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6. Stationary Dilations

The results of the last section play a key role in showing that each weakly harmonizable random field has a stationary dilation. It is a consequence of the preceding work that for any stationary field $Y : G \to L_0^2(P)$ with G an LCA group, and each orthogonal projection $Q : L_0^2(P) \to L_0^2(P)$, the new random field $X(g) = QY(g), g \in G$, giving $X : G \to L_0^2(P)$, is shown to be weakly harmonizable. The dilation result yields the reverse implication. A "concrete" version of this is given by the following theorem and an operator version will be obtained later from it.

THEOREM 6.1. *Let G be an LCA group, $X : G \to L_0^2(P) = \mathscr{H}$ a weakly harmonizable random field. Then there is a super (or extension) Hilbert space $\mathscr{K} \supset \mathscr{H}$, a probability measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ with $\mathscr{K} = L_0^2(\tilde{P})$, and a stationary random field $Y : G \to L_0^2(\tilde{P})$, such that $X(g) = QY(g), g \in G$, where $Q : L_0^2(\tilde{P}) \to L_0^2(\tilde{P})$ is the orthogonal projection with range $L_0^2(P)$. If moreover, $\mathscr{H} = \overline{sp}\{X(g), g \in G\}$, then Y determines $\mathscr{K}$ in the sense that $\mathscr{K} = \overline{sp}\{Y(g), g \in G\}$. [Thus $\mathscr{K}$ is the minimal super space for $\mathscr{H}$.]*

Proof. The "consequence" above is easily proved. In fact, if $Y : G \to L_0^2(P)$ is stationary, then Theorem 3.3 implies

$$Y(g) = \int_{\hat{G}} \langle g, s\rangle \, Z(ds), \qquad g \in G, \tag{63}$$

for a vector measure Z on $\hat{G}$ into $\mathscr{K} = L_0^2(P)$, with orthogonal increments (also called orthogonally scattered) where $\hat{G}$ is the dual group of the LCA group G, and $\langle \cdot, s\rangle$ is a character of G. If $Q : \mathscr{K} \to \mathscr{K}$ is any orthogonal projection, then $\tilde{Z} = Q \circ Z$ is a stochastic measure on $\tilde{G}$ into $\mathscr{K}$. Indeed,

$$\begin{aligned}
\| \tilde{Z} \|^2(\hat{G}) &= \sup \{\| \sum_{i=1}^{n} a_i \tilde{Z}(A_i) \|_2^2 : | a_i | \leqslant 1, A_i \subset \hat{G} \text{ disjoint Borel}, n \geqslant 1\} \\
&= \sup \{\| Q \sum_{i=1}^{n} a_i Z(A_i) \|_2^2 : | a_i | \leqslant 1, A_i \subset \hat{G}, \text{ as above}\} \\
&\leqslant \| Q \|^2 \sup \{\| \sum_{i=1}^{n} a_i Z(A_i) \|_2^2 : | a_i | \leqslant 1, A_i \subset \hat{G}, \text{ as before}\} \\
&= \| Q \|^2 \sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j} \, F(A_i \cap A_j) : | a_i | \leqslant 1, A_i \subset \hat{G} \text{ as before}\} \\
&\qquad\qquad \text{where } F(A_i \cap A_j) = (Z(A_i), Z(A_j)) \;, \\
&= \| Q \|^2 \, | F | (\hat{G}) \leqslant F(\hat{G}) < \infty ,
\end{aligned} \tag{64}$$

since F is the spectral measure of Z and so is finite and Q is a contraction. Hence $\tilde{Z}$ has finite semivariation and is clearly σ-additive, so that it is a stochastic measure. By Theorem 3.3, X given by $X(g) = QY(g) = \int_{\hat{G}} \langle g, s\rangle \tilde{Z}(ds), g \in G$, is weakly harmonizable. (Note that the same conclusion holds if Q is replaced by any bounded linear operator on $\mathscr{K}$. If the range of the projection Q is not finite dimensional, then X need *not* be strongly harmonizable!)

