Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. III

Autor: Norton, Karl K.

Kapitel: §1. Introduction

DOI: https://doi.org/10.5169/seals-52232

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 09.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

ON THE NUMBER OF RESTRICTED PRIME FACTORS OF AN INTEGER. III

by Karl K. NORTON

§1. INTRODUCTION

Let P be the set of all (positive rational) prime numbers, and let E be an arbitrary nonempty subset of P. Throughout this paper, let p denote a general member of P, and for non-negative integers a, write $p^a \parallel n$ if $p^a \mid n$ and $p^{a+1} \nmid n$. For each positive integer n, define

$$\omega(n; E) = \sum_{p \mid n,\, p \in E} 1, \qquad \Omega(n; E) = \sum_{p^a \parallel n,\, p \in E} a .$$

We usually write $\omega(n; P) = \omega(n), \Omega(n; P) = \Omega(n)$. In this paper, we shall estimate the functions

$$\begin{aligned} S(x, y; E, \omega) &= \text{card}\,\{n \leqslant x : \omega(n; E) > y\}, \\ S(x, y; E, \Omega) &= \text{card}\,\{n \leqslant x : \Omega(n; E) > y\} \end{aligned} \tag{1.1}$$

when y is appreciably larger than the normal order of $\omega(n; E)$ and $\Omega(n; E)$; y may even be as large as the maximum order of $\omega(n; E)$ or $\Omega(n; E)$, respectively. (Here and throughout, card B means the number of members of the set B, and if $Q(n)$ is a statement about the integer n, we often write $\{n \leqslant x : Q(n)\}$ instead of $\{n : 1 \leqslant n \leqslant x \text{ and } Q(n)\}$.)

Define

$$E(x) = \sum_{p \leqslant x,\, p \in E} p^{-1} \qquad (x \text{ real}). \tag{1.2}$$

In [13], it was observed that if $E(x) \to +\infty$ as $x \to +\infty$, then both the average order and the normal order of $\omega(n; E)$ are equal to $E(n)$, and the same statement holds for $\Omega(n; E)$. In [13], we obtained sharp inequalities for the functions (1.1) when $0 < y < 2E(x)$, roughly. In [14], we gave asymptotic formulas for the same functions when $E(x) \to +\infty$ and $y = E(x) + o(E(x))$ as $x \to +\infty$. It is well-known, however, that

$$E(x) \leqslant \log \log x + O(1) \qquad \text{for} \quad x \geqslant 2,$$

1980 Mathematics Subject Classification. Primary 10H15, 10H25. Secondary 10A20, 10A21.

whereas if x is large, $\omega(n; E)$ and $\Omega(n; E)$ may be much larger than $\log\log x$ for some values of $n \leqslant x$. For example, the method of [6, pp. 262-263, 359] shows that

$$\limsup_{n\to+\infty} \frac{\omega(n)\log\log n}{\log n} = 1, \tag{1.3}$$

and a more precise version of (1.3) was obtained in [12, pp. 96-100]. (See also the remarks at the beginning of §3 below.) Before stating estimates for the functions (1.1) when y is large, it seems worthwhile to generalize results like (1.3) to $\omega(n; E)$. First define

$$\pi(x; E) = \sum_{p\leqslant x,\, p\in E} 1 \qquad (x \text{ real}), \tag{1.4}$$

and write

$$\log_2 x = \log\log x, \qquad \log_r x = \log(\log_{r-1} x) \quad \text{for} \quad r = 3, 4, \dots . \tag{1.5}$$

THEOREM 1.6. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that*

$$\pi(x; E) = \gamma(E)(x/\log x)\{1 + O_E(1/\log x)\} \quad \text{for all} \quad x \geqslant 2. \tag{1.7}$$

Then for each $n \geqslant 3$, *we have*

$$\omega(n; E) \leqslant \frac{\log n}{\log_2 n} + \frac{\{1 + \log\gamma(E)\}\log n}{(\log_2 n)^2} + O_E\left(\frac{\log n}{(\log_2 n)^3}\right), \tag{1.8}$$

with equality for infinitely many n.

Here and throughout, the notation $O_{\delta,\varepsilon,\dots}$ implies a constant depending at most on $\delta, \varepsilon, \dots$, while O without subscripts implies an *absolute* constant. Likewise, for $i = 1, 2, \dots$, we shall write $c_i(\delta, \varepsilon, \dots)$ for a positive number depending at most on $\delta, \varepsilon, \dots$, while c_i will mean a positive absolute constant.

It is interesting to observe that a much weaker hypothesis than (1.7) still implies that the maximum order of $\omega(n; E)$ is approximately $(\log n)(\log_2 n)^{-1}$. See the remarks after the proof of Theorem 1.6 in §3.

