

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 28 (1982)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** THE REPRESENTATION THEORY OF  $SL(2, \mathbb{R})$ , A NON-INFINITESIMAL APPROACH  
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**Kapitel:** 3. The irreducible subquotient representations of the principal series  
**DOI:** <https://doi.org/10.5169/seals-52233>

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of rank 1 (i.e.,  $\dim(A) = 1$ ) can be written as Jacobi functions of certain order (cf. HARISH-CHANDRA [23, §13]). This motivated FLENSTED-JENSEN [14] to study harmonic analysis for Jacobi function expansions of quite general order  $(\alpha, \beta)$ ,  $\alpha \geq \beta \geq -\frac{1}{2}$ . This research was continued in several papers by Flensted-Jensen and the author.

### 3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

#### 3.1. SUBQUOTIENT REPRESENTATIONS

We start with the definition and some general properties and next derive an irreducibility criterium (Theorem 3.2) and a decomposition theorem 3.3.

Let  $G$  be a lcsc. group and let  $\tau$  be a Hilbert representation of  $G$ . Let  $\mathcal{H}_0$  be a closed subspace of  $\mathcal{H}(\tau)$  and let  $P_0$  be the orthogonal projection from  $\mathcal{H}(\tau)$  onto  $\mathcal{H}_0$ . Define

$$(3.1) \quad \tau_0(g)v := P_0\tau(g)v, \quad g \in G, v \in \mathcal{H}_0.$$

Then  $\tau(g) \in \mathcal{L}(\mathcal{H}_0)$  for each  $g \in G$ ,  $\tau_0(e) = id.$ , and  $g \rightarrow \tau_0(g)v: G \rightarrow \mathcal{H}_0$  is continuous for each  $v \in \mathcal{H}_0$ . If also

$$(3.2) \quad \tau_0(g_1g_2) = \tau_0(g_1)\tau_0(g_2), \quad g_1, g_2 \in G,$$

then  $\tau_0$  is a Hilbert representation of  $G$  on  $\mathcal{H}_0$  and it is called a *subquotient representation* of  $\tau$ . Formula (3.2) is clearly valid if  $\mathcal{H}_0$  is an *invariant subspace* of  $\mathcal{H}(\tau)$ , i.e., if  $\tau(g)v \in \mathcal{H}_0$  for all  $g \in G$ ,  $v \in \mathcal{H}_0$ . In that case,  $\tau_0$  is called a *subrepresentation* of  $\tau$ .

LEMMA 3.1. Let  $\mathcal{H}_0$  be a closed subspace of  $\mathcal{H}(\tau)$ , let  $\mathcal{H}_2$  be the closed  $G$ -invariant subspace of  $\mathcal{H}(\tau)$  which is generated by  $\mathcal{H}_0$  and let  $\mathcal{H}_1 := \mathcal{H}_2 \cap \mathcal{H}_0^\perp$ . Then  $\tau_0$  is a subquotient representation if and only if  $\mathcal{H}_1$  is  $G$ -invariant.

*Proof.* Let  $P_0$  and  $P_1$  denote the orthogonal projections on  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , respectively. It follows from (3.1) that

$$\begin{aligned} & \tau_0(g_1 g_2)v - \tau_0(g_1)\tau_0(g_2)v \\ &= P_0 \tau(g_1) P_1 \tau(g_2)v, \quad g_1, g_2 \in G, v \in \mathcal{H}_0. \end{aligned}$$

$\mathcal{H}_1$  is the closed linear span of all elements  $P_1 \tau(g_2)v$ ,  $g_2 \in G$ ,  $v \in \mathcal{H}_0$ . So (3.2) holds iff  $P_0 \tau(g_1)w = 0$  for all  $g_1 \in G$ ,  $w \in \mathcal{H}_1$ .  $\square$

