

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 28 (1982)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE REPRESENTATION THEORY OF  $SL(2, \mathbb{R})$ , A NON-INFINITESIMAL APPROACH  
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**Kapitel:** 3.2. The case  $SU(1, 1)$   
**DOI:** <https://doi.org/10.5169/seals-52233>

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and

$$\tau(k)v = \tau_0(k)v.$$

(b)  $\Rightarrow$  (a): Let  $0 \neq v \in \mathcal{H}(\tau_0)$ . Let  $\mathcal{H}_1$  be the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $v$ . Then  $\phi_\gamma \in \mathcal{H}_1$  for some  $\gamma \in \mathcal{M}(\tau_0)$ . Now, for some  $g \in G$ ,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0,$$

so  $\tau_0(g)\phi_\gamma$  and  $\phi_\delta$  are in  $\mathcal{H}_1$ . For each  $\beta \in \mathcal{M}(\tau_0)$  we have  $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$  for some  $g \in G$ . Thus  $\phi_\beta \in \mathcal{H}_1$  for all  $\beta \in \mathcal{M}(\tau_0)$ , so  $\mathcal{H}_1 = \mathcal{H}(\tau_0)$ .

(a)  $\Rightarrow$  (c): Suppose  $\tau_{\gamma\delta} = 0$  for some  $\gamma, \delta \in \mathcal{M}(\tau_0)$ . Then, for all  $g \in G$ ,  $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$ . Hence, the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $\phi_\delta$  is orthogonal to  $\phi_\gamma$ , so  $\tau_0$  is not irreducible.

(c)  $\Rightarrow$  (b): Clear. □

Let  $\tau$  be  $K$ -multiplicity free,  $K$  being compact abelian. Define a relation  $<$  on  $\mathcal{M}(\tau)$  by:  $\gamma < \delta$  iff  $\tau_{\gamma,\delta} \neq 0$ . Then  $\gamma < \delta$  iff  $\phi_\gamma$  is in the  $\tau$ -invariant subspace of  $\mathcal{H}(\tau)$  generated by  $\phi_\delta$ . It follows that

$$\beta < \gamma \text{ and } \gamma < \delta \Rightarrow \beta < \delta$$

Define a relation  $\sim$  on  $\mathcal{M}(\tau)$  by:  $\gamma \sim \delta$  iff  $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$ . It follows that  $\sim$  is an equivalence relation on  $\mathcal{M}(\tau)$  and that, if  $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$  then  $\tau_{\alpha,\beta} \neq 0$ . It follows that, for a given equivalence set, we can partition  $\mathcal{M}(\tau)$  into three parts, the first part being the equivalence set, such that the corresponding  $3 \times 3$  block matrix for  $(\tau_{\gamma\delta}(g))$  takes the form (3.3). In view of Theorem 3.2 this proves:

**THEOREM 3.3.** *Let  $G$  be a lcsc. group with compact abelian subgroup  $K$  and let  $\tau$  be a  $K$ -multiplicity free representation of  $G$ . Then there is a unique orthogonal decomposition of  $\mathcal{H}(\tau)$  into subspaces  $\mathcal{H}(\tau_i)$ , where the  $\tau_i$ 's are precisely the irreducible subquotient representations of  $\tau$ .*

### 3.2. THE CASE $SU(1, 1)$

For  $\lambda \in \mathbb{C}$ ,  $\xi = 0$  or  $\frac{1}{2}$ , the representation  $\pi_{\xi,\lambda}$  of  $G = SU(1, 1)$  on  $L^2_\xi(K)$  (cf. (2.8)) is  $K$ -multiplicity free with  $K$ -content given by (2.13). By inspecting (2.29) for small but nonzero  $t$  and by using (2.24) it follows that

$$(3.4) \quad \pi_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \pi_{\xi, \lambda, m, n}|_A \neq 0 \Leftrightarrow c_{\xi, \lambda, m, n} \neq 0,$$

where  $c_{\xi, \lambda, m, n}$  is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

**THEOREM 3.4.** *Depending on  $\xi$  and  $\lambda$ , the representation  $\pi_{\xi, \lambda}$  of  $SU(1, 1)$  has the following irreducible subquotient representations:*

(a)  $\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}$ :

$\pi_{\xi, \lambda}$  is irreducible itself.