To go in the reverse direction, the (possibly) augmented space $\mathscr{K} \supset \mathscr{H}$ has to be constructed. Consider $X : G \to \mathscr{H} = L_0^2(P)$, the given weakly harmonizable random field. In order to get simultaneously the additional structure demanded in the last part, let $\mathscr{H} = \overline{sp}\{X(g), g \in G\}$ also. Then, as before, there is a stochastic measure on $\hat{G}$ into $\mathscr{H}$ such that

$$X(g) = \int_{\hat{G}} \langle g, s\rangle Z(ds) \in \mathscr{H}, \qquad g \in G. \tag{65}$$

By Theorem 5.5, with $\mathscr{Y} = \mathscr{H}$, there exists a finite Radon (= regular Borel) measure μ on $\hat{G}$ such that

$$\| \int_{\hat{G}} f(t)Z(dt) \|_2^2 \leqslant \int_{\hat{G}} | f(t) |^2 \mu(dt), \qquad f \in C_0(\hat{G}). \tag{66}$$

Now define a mapping $v : \mathscr{B}(\hat{G} \times \hat{G}) \to \mathbf{R}^+$ by the equation

$$v(A, B) = \mu(A \cap B), A, B \in \mathscr{B}(\hat{G}), \tag{67}$$

where $\mathscr{B}(\hat{G})$ is the Borel σ-ring of $\hat{G}$ and similarly $\mathscr{B}(\hat{G} \times \hat{G})$. Then v is a bimeasure of finite Vitali variation on $\mathscr{B}(\hat{G}) \times \mathscr{B}(\hat{G})$ and since this ring generates $\mathscr{B}(\hat{G} \times \hat{G})$, v extends to a Radon measure on the latter σ-ring. Morevoer, it is clear that v concentrates on the diagonal of the product space $\hat{G} \times \hat{G}$. If $C_b(\hat{G})$ denotes the Banach space of bounded continuous scalar functions on $\hat{G}$ with uniform norm, then

$$\int_{\hat{G}} \int_{\hat{G}} f(s, t)v(ds, dt) = \int_{\hat{G}} f(s, s)\mu(ds), \qquad f \in C_b(\hat{G} \times \hat{G}). \tag{68}$$

Let $F(A, B) = (Z(A), Z(B))$ so that $F : \mathscr{B}(\hat{G} \times \hat{G}) \to \mathbf{C}$ is a bimeasure of finite semivariation, from (65). Thus using the D-S and MT-integration techniques as before,

$$0 \leqslant \| \int_{\hat{G}} f(s)Z(ds) \|_2^2 = \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)}F(ds, dt), \qquad f \in C_b(\hat{G}). \tag{69}$$

Letting $f(s, t) = f(s) \cdot f(t)$ in (68), $\alpha = v - F$ one has from (66)-(69),
$$0 \leqslant \int_{\hat{G}} | f(s) |^2 \mu(ds) - \| \int_{\hat{G}} f(s)Z(ds) \|_2^2$$
$$= \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)} [v(ds, dt) - F(ds, dt)]$$
$$= \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)}\alpha(ds, dt), \qquad f \in C_b(\hat{G}). \tag{70}$$

So α is positive semi-definite and $\alpha = 0$ iff $v = F$, i.e., if F concentrates on the diagonal. This corresponds to X being stationary itself. Excluding this trivial case, $\alpha \not\equiv 0$, and (70) is strictly positive, if $f = 1$. It follows from (70) that $[\cdot, \cdot]' : C_b(\hat{G}) \times C_b(\hat{G}) \to \mathbf{C}$ defines a nontrivial semi-inner product, where

$$[f, g]' = \int_{\hat{G}} \int_{\hat{G}} f(s)\bar{g}(t)\alpha(ds, dt), \qquad f, g \in C_b(\hat{G}) . \tag{71}$$

