Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\delta}$ COHOMOLOGY SPACES

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Kapitel: 5. FURTHER NOTES

DOI: https://doi.org/10.5169/seals-52231

since by (3.15) the Lie algebra cohomology space in Theorem 3.16 is a formal harmonic space. It is even true as a matter of fact that under reasonable conditions the formal harmonic space associated to a polarization coincides with a Lie algebra cohomology space; see Penney's Theorem 2 in [68]. The latter cohomology spaces have the form $H^j(p \cap \operatorname{Ker} \Lambda, \pi^{\infty}_{-\Lambda})$ where π^{∞} is the space of C^{∞} vectors in a representation π and Λ is considered also as a linear functional on $g^{\mathbf{C}}$. By very clever means these spaces are shown to vanish for all j except $j =$ the negativity index $q(p, \Lambda)$ of the polarization (see 4.1). Moreover $H^{q(p,\Lambda)}(p \cap \operatorname{Ker} \Lambda, \pi^{\infty}_{-\Lambda})$ is one-dimensional; see Rosenberg's Theorem 2.4 in [74]; also see Penney [69]. With these remarks in mind an application of Theorem 4.3 gives

THEOREM 4.5 (J. Rosenberg–R. Penney 1979). *Let G be a connected, simply connected nilpotent Lie group and let p be a relatively ideal complex polarization at $\Lambda \in g^*$, $g =$ Lie algebra of G. Let $\pi^j(\Lambda, p, G)$ be the j-th harmonically induced representation defined in (4.2). Then $\pi^j(\Lambda, p, G)$ vanishes for $j \neq$ the negativity index $q(p, \Lambda)$ (see (4.1)). Moreover $\pi^{q(p,\Lambda)}(\Lambda, p, G)$ is irreducible and unitarily equivalent to the Kirillov representation π_{Λ}.*

Theorem 4.5 is clearly analogous to Theorem 3.11 and thus it represents the confirmation of a version of the Kostant-Langlands conjecture for nilpotent Lie groups. One may add that as a matter of fact the distinguished integer q_{Λ} in (3.9) is indeed the negativity index of a complex polarization-namely the polarization is a Borel subalgebra at a regular point.

5. FURTHER NOTES

1. We have pointed out earlier that in addition to Schmid's thesis work, early efforts towards proving the Kostant-Langlands conjecture were made by Narasimhan and Okamoto. The latter authors considered the special case when G/K admits a G invariant complex structure [1]). They constructed unitary representations $\pi_{\Lambda}^{0,j}$ of G on L_2-cohomology spaces associated to holomorphic vector bundles E_{Λ} over G/K induced by an irreducible unitary representation of K with highest weight Λ; compare remarks following (3.10). The $\pi_{\Lambda}^{0,j}$ are shown to be subject to an important *alternating sum formula* which, roughly stated, says that

$$\sum_{j=0}^{n} (-1)^j \text{ character of } \pi_{\Lambda}^{0,j} = (-1)^{q_{\Lambda}} \text{ character of } \pi_{\Lambda}^{*} \tag{5.1}$$

[1]) Here G, K are as in section 3.

where π_Λ^* is the contragredient to the discrete class π_Λ given in Theorem 3.5, q_Λ is the number of non-compact positive roots α such that $(\Lambda+\delta, \alpha) > 0$ (compare (3.9)), and $n = \frac{1}{2}\dim_R G/K$. A precise statement of (5.1) is given in Theorem 1 of [60]. Once one has an alternating sum formula the class π_Λ^* can be realized by $\pi_\Lambda^{0,q_\Lambda}$ if a *vanishing theorem* $H^{0,j}_{\bar{\partial},2}(G/K, E_\Lambda) = 0$ for $j \neq q_\Lambda$ is proved. The methods of Narasimhan and Okamoto of establishing an alternating sum formula and a vanishing theorem served as a prototype for the work of later authors; see for example [59], [65], [99], [64], [98]. The vanishing theorems in [60] are improved by Parthasarathy in [64].

2. The ambitious program of decomposing a complex flag manifold under the action of a real group and of using the real group orbits as the setting for the geometric realization of unitary representations of semisimple (even reductive) Lie groups is carried out in the profound work of J. Wolf in [97], [99], [96]. A family of unitary representations which support the Plancherel measure are realized on *partially holomorphic* cohomology spaces. This family clearly contains many non-discrete classes. The realizations are similar to realizations by the Kostant-Kirillov method where one uses polarizations of semisimple orbits. However some interesting differences occur in the case when the reductive group has non-commutative Cartan subgroups; see [86], [98].

3. After (3.9) we remarked on the similarity in appearance of the Weyl and Harish-Chandra character formulas of Theorems 2.22 and 3.5. There is however a vast difference of roles which these formulas play. For example Weyl's formula determines the character *on all of the group* (since a compact group is covered by conjugates of a maximal torus) whereas conjugates of H in section 3 certainly do not cover G. It seems to be an extremely difficult problem (and perhaps an impossible one to solve) to obtain the character formula explicitely on an *arbitrary* Cartan subgroup H of a non-compact semisimple group.

4. A new proof of Harish-Chandra's regularity theorem for invariant eigendistributions (cf. remarks following (3.4) due to Atiyah and Schmid is now available; cf. [4], [7], [81]. [6] contains a new and largely self-contained account of the principal theory of the discrete series including existence theory, exhaustion, geometric realization, character formulae, and character behavior. These new methods rely on the Atiyah-Singer L_2-index theorem [3], [31], [32], [81], [83], [84].

5. It is possible to formulate Frobenius reciprocity for unitary representations on a Hilbert space $\mathscr{H}(D)$ of L_2-solutions of an invariant elliptic differential operator D on homogeneous bundles over a homogeneous space G/H whose isotropy subgroup H is compact modulo the center of G. Here G is a connected unimodular Lie group (not necessarily semisimple) subject to some mild structural constraints. In [33] Connes and Moscovici show that $\mathscr{H}(D)$ decomposes as a finite direct sum of irreducible unitary representations all of which are square-integrable modulo the center of G and occur with finite multiplicity. They derive for $\mathscr{H}(D)$ a reciprocity analogous to that expressed for the L_2-cohomology spaces in Theorem 3.15 and Theorem 4.3.

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