Let  $K$  be a compact subgroup of  $G$  and suppose that  $\tau$  is  $K$ -unitary. Let  $\tau_0$  be a subquotient representation of  $\tau$  on  $\mathcal{H}_0$  and let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be as in Lemma 3.1. Then  $\mathcal{H}_2$  and  $\mathcal{H}_1$  are  $G$ -invariant subspaces, so  $\mathcal{H}_0 = \mathcal{H}_2 \cap \mathcal{H}_1^\perp$  is  $K$ -invariant. It follows that  $\tau_0$  is  $K$ -unitary and that  $\tau_0(k)v = \tau(k)v$ ,  $k \in K$ ,  $v \in \mathcal{H}_0$ . If  $K$  is compact abelian and if  $\tau$  is  $K$ -multiplicity free then  $\tau_0$  is also  $K$ -multiplicity free,  $\mathcal{M}(\tau_0) \subset \mathcal{M}(\tau)$  and  $\tau_{0, \gamma, \delta}(g) = \tau_{\gamma, \delta}(g)$  for  $\gamma, \delta \in \mathcal{M}(\tau_0)$ ,  $g \in G$ .

Let again  $K$  be a compact abelian subgroup of  $G$  and  $\tau$  a  $K$ -multiplicity free Hilbert representation of  $G$ . Let  $\mathcal{H}_0$  be a  $K$ -invariant closed subspace of  $\mathcal{H}(\tau)$ . Then, by Lemma 3.1,  $\tau_0$  defined by (3.1) is a subquotient representation if and only if we can partition the  $K$ -basis for  $\mathcal{H}(\tau)$  into three parts, the first part providing a basis for  $\mathcal{H}_0$ , such that, for each  $g \in G$ , the corresponding  $3 \times 3$  block matrix of  $(\tau_{\gamma\delta}(g))$  takes the form

$$(3.3) \quad \begin{pmatrix} * & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}.$$

**THEOREM 3.2.** *Let  $K$  be a compact abelian subgroup of the lcsc. group  $G$  and let  $\tau$  be a  $K$ -multiplicity free Hilbert representation of  $G$ . Let  $\tau_0$  be a subquotient representation of  $\tau$ . Then the following three statements are equivalent:*

- (a)  $\tau_0$  is irreducible.
- (b) For some  $\delta \in \mathcal{M}(\tau_0)$  we have  $\tau_{\gamma\delta} \neq 0 \neq \tau_{\delta\gamma}$  for all  $\gamma \in \mathcal{M}(\tau_0)$ .
- (c) For all  $\gamma, \delta \in \mathcal{M}(\tau_0)$  we have  $\tau_{\gamma\delta} \neq 0$ .

*Proof.* First note: if  $v \in \mathcal{H}(\tau_0)$  and  $(v, \phi_\gamma) \neq 0$  for some  $\gamma \in \mathcal{M}(\tau_0)$  then  $\phi_\gamma$  (element of the  $K$ -basis) belongs to the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $v$ . Indeed,

$$(v, \phi_\gamma)\phi_\gamma = \int_K \gamma(k^{-1})\tau(k)v \, dv$$

and

$$\tau(k)v = \tau_0(k)v.$$

(b)  $\Rightarrow$  (a): Let  $0 \neq v \in \mathcal{H}(\tau_0)$ . Let  $\mathcal{H}_1$  be the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $v$ . Then  $\phi_\gamma \in \mathcal{H}_1$  for some  $\gamma \in \mathcal{M}(\tau_0)$ . Now, for some  $g \in G$ ,

$$(\tau_0(g)\phi_\gamma, \phi_\delta) = \tau_{0,\delta,\gamma}(g) = \tau_{\delta,\gamma}(g) \neq 0,$$

so  $\tau_0(g)\phi_\gamma$  and  $\phi_\delta$  are in  $\mathcal{H}_1$ . For each  $\beta \in \mathcal{M}(\tau_0)$  we have  $(\tau_0(g)\phi_\delta, \phi_\beta) = \tau_{\beta\delta}(g) \neq 0$  for some  $g \in G$ . Thus  $\phi_\beta \in \mathcal{H}_1$  for all  $\beta \in \mathcal{M}(\tau_0)$ , so  $\mathcal{H}_1 = \mathcal{H}(\tau_0)$ .