If $\mathcal{N}_0 = \{f : [f,f]' = 0, f \in C_b(\hat{G})\}$, and $\mathcal{H}_1 = C_b(\hat{G})/\mathcal{N}_0$ is the factor space, let $[\cdot, \cdot] : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbf{C}$ be defined by

$$[(f), (g)] = [f, g]', \qquad f \in (f) \in \mathcal{H}_1, g \in (g) \in \mathcal{H}_1 . \tag{72}$$

Then $[\cdot, \cdot]$ is an inner product on $\mathcal{H}_1$ and define $\mathcal{H}_0$ as its completion in $[\cdot, \cdot]$. Let $\pi_0 : C_b(\hat{G}) \to \mathcal{H}_0$ be the canonical projection. Thus $\mathcal{H}_0$ is nontrivial and need not be separable. Now let us replace $\mathcal{H}_0$ by $L_0^2(P')$ on a probability space (Ω', Σ', P'). This can be done based on the Fubini-Jessen theorem where P' can even be taken to be a Gaussian measure (for the real $\mathcal{H}$, see [36], pp. 414-415). The complex case is similar. A quick outline is as follows: Let $\{h_i, i \in I\} \subset \mathcal{H}_0$ be a CON set. If $(\Omega_i, \Sigma_i, P_i)$ is a probability space determined by a complex Gaussian variable, so that one can take $\Omega_i = \mathbf{C}$, $\Sigma_i =$ Borel σ-algebra of $\mathbf{C}$, and

$$P_i(A) = (2\pi)^{-1} \int_A \exp\left(-\frac{|t|^2}{2}\right) dt_1 dt_2, A \in \Sigma_i, (t = t_1 + \sqrt{-1}\, t_2),$$

let $(\Omega', \Sigma', P') = \bigotimes_{i \in I} (\Omega_i, \Sigma_i, P_i)$ the product space given by the Fubini-Jessen theorem. If $X_i(\omega) = \omega(i), \omega \in \Omega' = \mathbf{C}^I$, the coordinate function, then $E(X_i) = 0$ and $E(|X_i|^2) = 1$. Also $\{X_i, i \in I\}$ forms a CON basis of $\mathcal{L} = \overline{\mathrm{sp}}\{X_i, i \in I\} \subset L_0^2(P')$. The correspondence $\tau : h_i \to X_i$, extended linearly, sets up an isomorphism of $\mathcal{H}_0$ onto $\mathcal{L}$, and

$$\| \tau(h_i) \|_2^2 = E(|X_i|^2) = 1 = [h_i, h_i], \qquad i \in I .$$

Then by polarization one has $[h_i, h_j] = E(\tau(h_i)\overline{\tau(h_j)})$, so that τ is an isometric isomorphism of $\mathcal{H}_0$ onto $\mathcal{L} \subset L_0^2(P')$, as desired.

If $\pi = \tau \circ \pi_0 : f \mapsto \tau(\pi_0(f)) \in \mathcal{H}' \subset L_0^2(P')$, $f \in C_b(\hat{G})$, is the composite (canonical) mapping, let $X_1(t) = \pi(e_t(\cdot)) \in \mathcal{H}'$ where $e_t : s \mapsto (t, s)$, is a character of G at $t \in G$. Note that $e_0 = 1 \notin \mathcal{N}_0$, so $\pi_0(1)$ can be identified with the constant $1 \in C_b(\hat{G})$. Thus