After (1.3) and Theorem 1.6, it is natural to ask how often $\omega(n; E)$ and $\Omega(n; E)$ assume values appreciably larger than their normal order $E(n)$. It appears that rather little was known about this problem until very recently. The earliest contribution was by Hardy and Ramanujan [5] (reprinted in [15, pp. 262-275]), whose estimate for card $\{n \leqslant x : \omega(n) = m\}$ leads easily to a good upper bound for $S(x, y; P, \omega)$ (essentially the same as the bound given in Theorem 1.14 below). However, they did not state explicitly a result of the latter type. For arbitrary E, much weaker upper bounds for $S(x, y; E, \omega)$ and $S(x, y; E, \Omega)$ can be derived from a general theorem of Turán [19] on the distribution of values of additive functions. (See also Turán [18] or Hardy and Wright [6, pp. 356-358] for the case $E = P$, and see [13, §§1, 3] and [14, pp. 18-19] for remarks on all of this early work.) For the particular functions $\omega(n; E)$ and $\Omega(n; E)$, Turán's bounds were improved considerably in the author's paper [13; (5.16), (5.15), (1.11)], where it was observed that for any set E,

$$S(x, \alpha E(x); E, \omega) \leqslant x \exp\{(\alpha - 1 - \alpha \log \alpha) E(x)\} \tag{1.9}$$

for real $x \geqslant 1$, $\alpha \geqslant 1$, where $E(x)$ is defined by (1.2). A similar (slightly less precise) result was stated for $\Omega(n; E)$ when $1 \leqslant \alpha < p_1$, where p_1 is the smallest member of E. No lower bound was obtained in either case for $\alpha \geqslant 2$, so that the precision of (1.9) for large α was not clear. In a later paper [2], Erdös and Nicolas obtained a rather good estimate in the special case $E = P$. They showed that for any fixed α with $0 < \alpha < 1$,

$$\operatorname{card}\{n \leqslant x : \omega(n) > \alpha (\log x)(\log_2 x)^{-1}\} = x^{1-\alpha+o(1)} \tag{1.10}$$

as $x \to +\infty$. (In fact, they obtained a somewhat more precise result resembling Theorem 4.13 below.) However, they did not get an analogous result for $\Omega(n)$, nor did they generalize to $\omega(n; E)$ or $\Omega(n; E)$. Furthermore, their method did not give good upper estimates for $S(x, y; P, \omega)$ when y is appreciably smaller than $(\log x)(\log_2 x)^{-1}$. We propose to remedy all of these drawbacks to some extent. First, we obtain the following lower bound by a refinement of the Erdös-Nicolas method:

THEOREM 1.11. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. Let* $\varepsilon > 0$, *and suppose that* $x \geqslant c_1(E, \varepsilon)$ *and*

$$c_2(E) \leqslant y \leqslant (\log x)(\log_2 x)^{-1} + \{1 + \log \gamma(E) - \varepsilon\}(\log x)(\log_2 x)^{-2}. \tag{1.12}$$

Then

$$S(x, y; E, \omega) \geqslant x \exp\{-y(\log y + \log_2 y - \log \gamma(E) - 1) + O_E(y(\log_2 y)/\log y)\}. \tag{1.13}$$

(1.8) shows that only a very small weakening of the hypothesis (1.12) would be of any interest. In Theorem 3.20, we assume much less than (1.7) and derive a result similar to Theorem 1.11 (but somewhat weaker).

Concerning upper bounds for $S(x, y; E, \omega)$, we have obtained only a modest improvement of (1.9); see Theorem 4.8 and Corollary 4.12. It should be emphasized that (1.9) and Theorem 4.8 hold for an arbitrary set E (without the assumption (1.7)). Using the same methods, we deduce

THEOREM 1.14. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. If* $x \geqslant 3$ *and* $y \geqslant \gamma(E) \log_2 x$, *then*

$$S(x, y; E, \omega) \leqslant x \exp\{-y(\log y - \log_3 x - \log \gamma(E) - 1) - \gamma(E) \log_2 x + O_E(y/\log_2 x)\}. \tag{1.15}$$

Although there is a considerable gap between (1.13) and (1.15), the results are more general and somewhat sharper than those of Erdös and Nicolas [2]. In particular, we get a generalization of (1.10) (see Theorem 4.13). Theorems 1.11 and 1.14 also yield immediately the following result which could not be obtained by the Erdös-Nicolas method:

COROLLARY 1.16. *Suppose that there exists a real number* $\gamma(E) > 0$ *such that* (1.7) *holds. If* $0 < \alpha < 1$ *and* $x \geqslant c_3(E, \alpha)$, *then*

$$S(x, (\log x)^\alpha; E, \omega) = x \exp\{-\alpha(\log x)^\alpha \log_2 x + O((\log x)^\alpha \log_3 x)\}.$$

