

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 28 (1982)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE REPRESENTATION THEORY OF  $SL(2, \mathbb{R})$ , A NON-INFINITESIMAL APPROACH  
**Autor:** Koornwinder, Tom H.  
**Kapitel:** 3.1. Subquotient representations  
**DOI:** <https://doi.org/10.5169/seals-52233>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 10.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

of rank 1 (i.e.,  $\dim(A) = 1$ ) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order  $(\alpha, \beta)$ ,  $\alpha \geq \beta \geq -\frac{1}{2}$ . This research was continued in several papers by Flensted-Jensen and the author.

### 3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

#### 3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let  $G$  be a lcsc. group and let  $\tau$  be a Hilbert representation of  $G$ . Let  $\mathcal{H}_0$  be a closed subspace of  $\mathcal{H}(\tau)$  and let  $P_0$  be the orthogonal projection from  $\mathcal{H}(\tau)$  onto  $\mathcal{H}_0$ . Define

$$(3.1) \quad \tau_0(g)v := P_0\tau(g)v, \quad g \in G, v \in \mathcal{H}_0.$$

Then  $\tau(g) \in \mathcal{L}(\mathcal{H}_0)$  for each  $g \in G$ ,  $\tau_0(e) = id.$ , and  $g \rightarrow \tau_0(g)v : G \rightarrow \mathcal{H}_0$  is continuous for each  $v \in \mathcal{H}_0$ . If also

$$(3.2) \quad \tau_0(g_1g_2) = \tau_0(g_1)\tau_0(g_2), \quad g_1, g_2 \in G,$$

then  $\tau_0$  is a Hilbert representation of  $G$  on  $\mathcal{H}_0$  and it is called a *subquotient representation* of  $\tau$ . Formula (3.2) is clearly valid if  $\mathcal{H}_0$  is an *invariant subspace* of  $\mathcal{H}(\tau)$ , i.e., if  $\tau(g)v \in \mathcal{H}_0$  for all  $g \in G$ ,  $v \in \mathcal{H}_0$ . In that case,  $\tau_0$  is called a *subrepresentation* of  $\tau$ .

LEMMA 3.1. *Let  $\mathcal{H}_0$  be a closed subspace of  $\mathcal{H}(\tau)$ , let  $\mathcal{H}_2$  be the closed  $G$ -invariant subspace of  $\mathcal{H}(\tau)$  which is generated by  $\mathcal{H}_0$  and let  $\mathcal{H}_1 := \mathcal{H}_2 \cap \mathcal{H}_0^\perp$ . Then  $\tau_0$  is a subquotient representation if and only if  $\mathcal{H}_1$  is  $G$ -invariant.*

*Proof.* Let  $P_0$  and  $P_1$  denote the orthogonal projections on  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , respectively. It follows from (3.1) that

$$\begin{aligned} \tau_0(g_1 g_2)v &= \tau_0(g_1)\tau_0(g_2)v \\ &= P_0\tau(g_1)P_1\tau(g_2)v, \quad g_1, g_2 \in G, v \in \mathcal{H}_0. \end{aligned}$$

$\mathcal{H}_1$  is the closed linear span of all elements  $P_1\tau(g_2)v$ ,  $g_2 \in G$ ,  $v \in \mathcal{H}_0$ . So (3.2) holds iff  $P_0\tau(g_1)w = 0$  for all  $g_1 \in G$ ,  $w \in \mathcal{H}_1$ .  $\square$

Let  $K$  be a compact subgroup of  $G$  and suppose that  $\tau$  is  $K$ -unitary. Let  $\tau_0$  be a subquotient representation of  $\tau$  on  $\mathcal{H}_0$  and let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be as in Lemma 3.1. Then  $\mathcal{H}_2$  and  $\mathcal{H}_1$  are  $G$ -invariant subspaces, so  $\mathcal{H}_0 = \mathcal{H}_2 \cap \mathcal{H}_1^\perp$  is  $K$ -invariant. It follows that  $\tau_0$  is  $K$ -unitary and that  $\tau_0(k)v = \tau(k)v$ ,  $k \in K$ ,  $v \in \mathcal{H}_0$ . If  $K$  is compact abelian and if  $\tau$  is  $K$ -multiplicity free then  $\tau_0$  is also  $K$ -multiplicity free,  $\mathcal{M}(\tau_0) \subset \mathcal{M}(\tau)$  and  $\tau_{0,\gamma,\delta}(g) = \tau_{\gamma,\delta}(g)$  for  $\gamma, \delta \in \mathcal{M}(\tau_0)$ ,  $g \in G$ .