$$X_1(0) = \tau(1), E(|\tau(1)|^2) = 1 .$$

Let $\mathcal{H}'' = \overline{\mathrm{sp}}\{X_1(t), t \in G\} \subset \mathcal{H}'$. Then there exists a probability space $(\Omega'', \Sigma'', P'')$, as above, such that $\mathcal{H}'' \subset L^2(P'')$. Finally set $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}''$, in the direct sum of Hilbert spaces $L_0^2(P)$ and $L_0^2(P'')$. If $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) = (\Omega, \Sigma, P) \otimes (\Omega'', \Sigma'', P'')$ then one can identify, in a natural way, $\mathscr{K} \subset L_0^2(\tilde{P})$. Define $Y(t) = X(t) + X_1(t)$, $t \in G$, so that $(X(t), X_1(t)) = 0$ since $\mathscr{H} \perp \mathscr{H}''$ in $\mathscr{K}$. Then $\{Y(t), t \in G\} \subset \mathscr{K} \subset L_0^2(\tilde{P})$, and if $Q: \mathscr{K} \to \mathscr{H} = \{\mathscr{H} \oplus \{0\}\}$ is the orthogonal projection, one has $X(t) = QY(t)$, $t \in G$. It remains to show that $Y: G \to L_0^2(\tilde{P})$ is stationary. By construction $Y(0) = X(0) + X_1(0)$ and this is $X(0)$ only when $X_1(0) = 0$ which can happen iff $\mathscr{H}'' = \{0\}$, i.e., when no enlargement is needed.

To verify stationarity, consider

$$\begin{aligned} r(s, t) &= (Y(s), Y(t)) = (X(s), X(t)) + (X_1(s), X_1(t)) \text{ since } X \perp X_1, \\ &= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} F(d\lambda, d\lambda') + \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} \alpha(d\lambda, d\lambda'), \\ &\qquad \text{by (69) and (72) and these are MT-integrals,} \\ &= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} v(d\lambda, d\lambda'), \text{ since } \alpha = v - F \\ &= \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda)} \mu(d\lambda), \text{ by (68),} \\ &= \int_{\hat{G}} (s-t, \lambda) \mu(d\lambda), \text{ by the composition of characters.} \end{aligned} \tag{73}$$

Since μ is a finite positive measure, (73) implies

$$r(s+h, t+h) = r(s, t) = \tilde{r}(s-t),$$

and so the $Y: G \to L_0^2(\tilde{P})$ is stationary. The construction also implies that $\overline{\text{sp}}\{Y(t), t \in G\} = \mathscr{K}$ in the case that $\mathscr{H} = \overline{\text{sp}}\{X(t), t \in G\}$. This completes the proof.

The following is a useful deduction:

COROLLARY 6.2. *Every vector measure* $\nu: \mathscr{B}(G) \to \mathscr{H}$ *where* G *is an LCA group,* $\mathscr{B}(G)$ *being its Borel algebra, and* $\mathscr{H}$ *is a Hilbert space, has an orthogonally scattered dilation.*

Proof. Since $G = \hat{\hat{G}}$ consider the mapping $X: \hat{G} \to \mathscr{H}$ defined as the D-S integral $X(\hat{g}) = \int_G \langle \hat{g}, \lambda \rangle \nu(d\lambda)$. Then X is V-bounded; so it is weakly harmonizable. By the above theorem there are an extension Hilbert space $\mathscr{K} \supset \mathscr{H}$, an orthogonal projection $Q: \mathscr{K} \to \mathscr{K}$, with range $\mathscr{H}$, and a stationary field $Y: \hat{G} \to \mathscr{K}$ such that $X(\hat{g}) = QY(\hat{g})$. Let Z be the stochastic measure representing Y, (cf. Theorem 3.3). Hence for each $h \in \mathscr{H}$ one has $(Z: \mathscr{B}(\hat{G}) \to \mathscr{K})$

$$\int_G (\hat{g}, \lambda)(\nu(d\lambda), h) = (X(\hat{g}), h) = (QY(\hat{g}), h) = \int_{\hat{G}} (\hat{g}, \lambda)(Q \circ Z(d\lambda), h).$$

These are now scalar (Lebesgue-Stieltjes) integrals. By the classical uniqueness theorem of Fourier analysis for such integrals, one has

$$(\nu(A) - Q \circ Z(A), h) = 0, A \in \mathscr{B}(G), h \in \mathscr{H}.$$

Hence $\nu = Q \circ Z$. Since Z is orthogonally scattered by virtue of the fact that Y is stationary, the result follows.