(a)  $\Rightarrow$  (c): Suppose  $\tau_{\gamma\delta} = 0$  for some  $\gamma, \delta \in \mathcal{M}(\tau_0)$ . Then, for all  $g \in G$ ,  $(\tau_0(g)\phi_\delta, \phi_\gamma) = 0$ . Hence, the  $\tau_0$ -invariant subspace of  $\mathcal{H}(\tau_0)$  generated by  $\phi_\delta$  is orthogonal to  $\phi_\gamma$ , so  $\tau_0$  is not irreducible.

(c)  $\Rightarrow$  (b): Clear. □

Let  $\tau$  be  $K$ -multiplicity free,  $K$  being compact abelian. Define a relation  $<$  on  $\mathcal{M}(\tau)$  by:  $\gamma < \delta$  iff  $\tau_{\gamma,\delta} \neq 0$ . Then  $\gamma < \delta$  iff  $\phi_\gamma$  is in the  $\tau$ -invariant subspace of  $\mathcal{H}(\tau)$  generated by  $\phi_\delta$ . It follows that

$$\beta < \gamma \text{ and } \gamma < \delta \Rightarrow \beta < \delta$$

Define a relation  $\sim$  on  $\mathcal{M}(\tau)$  by:  $\gamma \sim \delta$  iff  $\tau_{\gamma,\delta} \neq 0 \neq \tau_{\delta,\gamma}$ . It follows that  $\sim$  is an equivalence relation on  $\mathcal{M}(\tau)$  and that, if  $\tau_{\gamma,\delta} \neq 0, \alpha \sim \gamma, \beta \sim \delta$  then  $\tau_{\alpha,\beta} \neq 0$ . It follows that, for a given equivalence set, we can partition  $\mathcal{M}(\tau)$  into three parts, the first part being the equivalence set, such that the corresponding  $3 \times 3$  block matrix for  $(\tau_{\gamma\delta}(g))$  takes the form (3.3). In view of Theorem 3.2 this proves:

**THEOREM 3.3.** *Let  $G$  be a lcsc. group with compact abelian subgroup  $K$  and let  $\tau$  be a  $K$ -multiplicity free representation of  $G$ . Then there is a unique orthogonal decomposition of  $\mathcal{H}(\tau)$  into subspaces  $\mathcal{H}(\tau_i)$ , where the  $\tau_i$ 's are precisely the irreducible subquotient representations of  $\tau$ .*

### 3.2. THE CASE $SU(1, 1)$

For  $\lambda \in \mathbb{C}$ ,  $\xi = 0$  or  $\frac{1}{2}$ , the representation  $\pi_{\xi,\lambda}$  of  $G = SU(1, 1)$  on  $L^2_\xi(K)$  (cf. (2.8)) is  $K$ -multiplicity free with  $K$ -content given by (2.13). By inspecting (2.29) for small but nonzero  $t$  and by using (2.24) it follows that

$$(3.4) \quad \pi_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \pi_{\xi, \lambda, m, n}|_A \neq 0 \Leftrightarrow c_{\xi, \lambda, m, n} \neq 0,$$

where  $c_{\xi, \lambda, m, n}$  is given by (2.30). Combination of (3.4) with Theorems 3.2 and 3.3 yields:

**THEOREM 3.4.** *Depending on  $\xi$  and  $\lambda$ , the representation  $\pi_{\xi, \lambda}$  of  $SU(1, 1)$  has the following irreducible subquotient representations:*

$$(a) \quad \underline{\lambda + \xi \notin \mathbf{Z} + \frac{1}{2}}:$$

$\pi_{\xi, \lambda}$  is irreducible itself.

$$(b) \quad \underline{\lambda = 0, \xi = \frac{1}{2}}:$$

$\pi_{1/2, 0}^+$  on  $\text{Cl Span } \{\phi_{1/2}, \phi_{3/2}, \dots\}$ ,

$\pi_{1/2, 0}^-$  on  $\text{Cl Span } \{\dots, \phi_{-3/2}, \phi_{-1/2}\}$ .

