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Artikel: THE REPRESENTATION THEORY OF SL(2, R), A NON-INFINITESIMAL APPROACH
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and

$$\tau(k)v = \tau_0(k)v.$$

(b) ⇒ (a): Let $0 \neq v \in \mathscr{H}(\tau_0)$. Let $\mathscr{H}_1$ be the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by v. Then $\phi_\gamma \in \mathscr{H}_1$ for some $\gamma \in \mathscr{M}(\tau_0)$. Now, for some $g \in G$,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0,$$

so $\tau_0(g)\phi_\gamma$ and ϕ_δ are in $\mathscr{H}_1$. For each $\beta \in \mathscr{M}(\tau_0)$ we have $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$ for some $g \in G$. Thus $\phi_\beta \in \mathscr{H}_1$ for all $\beta \in \mathscr{M}(\tau_0)$, so $\mathscr{H}_1 = \mathscr{H}(\tau_0)$.

(a) ⇒ (c): Suppose $\tau_{\gamma\delta} = 0$ for some $\gamma, \delta \in \mathscr{M}(\tau_0)$. Then, for all $g \in G$, $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$. Hence, the τ_0-invariant subspace of $\mathscr{H}(\tau_0)$ generated by ϕ_δ is orthogonal to ϕ_γ, so τ_0 is not irreducible.

(c) ⇒ (b): Clear. □

Let τ be K-multiplicity free, K being compact abelian. Define a relation $\prec$ on $\mathscr{M}(\tau)$ by: $\gamma \prec \delta$ iff $\tau_{\gamma,\delta} \neq 0$. Then $\gamma \prec \delta$ iff ϕ_γ is in the τ-invariant subspace of $\mathscr{H}(\tau)$ generated by ϕ_δ. It follows that

$$\beta \prec \gamma \text{ and } \gamma \prec \delta \Rightarrow \beta \prec \delta$$

Define a relation $\sim$ on $\mathscr{M}(\tau)$ by: $\gamma \sim \delta$ iff $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$. It follows that $\sim$ is an equivalence relation on $\mathscr{M}(\tau)$ and that, if $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$ then $\tau_{\alpha,\beta} \neq 0$. It follows that, for a given equivalence set, we can partition $\mathscr{M}(\tau)$ into three parts, the first part being the equivalence set, such that the corresponding 3×3 block matrix for $(\tau_{\gamma\delta}(g))$ takes the form (3.3). In view of Theorem 3.2 this proves:

THEOREM 3.3. *Let G be a lcsc. group with compact abelian subgroup K and let τ be a K-multiplicity free representation of G. Then there is a unique orthogonal decomposition of $\mathscr{H}(\tau)$ into subspaces $\mathscr{H}(\tau_i)$, where the τ_i's are precisely the irreducible subquotient representations of τ.*

3.2. THE CASE $SU(1, 1)$

For $\lambda \in \mathbf{C}$, $\xi = 0$ or $\frac{1}{2}$, the representation $\pi_{\xi,\lambda}$ of $G = SU(1, 1)$ on $L^2_\xi(K)$ (cf. (2.8)) is K-multiplicity free with K-content given by (2.13). By inspecting (2.29) for small but nonzero t and by using (2.24) it follows that

$$(3.4)\qquad \pi_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow \pi_{\xi,\lambda,m,n}|_A \neq 0 \Leftrightarrow c_{\xi,\lambda,m,n} \neq 0\,,$$

where $c_{\xi,\lambda,m,n}$ is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

THEOREM 3.4. *Depending on* ξ *and* λ, *the representation* $\pi_{\xi,\lambda}$ *of* $SU(1, 1)$ *has the following irreducible subquotient representations:*

(a) $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$:

$\pi_{\xi,\lambda}$ *is irreducible itself.*

(b) $\lambda = 0, \xi = \frac{1}{2}$:

$\pi^+_{1/2,0}$ *on* $\mathrm{Cl\,Span}\,\{\phi_{1/2}, \phi_{3/2}, \ldots\}$,
$\pi^-_{1/2,0}$ *on* $\mathrm{Cl\,Span}\,\{\ldots, \phi_{-3/2}, \phi_{-1/2}\}$.