With the last theorem, a more perspicuous version of the dilation problem for a weakly harmonizable random field can be given. This, however, depends also on an interesting theorem of Sz.-Nagy [41] and will be presented. Recall from the classical theory of stationary processes ([6], p. 512 and p. 638) every such process $\{Y_t, t \in \mathbf{R}\} \subset L_0^2(P)$, can be expressed as $Y_t = U_t Y_0$, where $\{U_t, t \in \mathbf{R}\}$ is a group of unitary operators acting on $L_0^2(P)$ (first on $\overline{\mathrm{sp}}\{Y_t, t \in \mathbf{R}\}$ and then, for instance, define each U_t as an identity on the orthogonal complement of this subspace). The spectral theory of U_t then yields immediately the corresponding integral representation of Y_t's. The same result holds if $\mathbf{R}$ is replaced by an LCA group G. The corresponding operator representation for harmonizable processes (or fields) is not so simple. Its solution will be presented in the following theorem. Recall that a family $T : G \to B(\mathscr{X})$, $\mathscr{X}$ a Hilbert space, is of positive type if $T(-g) = T(g)^*$ (adjoint operator) and for each finite set $\{x_{s_1}, ..., x_{s_n}\}$ of $\mathscr{X}$ indexed by $J = \{s_1, s_2, ..., s_n\} \subset G$, one has

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (T(s_j^{-1} s_i) x_{s_i}, x_{s_j}) \geqslant 0. \tag{74}$$

THEOREM 6.3. *Let G be an LCA group and $X : G \to L_0^2(P) = \mathscr{X}$, a Hilbert space, be weakly harmonizable. Then there exists a super Hilbert space $\mathscr{H} = L_0^2(\tilde{P}) \supset \mathscr{X}$ on an enlarged probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, a random variable $Y_0 \in \mathscr{H}$ a weakly continuous family $\{T(g), g \in G\}$ of contractive linear operators from $\mathscr{H}$ to $\mathscr{X}$ with $T(0)$ as the identity on $\mathscr{X}$ (0 being the neutral element of G), such that, when its domain is restricted to $\mathscr{X}$, it is of positive type, in terms of which $X(g) = T(g)Y_0, g \in G$. Conversely every weakly continuous contractive family $\{T(g), g \in G\}$ of the above type from any super Hilbert space $\mathscr{H} \supseteq \mathscr{X}$ into $\mathscr{X}$ which, when restricted to $\mathscr{X}$ is of positive type, defines a weakly harmonizable process $X : G \to \mathscr{X}$, by the equation $X(g) = T(g)Y_0$ for any $Y_0 \in \mathscr{X}$, $T(0)$ being identity on $\mathscr{X}$.*

Proof. The direct part is an operator-theoretic reformulation of Theorem 6.1. Briefly, let $X : G \to L_0^2(P) = \mathscr{X}$ be weakly harmonizable. Then there exist a $\mathscr{H} = L_0^2(\tilde{P}) \supset \mathscr{X}$ and a stationary $Y: G \to \mathscr{H}$ such that $X(g) = QY(g), g \in G$, by Theorem 6.1 with Q as the orthogonal projection on $\mathscr{H}$ and range $\mathscr{X}$. But $Y(g) = U(g)Y(0)$ where $\{U(g), g \in G\}$ is a (strongly) continuous group of unitary operators on $\mathscr{H}$. Let $T(g) = QU(g), g \in G$. It is asserted that $\{T(g), g \in G\}$ is the desired family.