*These are also subrepresentations.*

$$(c) \quad \underline{\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda > 0}:$$

$\pi_{\xi, \lambda}^+$  on  $\text{Cl Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots\}$ ,

$\pi_{\xi, \lambda}^-$  on  $\text{Cl Span } \{\dots, \phi_{-\lambda-3/2}, \phi_{-\lambda-1/2}\}$ ,

$\pi_{\xi, \lambda}^0$  on  $\text{Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots, \phi_{\lambda-1/2}\}$ .

*Among these  $\pi_{\xi, \lambda}^+$  and  $\pi_{\xi, \lambda}^-$  are subrepresentations.*

$$(d) \quad \underline{\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda < 0}:$$

$\pi_{\xi, \lambda}^+$  on  $\text{Cl Span } \{\phi_{-\lambda+1/2}, \phi_{-\lambda+3/2}, \dots\}$ ,

$\pi_{\xi, \lambda}^-$  on  $\text{Cl Span } \{\dots, \phi_{\lambda-3/2}, \phi_{\lambda-1/2}\}$ ,

$\pi_{\xi, \lambda}$  on  $\text{Span } \{\phi_{\lambda+1/2}, \phi_{\lambda+3/2}, \dots, \phi_{-\lambda-1/2}\}$ .

*Among these  $\pi_{\xi, \lambda}^0$  is a subrepresentation.*

*Proof.*

$$(a) \quad c_{\xi, \lambda, m, n} \neq 0.$$

$$(b) \quad c_{1/2, 0, m, n} \neq 0 \Leftrightarrow m, n \leq -\frac{1}{2} \text{ or } m, n \geq \frac{1}{2}.$$

$$(c) \quad c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2} \\ \text{or } m, n \leq -\lambda - \frac{1}{2} \text{ or } m, n \geq \lambda + \frac{1}{2}.$$

Thus  $c_{\xi, \lambda, m, n}$  has block matrix

$$\begin{array}{l} m \leq -\lambda - \frac{1}{2} \\ -\lambda + \frac{1}{2} \leq m \leq \lambda - \frac{1}{2} \\ m \geq \lambda + \frac{1}{2} \end{array} \quad \begin{array}{ccc} n \leq -\lambda - \frac{1}{2} & -\lambda + \frac{1}{2} \leq n \leq \lambda - \frac{1}{2} & n \geq \lambda + \frac{1}{2} \\ \left( \begin{array}{ccc} * & * & 0 \\ 0 & * & 0 \\ 0 & * & * \end{array} \right) \end{array}$$

where each starred block has all entries nonzero.

$$(d) \quad c_{\xi, \lambda, m, n} \neq 0 \Leftrightarrow \lambda + \frac{1}{2} \leq m \leq -\lambda - \frac{1}{2} \text{ or } m, n \leq \lambda - \frac{1}{2} \\ \text{or } m, n > -\lambda + \frac{1}{2}. \quad \square$$

The finite-dimensional representation occurring in the above classification are the representations  $\pi_{\xi, \lambda}^0(\lambda + \xi \in \mathbf{Z} + \frac{1}{2}, \lambda \neq 0)$ .

### 3.3. NOTES

3.3.1. In the case of the unitary principal series ( $\lambda$  imaginary), Theorem 3.4 was first proved by BARGMANN [2, sections 6 and 7]. See van DIJK [9, Theorem 4.1] for the statement and (infinitesimal) proof of our Theorem 3.4 in the general case. A proof of Theorem 3.4 similar to our proof was earlier given by BARUT & PHILLIPS [3, §II (4)].

