator. Let $v \in \mathcal{D}(A)$, $g \in G$. Then, by (4.4) and the definition of Av:

$$c_{\gamma}(\sigma(g)v, \, \varphi_{\gamma}) = c_{\gamma} \Sigma_{\delta}(v, \, \varphi_{\delta}) \sigma_{\gamma, \, \delta}(g)$$

= $\Sigma_{\delta} c_{\delta}(v, \, \varphi_{\delta}) \tau_{\gamma, \, \delta}(g) = (\tau(g) A v, \, \psi_{\gamma}).$

Hence

$$\Sigma_{\gamma} | c_{\gamma}(\sigma(g)v, \phi_{\gamma}) |^2 = || \tau(g)Av ||^2 < \infty.$$

So $\sigma(g)v \in \mathcal{D}(A)$ and $A\sigma(g)v = \tau(g)Av$. Now apply Lemma 4.4.

 $\underline{(c) \Rightarrow (b)}$: $(\sigma, \tau \text{ irreducible})$: We will first show that $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and, for each $\beta \in \mathcal{M}(\sigma)$, $\tau_{\gamma, \beta} = C_{\gamma, \beta}\sigma_{\gamma, \beta}$ and $\tau_{\beta, \delta} = C_{\beta, \delta}\sigma_{\beta, \delta}$ for some nonzero complex $C_{\gamma, \beta}$ and $C_{\beta, \delta}$. It follows from (4.4) evaluated for $g = g_1kg_2$ that

$$\begin{split} &\sum_{\beta \in \mathcal{M} \ (\tau)} \beta(k) \tau_{\gamma, \ \beta}(g_1) \tau_{\beta, \ \delta}(g_2) \\ &= C_{\delta, \ \gamma} \sum_{\beta \in \mathcal{M} \ (\sigma)} \beta(k) \sigma_{\gamma, \ \beta}(g_1) \sigma_{\beta, \ \delta}(g_2) \ , \quad g_1, \ g_2 \in G, \ k \in K \ . \end{split}$$

Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Because of Theorem 3.2 and the irreducibility of σ and τ , for each $\beta \in \mathcal{M}(\tau)$ respectively $\beta \in \mathcal{M}(\sigma)$ the Fourier coefficient at the left respectively right hand side does not vanish identically in g_1, g_2 . Hence $\mathcal{M}(\sigma) = \mathcal{M}(\tau)$ and

$$\tau_{\gamma, \beta}(g_1)\tau_{\beta, \delta}(g_2) = C_{\gamma, \delta}\sigma_{\gamma, \beta}(g_1)\sigma_{\beta, \delta}(g_2).$$

This implies

$$\tau_{\gamma, \beta} = C_{\gamma, \beta} \sigma_{\gamma, \beta}$$
 and $\tau_{\beta, \delta} = C_{\beta, \delta} \sigma_{\beta, \delta}$ with $C_{\gamma, \beta} C_{\beta, \delta} = C_{\gamma, \delta}$.

By repeating this argument we prove that $\tau_{\alpha, \beta} = C_{\alpha, \beta} \sigma_{\alpha, \beta}$ for all $\alpha, \beta \in \mathcal{M}(\sigma)$ and that $C_{\alpha, \beta} C_{\beta, \delta} = C_{\alpha, \delta}$, i.e. $C_{\alpha, \beta} = C_{\alpha, \delta} / C_{\beta, \delta}$.

COROLLARY 4.6. Let G be an lcsc. group with compact abelian subgroup K. Then Naimark relatedness is an equivalence relation in the class of K-multiplicity free representations of G.