It should be mentioned that when $E = P$ (the set of all primes) and $y/\log_2 x$ is bounded and not too close to 1, Theorems 1.11 and 1.14 can be replaced by a striking asymptotic formula which was recently obtained by H. Delange (for the proof, see [2]):

THEOREM 1.17 (Delange). *Let* x, α, r_1, r_2 *be real with* $x \geqslant 3$, $1 < r_1 \leqslant \alpha \leqslant r_2$. *Then*

$$S(x, \alpha \log_2 x; P, \omega) = \frac{F(\alpha)\, \alpha^{1/2 + \alpha \log_2 x - [\alpha \log_2 x]}}{(2\pi)^{1/2}(\alpha - 1)} \cdot \frac{x}{(\log x)^{1 - \alpha + \alpha \log \alpha} (\log_2 x)^{1/2}} \left\{1 + O_{r_1, r_2}\left(\frac{1}{\log_2 x}\right)\right\},$$

where $[z]$ *means the largest integer* $\leqslant z$ *and*

$$F(\alpha) = \frac{1}{\Gamma(\alpha+1)} \prod_p \left(1 + \frac{\alpha}{p-1}\right)\left(1 - \frac{1}{p}\right)^{\alpha}.$$

Delange obtained a similar result for card $\{n \leqslant x : \omega(n) \leqslant \alpha \log_2 x\}$ when $x \geqslant 3$, $(\log_2 x)^{-1} \leqslant \alpha \leqslant r_3 < 1$ (see [2]). In this connection, it is interesting to note the estimate

$$F(\alpha) = \exp\{-\alpha \log \alpha - \alpha \log_2 \alpha + (1-\gamma)\alpha + O(\alpha/\log \alpha)\}$$

for real $\alpha \geqslant 2$, where γ is Euler's constant. (Some effort is required to show this, and we omit the proof.)

For values of α near 1, Kubilius [8, Theorem 9.2] proved a result on the distribution of $\omega(n)$ which is similar to Theorem 1.17. His theorem was later extended by himself [9] and Laurinčikas [10] to somewhat more general additive functions, and it was generalized to $\omega(n; E)$ and $\Omega(n; E)$ by Norton [14]. The estimates for $S(x, y; E, \omega)$ derived in the present paper are not as precise as Theorem 1.17 or the earlier work cited, but they are more general with respect to E (except for [14]), and they hold for much larger values of y.

We now consider the function $\Omega(n; E)$. Here we assume that E is any nonempty set of primes; in particular, nothing like (1.7) is assumed. For completeness, we begin by stating the following easy result:

THEOREM 1.18. *Let* p_1 *be the smallest member of* E. *Then*

$$\Omega(n; E) \leqslant (\log n)(\log p_1)^{-1} \qquad \text{for all} \qquad n \geqslant 1, \tag{1.19}$$

with equality if and only if $n = p_1^a$ *for some integer* $a \geqslant 0$.

This follows from

$$n \geqslant \prod_{p^a \| n,\, p \in E} p^a \geqslant \prod_{p^a \| n,\, p \in E} p_1^a = p_1^{\Omega(n; E)}.$$

We now proceed to estimate $S(x, y; E, \Omega)$ (defined by (1.1)). For $y \geqslant 2E(x)$, rather little previous work has been done on this problem, and all of it was restricted to the special case $E = P$ (the set of all primes). Selberg [17, p. 87] stated without detailed proof the following asymptotic formula:

$$\text{card}\,\{n \leqslant x : \Omega(n) = m\} \sim A 2^{-m} x \log x$$

for integers m satisfying $(2+\varepsilon)\log_2 x \leqslant m \leqslant B \log_2 x$. (Here $\varepsilon > 0$ is arbitrarily small, while A and B are positive absolute constants; it is not clear from [17] how large B could be.) Selberg also gave an asymptotic formula for card $\{n \leqslant x : \omega(n) = m\}$ when $m/\log_2 x$ is bounded. His work was recently extended to considerably larger values of m (roughly $m < (\log x)^{3/5}$) by Kolesnik and Straus [7], whose theorems are quite complicated. These results, together with the formula

$$S(x, y; P, \Omega) = \sum_{y < m \leqslant Y} \text{card}\,\{n \leqslant x : \Omega(n) = m\} + S(x, Y; P, \Omega)$$

and different tools for estimating $S(x, Y; P, \Omega)$ from above, would yield some information about $S(x, y; P, \Omega)$. However, it appears that neither [17] nor [7] would thus lead to an estimate for $S(x, y; P, \Omega)$ which is both simple and reasonably precise when $y/\log_2 x$ is unbounded. To the best of our knowledge, the only previous result of the latter type is due to Erdös and Sárközy [3], who recently proved that

$$S(x, y; P, \Omega) \leqslant c_4\, y^4\, 2^{-y}\, x \log x \qquad \text{for} \qquad x \geqslant 3, y \geqslant 1. \tag{1.20}$$

We shall generalize their work to $S(x, y; E, \Omega)$ and get a sharper upper bound. Although the result could be phrased in terms of the function $E(x)$ (defined by (1.2)), it is more convenient to state it in terms of a real number v which in practice is taken to be an approximation to $E(x)$. (For example, if $E = P$, we could take $v = \log_2 x$.)