3.3.2. Theorem 3.4 in the case of imaginary and nonzero  $\lambda$  is contained in a general theorem by BRUHAT [5, Theorem 7; 2]: For  $\xi \in \hat{M}$ ,  $\lambda \in i\mathfrak{a}$ , the principal series representation  $\pi_{\xi, \lambda}$  of  $G$  (cf. (2.2)) is irreducible if  $s \cdot \lambda \neq \lambda$  for all  $s \neq e$  in the Weyl group for  $(G, K)$ .

3.3.3. GELFAND & NAIMARK [18, §5.4, Theorem 1] proved the irreducibility of the unitary principal series for  $SL(2, \mathbf{C})$  by a global method different from ours, working in a noncompact realization and calculating the "matrix elements" of the representation with respect to a (continuous)  $\bar{N}$ -basis.

3.3.4. Analogues of Theorems 3.2 and 3.3 can be formulated in the case of non-abelian  $K$ , cf. [27, Theorem 3.3]. In that case the canonical matrix elements  $\tau_{\gamma, \delta}$  are matrix-valued functions. By using this method, NAIMARK [34, Ch. 3, §9, No. 15] examined the irreducibility of the nonunitary principal series for  $SL(2, \mathbf{C})$ , see also KOSTERS [28].

3.3.5. Further applications of the irreducibility criterium in Theorem 3.2 can be found in MILLER [32, Lemmas 3.2 and 4.5] for the Euclidean motion group of  $\mathbf{R}^2$  and for the harmonic oscillator group, TAKAHASHI [39, §3.4] for the discrete series of  $SL(2, \mathbf{R})$  and [41, p. 560, Cor. 2] for the spherical principal series of  $F_{4(-20)}$ .

3.3.6. The method of this section does not show in an *a priori* way that a  $K$ -multiplicity free principal series representation has only finitely many irreducible subquotient representations. Actually, this property holds quite generally, cf. WALLACH [45, Theorem 8.13.3].

#### 4. EQUIVALENCES BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

##### 4.1. NAIMARK EQUIVALENCE

In this subsection we derive a criterium (Theorem 4.5) for Naimark equivalence of  $K$ -multiplicity free representations. Lemmas 4.3 and 4.4 are preparations for its proof.

Let  $G$  be an lcsc. group.

*Definition 4.1.* Let  $\sigma$  and  $\tau$  be Hilbert representations of  $G$ . The representation  $\sigma$  is called *Naimark related* to  $\tau$  if there is a closed (possibly unbounded) injective linear operator  $A$  from  $\mathcal{H}(\sigma)$  to  $\mathcal{H}(\tau)$  with domain  $\mathcal{D}(A)$  dense in  $\mathcal{H}(\sigma)$  and range  $\mathcal{R}(A)$  dense in  $\mathcal{H}(\tau)$  such that  $\mathcal{D}(A)$  is  $\sigma$ -invariant and  $A\sigma(g)v = \tau(g)Av$  for all  $v \in \mathcal{D}(A)$ ,  $g \in G$ . Then we use the notation  $\sigma \stackrel{A}{\simeq} \tau$  or  $\sigma \simeq \tau$ .

Naimark relatedness is not necessarily a transitive relation (cf. WARNER [48, p. 242]). However, we will see that it becomes an equivalence relation (called *Naimark equivalence*) when restricted to the class of unitary representations or of  $K$ -multiplicity free representations,  $K$  abelian.

Two unitary representations  $\sigma$  and  $\tau$  of  $G$  are called *unitarily equivalent* if there is an isometry  $A$  from  $\mathcal{H}(\sigma)$  onto  $\mathcal{H}(\tau)$  such that  $A\sigma(g)v = \tau(g)Av$  for all  $v \in \mathcal{H}(\sigma)$ ,  $g \in G$ . Clearly unitary equivalence is an equivalence relation.