These are also subrepresentations.

(c) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0$:

$\pi^+_{\xi,\lambda}$ *on* $\mathrm{Cl\,Span}\,\{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \ldots\}$,
$\pi^-_{\xi,\lambda}$ *on* $\mathrm{Cl\,Span}\,\{\ldots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$,
$\pi^0_{\xi,\lambda}$ *on* $\mathrm{Span}\,\{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \ldots, \phi_{\lambda-1/2}\}$.

Among these $\pi^+_{\xi,\lambda}$ *and* $\pi^-_{\xi,\lambda}$ *are subrepresentations.*

(d) $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0$:

$\pi^+_{\xi,\lambda}$ *on* $\mathrm{Cl\,Span}\,\{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \ldots\}$,
$\pi^-_{\xi,\lambda}$ *on* $\mathrm{Cl\,Span}\,\{\ldots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$,
$\pi_{\xi,\lambda}$ *on* $\mathrm{Span}\,\{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \ldots, \phi_{-\lambda-1/2}\}$.

Among these $\pi^0_{\xi,\lambda}$ *is a subrepresentation.*

Proof.

(a) $c_{\xi,\lambda,m,n} \neq 0$.

(b) $c_{1/2,0,m,n} \neq 0 \Leftrightarrow m, n \leqslant -\frac{1}{2}$ or $m, n \geqslant \frac{1}{2}$.

(c) $c_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leqslant n \leqslant \lambda - \frac{1}{2}$ or $m, n \leqslant -\lambda - \frac{1}{2}$ or $m, n \geqslant \lambda + \frac{1}{2}$.

Thus $c_{\xi,\lambda,m,n}$ has block matrix

$$\begin{array}{l|ccc} & n\leqslant -\lambda-\frac{1}{2} & -\lambda+\frac{1}{2}\leqslant n\leqslant \lambda-\frac{1}{2} & n\geqslant \lambda+\frac{1}{2} \\ \hline m\leqslant -\lambda-\frac{1}{2} & * & * & 0 \\ -\lambda+\frac{1}{2}\leqslant m\leqslant \lambda-\frac{1}{2} & 0 & * & 0 \\ m\geqslant \lambda+\frac{1}{2} & 0 & * & * \end{array}$$

where each starred block has all entries nonzero.

(d) $c_{\xi,\lambda,m,n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leqslant m \leqslant -\lambda - \frac{1}{2}$ or $m, n \leqslant \lambda - \frac{1}{2}$ or $m, n > -\lambda + \frac{1}{2}$. □

The finite-dimensional representation occurring in the above classification are the representations $\pi^0_{\xi,\lambda}(\lambda+\xi\in\mathbf{Z}+\frac{1}{2}, \lambda\neq 0)$.

3.3. Notes

3.3.1. In the case of the unitary principal series (λ imaginary), Theorem 3.4 was first proved by Bargmann [2, sections 6 and 7]. See van Dijk [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by Barut & Phillips [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero λ is contained in a general theorem by Bruhat [5, Theorem 7; 2]: For $\xi \in \hat{M}$, $\lambda \in i\mathfrak{a}$, the principal series representation $\pi_{\xi,\lambda}$ of G (cf. (2.2)) is irreducible if $s \cdot \lambda \neq \lambda$ for all $s \neq e$ in the Weyl group for (G, K).

3.3.3. Gelfand & Naimark [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for $SL(2, \mathbf{C})$ by a global method different from ours, working in a noncompact realization and calculating the "matrix elements" of the representation with respect to a (continuous) $\overline{N}$-basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian K, cf. [27, Theorem 3.3]. In that case the canonical matrix elements $\tau_{\gamma,\delta}$ are matrix-valued functions. By using this method, Naimark [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for $SL(2, \mathbf{C})$, see also Kosters [28].