THEOREM 1.21. *Let* x, v, y *be real with* $x \geqslant 1, v \geqslant 1,$ *and* $y \geqslant 0$. *Let* p_1 *be the smallest member of* E, *and define*

$$\Lambda = \Lambda(x, v; E) = \max\{2, |E(x) - v|\}. \tag{1.22}$$

Then

$$S(x, y; E, \Omega) \leqslant c_5(p_1)\, p_1^{-y}\, x v^{1/2}\, e^{(p_1 - 1)v + p_1 \Lambda}. \tag{1.23}$$

We remark that (1.23) is our best upper bound when $y > p_1 v - v^{1/2}$, but it can be improved for smaller values of y (see Lemma 5.3).

Concerning the problem of estimating $S(x, y; E, \Omega)$ from below, we shall state only the following simple result:

THEOREM 1.24. *Let* p_1 *be the smallest member of* E. *If* $x \geqslant p_1$ *and* $0 \leqslant y \leqslant (\log x)(\log p_1)^{-1} - 1$, *then*

$$S(x, y; E, \Omega) \geqslant (1/2)\, p_1^{-y-1}\, x.$$

To prove this, let $k = [y] + 1$ (so k is the smallest integer greater than y), and observe that the multiples n of p_1^k have the property that $\Omega(n; E) \geqslant k > y$.

There are just $[xp_1^{-k}]$ of these $n \leqslant x$, and since $[z] \geqslant z/2$ for $z \geqslant 1$, we get the result.

It is clear that Theorem 1.24 is essentially best possible in certain extreme cases (for example, if $E = \{p_1\}$, or if $x = p_1^a$ and $y = a - 1$).

When $E = P$ (the set of all primes), we can take $v = \log_2 x$. Then $\Lambda = O(1)$, and we have the following corollary of Theorems 1.21 and 1.24:

COROLLARY 1.25. *If* $x \geqslant e^e$ *and* $0 \leqslant y \leqslant (\log x)(\log 2)^{-1} - 1$, *then*

$$2^{-y-2}\, x \leqslant S(x, y; P, \Omega) \leqslant c_6\, 2^{-y}\, x (\log x)(\log_2 x)^{1/2}.$$

Corollary 1.25 should be compared with the Erdös-Sárközy result (1.20) and with the asymptotic formula of Selberg mentioned after Theorem 1.18. When $y < 2 \log_2 x$ (roughly), more precise estimates for $S(x, y; P, \Omega)$ can be obtained from [13] and [14].

In a later paper, we shall show that if p_1 is the smallest member of E and $\varepsilon > 0$ is fixed, then the precise order of magnitude of $S(x, y; E, \Omega)$ is

$$p_1^{-y}\, x \exp\{(p_1 - 1) E(x)\}$$

when $E(x)$ is sufficiently large and

$$p_1 E(x) \leqslant y \leqslant (1-\varepsilon)(\log x)(\log p_1)^{-1}.$$

This theorem is much more difficult to prove than Theorem 1.21. Its proof depends on Theorem 1.21 and on an extension of Halász's work [4] concerning the local distribution of $\Omega(n; E)$. Theorem 1.21 remains our best upper bound when y is close to $(\log x)(\log p_1)^{-1}$ (cf. Theorem 1.18), and it seems to be the most we can achieve by a fairly simple method.

§2. NOTATION

The symbols a, m, n always represent integers with $a \geqslant 0, m \geqslant 0, n > 0$. The letter p always denotes a prime, while $v, w, x, y, z, \alpha, \beta, \delta, \varepsilon, \sigma$ are real numbers. $[x]$ means the largest integer $\leqslant x$. The notation $\log_r x$ is defined by (1.5), and the notations $O, O_{\delta, \varepsilon, \ldots}, c_i, c_i(\delta, \varepsilon, \ldots)$ are explained after Theorem 1.6. If a condition such as "$x \geqslant c_i(\delta, \varepsilon, \ldots)$" is used as a hypothesis, it is to be understood that $c_i(\delta, \varepsilon, \ldots)$ is sufficiently large. We shall occasionally use the notations $\ll, \gg$ to imply constants which are *absolute*. (Thus $A = O(B)$ is equivalent to $A \ll B$.)