Indeed, $T(0) = Q$ (= identity on $\mathscr{X}$), and $\| T(g) \| \leqslant \| Q \| \| U(g) \| \leqslant 1$. The continuity of $U(g)$ on G clearly implies the weak continuity of $T(g)$'s. To verify the positive definiteness on $\mathscr{X}$, let $h_{s_1}, \ldots, h_{s_n}$ be a finite set in $\mathscr{X}$. Then letting $\tilde{T}(g) = T(g)|_{\mathscr{X}}$ one has $\tilde{T}(-g) = (\tilde{T}(g))^*$ since

$$\begin{aligned}(\tilde{T}(-g)h_{s_1}, h_{s_2}) &= (QU(-g)h_{s_1}, h_{s_2}) = (U^*(g)h_{s_1}, Qh_{s_2}) \\ &= (h_{s_1}, U(g)h_{s_2}), \text{ since } Qh_{s_i} = h_{s_i} \text{ and } U^{**}(g) = U(g), \\ &= (Qh_{s_1}, U(g)h_{s_2}) = (h_{s_1}, QU(g)h_{s_2}) \\ &= (h_{s_1}, \tilde{T}(g)h_{s_2}) = (\tilde{T}(g)^*h_{s_1}, h_{s_2}),\ h_{s_i} \in \mathscr{X},\ i = 1, 2\,. \end{aligned} \tag{75}$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} (\tilde{T}(s_j^{-1}s_i)h_{s_i}, h_{s_j}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} (QU(-s_j)U(s_i)h_{s_i}, h_{s_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} (U(s_j)^*U(s_i)h_{s_i}, h_{s_j}) \\ &= \Big\| \sum_{i=1}^{n} U(s_i)h_{s_i} \Big\|^2 \geqslant 0\,. \end{aligned} \tag{76}$$

The converse depends explicitly on an important theorem of Sz.-Nagy ([41], Thm. III; this is an extension of a classical result of Naĭmark). According to this result if $\tilde{T}(\cdot) = T(\cdot)|_{\mathscr{X}}$, then there is a super Hilbert space $\mathscr{K}_1 \supset \mathscr{X}$ ($\mathscr{K}_1$ may be quite different from $\mathscr{K}$) and a weakly (hence strongly) continuous group $\{V(g), g \in G\}$ of unitary operators on $\mathscr{K}_1$ such that $\tilde{T}(g) = Q_1V(g)|_{\mathscr{X}}$, Q_1 being the orthogonal projection of $\mathscr{K}_1$ onto $\mathscr{X}$. Here $\mathscr{K}_1$ can be chosen as $\mathscr{K}_1 = \overline{\text{sp}}\{V(g)\mathscr{X}, g \in G\}$. If $x_0 \in \mathscr{X}$ is arbitrary, then $x_0 \in \mathscr{K}_1 \cap \mathscr{K}$, and

$$T(g)x_0 = \tilde{T}(g)x_0 = Q_1V(g)x_0 = X(g)\,, \qquad \text{(say)},\ g \in G\,.$$

But $\{Y(g) = V(g)x_0, g \in G\} \subset \mathscr{K}_1$ is a stationary process so that by the first paragraph of the proof of Theorem 6.1, $\{X_0(g), g \in G\} \subset \mathscr{X}$ is weakly harmonizable. Thus for each $x_0 \in \mathscr{X}$, $\{T(g)x_0, g \in G\}$ is weakly harmonizable, and this completes the proof.

Remark. In the converse direction one can take $\mathscr{K} = \mathscr{X}$ However in the forward direction, it is not always possible to take Y_0 in $\mathscr{X}$, so that $X(0) = Y_0$, as the example following Definition 2.1 shows. Thus there is an inherent asymmetry in the statement of this theorem, and the mention of the super Hilbert space $\mathscr{K}$ in the enunciation cannot be avoided. It should also be noted that the above quoted theorem of Sz.-Nagy [41] can be deduced also from Naĭmark's theorem *and* Theorem 6.1. See [38] for a further discussion on this point.