4.2. The case SU(1, 1)

Consider irreducible subquotient representations of $\pi_{\xi,\lambda}$ as classified in Theorem 3.4. By comparing K-contents it follows that the only possible nontrivial Naimark equivalences are:

$$\pi_{\epsilon,\lambda} \simeq \pi_{\epsilon,\mu}(\lambda + \xi, \mu + \xi \notin \mathbb{Z} + \frac{1}{2}, \lambda \neq \mu)$$

and

Suppose that σ and τ are irreducible subquotient representations of $\pi_{\xi, \lambda}$ and $\pi_{\xi, \mu}$, respectively, and that $\phi_m \in \mathcal{H}(\sigma) \cap \mathcal{H}(\tau)$ for some $m \in \mathbb{Z} + \xi$. It follows from Theorem 4.5 that $\sigma \simeq \tau$ iff $\tau_{\xi, \lambda, m, m} = \pi_{\xi, \mu, m, m}$. This last identity already holds if it is valid for the restrictions to A. In view of (2.29) and (2.30) we have: $\sigma \simeq \tau$ iff

(4.5)
$$\phi_{2i\lambda}^{(0, 2m)}(t) = \phi_{2i\mu}^{(0, 2m)}(t), \quad t \in \mathbf{R}.$$

Formula (4.5) holds if $\lambda = \pm \mu$ (cf. (2.26)). Conversely, assume (4.5) and expand both sides of (4.5) as a power series in $-(sh\ t)^2$ by using (2.23) and (2.20). The coefficients of $-(sh\ t)^2$ yield the equality

$$(m+1+\lambda)(m+1-\lambda) = (m+1+\mu)(m+1-\mu)$$

Hence $\lambda = \pm \mu$. We have proved:

Theorem 4.7. Let σ and $\tau(\sigma \neq \tau)$ be irreducible subquotient representations of the principal series. Then σ is Naimark equivalent to τ in precisely the following situations (cf. the notation of Theorem 3.4):

(a)
$$\pi_{\xi,\lambda} \simeq \pi_{\xi,-\lambda}(\lambda + \xi \notin \mathbb{Z} + \frac{1}{2}, \lambda \neq 0)$$

$$(b) \qquad \quad \pi_{\xi,\;\lambda}^{+} \; \simeq \; \pi_{\xi,\;-\lambda}^{+}, \, \pi_{\xi,\;\lambda}^{0} \; \simeq \; \pi_{\xi,\;-\lambda}^{0}, \, \pi_{\xi,\;\lambda}^{-} \; \simeq \; \pi_{\xi,\;-\lambda}^{-} \; \; (\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \, \lambda \neq 0) \; .$$

Remark 4.8. It follows from Theorem 3.4 and Theorem 4.7 that each irreducible subquotient representation of some $\pi_{\xi, \lambda}$ is Naimark equivalent to some irreducible subrepresentation of some $\pi_{\xi, \lambda}$.

It follows from Theorems 4.7 and 4.5 that for each $\xi \in \{0, \frac{1}{2}\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ we have identities

$$\pi_{\xi, -\lambda, m, n} = C_{\xi, \lambda, m, n} \pi_{\xi, \lambda, m, n}$$

for certain nonzero complex constants $C_{\xi, \lambda, m, n}$, where $m, n \in \mathbb{Z} + \xi$ and, if $\lambda + \xi \in \mathbb{Z} + \frac{1}{2}$, we have the further restriction that $m, n \in (-\infty, -|\lambda| - \frac{1}{2}]$ or $m, n \in [-|\lambda| + \frac{1}{2}, |\lambda| - \frac{1}{2}]$ or $m, n \in [|\lambda| + \frac{1}{2}, \infty)$. Indeed, it follows from (2.29) and (2.26) that (4.6) holds with

(4.7)
$$C_{\xi, \lambda, m, n} = \frac{C_{\xi, -\lambda, m, n}}{C_{\xi, \lambda, m, n}}.$$