Let again  $K$  be a compact abelian subgroup of  $G$  and  $\tau$  a  $K$ -multiplicity free Hilbert representation of  $G$ . Let  $\mathcal{H}_0$  be a  $K$ -invariant closed subspace of  $\mathcal{H}(\tau)$ . Then, by Lemma 3.1,  $\tau_0$  defined by (3.1) is a subquotient representation if and only if we can partition the  $K$ -basis for  $\mathcal{H}(\tau)$  into three parts, the first part providing a basis for  $\mathcal{H}_0$ , such that, for each  $g \in G$ , the corresponding  $3 \times 3$  block matrix of  $(\tau_{\gamma\delta}(g))$  takes the form

$$(3.3) \quad \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} .$$

**THEOREM 3.2.** *Let  $K$  be a compact abelian subgroup of the lcsc. group  $G$  and let  $\tau$  be a  $K$ -multiplicity free Hilbert representation of  $G$ . Let  $\tau_0$  be a subquotient representation of  $\tau$ . Then the following three statements are equivalent :*

- (a)  $\tau_0$  is irreducible.
- (b) For some  $\delta \in \mathcal{M}(\tau_0)$  we have  $\tau_{\gamma\delta} \neq 0 \neq \tau_{\delta\gamma}$  for all  $\gamma \in \mathcal{M}(\tau_0)$ .
- (c) For all  $\gamma, \delta \in \mathcal{M}(\tau_0)$  we have  $\tau_{\gamma\delta} \neq 0$ .

*Proof.* First note: if  $v \in \mathcal{H}(\tau_0)$  and  $(v, \phi_\gamma) \neq 0$  for some  $\gamma \in \mathcal{M}(\tau_0)$  then  $\phi_\gamma$  (element of the  $K$ -basis) belongs to the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $v$ . Indeed,

$$(v, \phi_\gamma)\phi_\gamma = \int_K \gamma(k^{-1})\tau(k)v \, dv$$

and

$$\tau(k)v = \tau_0(k)v.$$

(b)  $\Rightarrow$  (a): Let  $0 \neq v \in \mathcal{H}(\tau_0)$ . Let  $\mathcal{H}_1$  be the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $v$ . Then  $\phi_\gamma \in \mathcal{H}_1$  for some  $\gamma \in \mathcal{M}(\tau_0)$ . Now, for some  $g \in G$ ,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0,$$

so  $\tau_0(g)\phi_\gamma$  and  $\phi_\delta$  are in  $\mathcal{H}_1$ . For each  $\beta \in \mathcal{M}(\tau_0)$  we have  $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$  for some  $g \in G$ . Thus  $\phi_\beta \in \mathcal{H}_1$  for all  $\beta \in \mathcal{M}(\tau_0)$ , so  $\mathcal{H}_1 = \mathcal{H}(\tau_0)$ .

(a)  $\Rightarrow$  (c): Suppose  $\tau_{\gamma\delta} = 0$  for some  $\gamma, \delta \in \mathcal{M}(\tau_0)$ . Then, for all  $g \in G$ ,  $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$ . Hence, the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $\phi_\delta$  is orthogonal to  $\phi_\gamma$ , so  $\tau_0$  is not irreducible.

(c)  $\Rightarrow$  (b): Clear. □

Let  $\tau$  be  $K$ -multiplicity free,  $K$  being compact abelian. Define a relation  $\prec$  on  $\mathcal{M}(\tau)$  by:  $\gamma \prec \delta$  iff  $\tau_{\gamma,\delta} \neq 0$ . Then  $\gamma \prec \delta$  iff  $\phi_\gamma$  is in the  $\tau$ -invariant subspace of  $\mathcal{H}(\tau)$  generated by  $\phi_\delta$ . It follows that

$$\beta \prec \gamma \text{ and } \gamma \prec \delta \Rightarrow \beta \prec \delta$$

Define a relation  $\sim$  on  $\mathcal{M}(\tau)$  by:  $\gamma \sim \delta$  iff  $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$ . It follows that  $\sim$  is an equivalence relation on  $\mathcal{M}(\tau)$  and that, if  $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$  then  $\tau_{\alpha,\beta} \neq 0$ . It follows that, for a given equivalence set, we can partition  $\mathcal{M}(\tau)$  into three parts, the first part being the equivalence set, such that the corresponding  $3 \times 3$  block matrix for  $(\tau_{\gamma\delta}(g))$  takes the form (3.3). In view of Theorem 3.2 this proves:

**THEOREM 3.3.** *Let  $G$  be a lcsc. group with compact abelian subgroup  $K$  and let  $\tau$  be a  $K$ -multiplicity free representation of  $G$ . Then there is a unique orthogonal decomposition of  $\mathcal{H}(\tau)$  into subspaces  $\mathcal{H}(\tau_i)$ , where the  $\tau_i$ 's are precisely the irreducible subquotient representations of  $\tau$ .*

### 3.2. THE CASE $SU(1, 1)$

For  $\lambda \in \mathbb{C}$ ,  $\xi = 0$  or  $\frac{1}{2}$ , the representation  $\pi_{\xi,\lambda}$  of  $G = SU(1, 1)$  on  $L^2_\xi(K)$  (cf. (2.8)) is  $K$ -multiplicity free with  $K$ -content given by (2.13). By inspecting (2.29) for small but nonzero  $t$  and by using (2.24) it follows that