(b)  $\lambda = 0, \xi = \frac{1}{2}$ :

$\pi_{1/2, 0}^+$  on  $\text{Cl Span } \{\phi_{1/2}, \phi_{3/2}, \dots\}$ ,

$\pi_{1/2, 0}^-$  on  $\text{Cl Span } \{\dots, \phi_{-3/2}, \phi_{-1/2}\}$ .

*These are also subrepresentations.*

(c)  $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0$ :

$\pi_{\xi, \lambda}^+$  on  $\text{Cl Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots\}$ ,

$\pi_{\xi, \lambda}^-$  on  $\text{Cl Span } \{\dots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$ ,

$\pi_{\xi, \lambda}^0$  on  $\text{Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots, \phi_{\lambda-1/2}\}$ .

*Among these  $\pi_{\xi, \lambda}^+$  and  $\pi_{\xi, \lambda}^-$  are subrepresentations.*

(d)  $\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0$ :

$\pi_{\xi, \lambda}^+$  on  $\text{Cl Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots\}$ ,

$\pi_{\xi, \lambda}^-$  on  $\text{Cl Span } \{\dots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$ ,

$\pi_{\xi, \lambda}$  on  $\text{Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots, \phi_{-\lambda-1/2}\}$ .

*Among these  $\pi_{\xi, \lambda}^0$  is a subrepresentation.*

*Proof.*

(a)  $c_{\xi, \lambda, m, n} \neq 0$ .

(b)  $c_{1/2, 0, m, n} \neq 0 \Leftrightarrow m, n \leq -\frac{1}{2}$  or  $m, n \geq \frac{1}{2}$ .

(c)  $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2}$   
or  $m, n \leq -\lambda - \frac{1}{2}$  or  $m, n \geq \lambda + \frac{1}{2}$ .

Thus  $c_{\xi, \lambda, m, n}$  has block matrix

$$\begin{array}{ccc}
 n \leq -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2} & n \geq \lambda + \frac{1}{2} \\
 \begin{array}{l} m \leq -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leq m \leq \lambda - \frac{1}{2} \\ m \geq \lambda + \frac{1}{2} \end{array} & \left( \begin{array}{ccc} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{array} \right) & 
 \end{array}$$

where each starred block has all entries nonzero.

- (d)  $c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leq m \leq -\lambda - \frac{1}{2}$  or  $m, n \leq \lambda - \frac{1}{2}$   
 or  $m, n > -\lambda + \frac{1}{2}$ . □

The finite-dimensional representation occurring in the above classification are the representations  $\pi_{\xi, \lambda}^0(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$ .

### 3.3. NOTES

3.3.1. In the case of the unitary principal series ( $\lambda$  imaginary), Theorem 3.4 was first proved by BARGMANN [2, sections 6 and 7]. See van DIJK [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by BARUT & PHILLIPS [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero  $\lambda$  is contained in a general theorem by BRUHAT [5, Theorem 7; 2]: For  $\xi \in \hat{M}$ ,  $\lambda \in ia$ , the principal series representation  $\pi_{\xi, \lambda}$  of  $G$  (cf. (2.2)) is irreducible if  $s \cdot \lambda \neq \lambda$  for all  $s \neq e$  in the Weyl group for  $(G, K)$ .

3.3.3. GELFAND & NAIMARK [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for  $SL(2, \mathbf{C})$  by a global method different from ours, working in a noncompact realization and calculating the “matrix elements” of the representation with respect to a (continuous)  $\overline{N}$ -basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian  $K$ , cf. [27, Theorem 3.3]. In that case the canonical matrix elements  $\tau_{\gamma, \delta}$  are matrix-valued functions. By using this method, NAIMARK [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for  $SL(2, \mathbf{C})$ , see also KOSTERS [28].