A calculation using (4.7) and (2.30) shows that

$$(4.8) C_{\xi, \lambda, m, n} = c_{\xi, \lambda, m}/c_{\xi, \lambda, n}$$

with

(4.9)
$$c_{\xi, \lambda, m} = \text{const.} \frac{\Gamma(-\lambda + m + \frac{1}{2})}{\Gamma(\lambda + m + \frac{1}{2})} = \text{const.} \frac{\Gamma(-\lambda - m + \frac{1}{2})}{\Gamma(\lambda - m + \frac{1}{2})}$$
$$= \text{const.} (-1)^{m-\xi} \Gamma(-\lambda + m + \frac{1}{2}) \Gamma(-\lambda - m + \frac{1}{2})$$
$$= \text{const.} \frac{(-1)^{m-\xi}}{\Gamma(\lambda + m + \frac{1}{2}) \Gamma(\lambda - m + \frac{1}{2})}.$$

If $\lambda + \xi \notin \mathbb{Z} + \frac{1}{2}$ then we can use all alternatives for $c_{\xi, \lambda, m}$, but if $\lambda + \xi \in \mathbb{Z} + \frac{1}{2}$ then we can use precisely one alternative. Now, by Theorem 4.5, we obtain:

PROPOSITION 4.9. Let $\sigma \simeq \tau$ be one of the equivalences of Theorem 4.7 with σ being a subquotient representation of $\pi_{\xi,\lambda}$. Then

$$(4.10) A \phi_m = c_{\xi, \lambda, m} \phi_m,$$

where $m \in \mathbb{Z} + \xi$ such that $\delta_m \in \mathcal{M}(\sigma)$ and $c_{\xi, \lambda, m}$ is given by (4.9).

4.3. Notes

- Definition 4.1 of Naimark relatedness goes back to Naimark [33]. He introduced this concept in the context of representations of the Lorentz group on a reflexive Banach space. Next he gave a much more involved definition in his book [34, Ch. 3, §9, No. 3]. Afterwards, many different versions of this definition appeared in literature, which all refer to [34]. We mention ZELOBENKO & NAIMARK [51, Def. 2] ("weak equivalence" for representations on locally convex spaces), Fell [13, §6] (Naimark relatedness for "linear system representations") and WARNER [48, p. 232 and p. 242]. Warner starts with the definition of Naimark relatedness for Banach representations of an associative algebra over C (this definition is similar to our Definition 4.1) and next he defines Naimark relatedness for Banach representations of an lcsc. group G in terms of Naimark relatedness for the corresponding representations of $M_c(G)$ or (equivalently) $C_c(G)$. Warner's definition seems to be standard now. Poulsen [35, Def. 33] gives Naimark's original definition [33] and he calls it weak equivalence. Fell [13] (see also Warner [48, Theorem 4.5.5.2]) proved that, for K-finite Banach representations of a connected unimodular Lie group, two representations are Naimark related iff they are infinitesimally equivalent.
- 4.3.2. Our implication $(c) \Rightarrow (a)$ in Theorem 4.5 is related to Wallach [44, Cor. 2.1]. Theorem 4.7 can be formulated for general semisimple Lie groups G. If $\pi_{\xi,\lambda}$ is an irreducible principal series representation and if $s \in W$ then $\pi_{\xi,\lambda} \simeq \pi_{\xi^s,s\cdot\lambda}$ (cf. Wallach [44, Theorem 3.1]). This yields part (a). Regarding part (b) see Lepowsky's [29, Theorem 9.8] result that $\pi_{\xi,\lambda}$ and $\pi_{\xi^s,s\cdot\lambda}$ have equivalent composition series.
- 4.3.3. Theorem 4.7 was first proved in the unitarizable cases by Bargmann [2]. He used infinitesimal methods. Takahashi [39] proved Theorem 4.7 (again in the unitarizable cases) by calculating the diagonal matrix elements $\pi_{\xi, \lambda, m, n}(a_t)$ and by observing that they are even in λ . Gelfand, Graev & Vilenkin [17, Ch. VII, §4] obtained Theorem 4.7 by working in the noncompact realization of the principal series and by explicitly constructing all possible intertwining operators.
- 4.3.4. Analogues of the results in $\S4.1$ hold for nonabelian K and (in Lemmas 4.3, 4.4 and Corollary 4.6) for K-finite representations, cf. [27, $\S4$].