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6. STATIONARY DILATIONS

The results of the last section play a key role in showing that each weakly harmonizable random field has a stationary dilation. It is a consequence of the preceding work that for any stationary field $Y: G \to L_0^2(P)$ with G an LCA group, and each orthogonal projection $Q: L_0^2(P) \to L_0^2(P)$, the new random field $X(g) = QY(g), g \in G$, giving $X: G \to L_0^2(P)$, is shown to be weakly harmonizable. The dilation result yields the reverse implication. A "concrete" version of this is given by the following theorem and an operator version will be obtained later from it.

Theorem 6.1. Let G be an LCA group, $X:G\to L_0^2(P)=\mathcal{H}$ a weakly harmonizable random field. Then there is a super (or extension) Hilbert space $\mathscr{K}\to \mathscr{H}$, a probability measure space $(\widetilde{\Omega},\widetilde{\Sigma},\widetilde{P})$ with $\mathscr{K}=L_0^2(\widetilde{P})$, and a stationary random field $Y:G\to L_0^2(\widetilde{P})$, such that $X(g)=QY(g),g\in G$, where $Q:L_0^2(\widetilde{P})\to L_0^2(\widetilde{P})$ is the orthogonal projection with range $L_0^2(P)$. If moreover, $\mathscr{H}=\overline{sp}\{X(g),g\in G\}$, then Y determines \mathscr{K} in the sense that $\mathscr{K}=\overline{sp}\{Y(g),g\in G\}$. [Thus \mathscr{K} is the minimal super space for \mathscr{H} .]

Proof. The "consequence" above is easily proved. In fact, if $Y: G \to L_0^2(P)$ is stationary, then Theorem 3.3 implies

$$Y(g) = \int_{\widehat{G}} \langle g, s \rangle Z(ds), \qquad g \in G, \qquad (63)$$

for a vector measure Z on \widehat{G} into $\mathscr{K} = L_0^2(P)$, with orthogonal increments (also called orthogonally scattered) where \widehat{G} is the dual group of the LCA group G, and $\langle \cdot, s \rangle$ is a character of G. If $Q : \mathscr{K} \to \mathscr{K}$ is any orthogonal projection, then $\widetilde{Z} = Q \circ Z$ is a stochastic measure on \widetilde{G} into \mathscr{K} . Indeed,

$$\|\tilde{Z}\|^{2}(\hat{G}) = \sup \{\|\sum_{i=1}^{n} a_{i}\tilde{Z}(A_{i})\|_{2}^{2} : |a_{i}| \leq 1, A_{i} \subset \hat{G} \text{ disjoint Borel, } n \geq 1\}$$

$$= \sup \{\|Q\sum_{i=1}^{n} a_{i}Z(A_{i})\|_{2}^{2} : |a_{i}| \leq 1, A_{i} \subset \hat{G}, \text{ as above}\}$$

$$\leq \|Q\|^{2} \sup \{\|\sum_{i=1}^{n} a_{i}Z(A_{i})\|_{2}^{2} : |a_{i}| \leq 1, A_{i} \subset \hat{G}, \text{ as before}\}$$

$$= \|Q\|^{2} \sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i}\overline{a_{j}} F(A_{i} \cap A_{j}) : |a_{i}| \leq 1, A_{i} \subset \hat{G} \text{ as before}\}$$

$$\text{where } F(A_{i} \cap A_{j}) = (Z(A_{i}), Z(A_{j})),$$

$$= \|Q\|^{2} \|F\|(\hat{G}) \leq F(\hat{G}) < \infty,$$

$$(64)$$

since F is the spectral measure of Z and so is finite and Q is a contraction. Hence \widetilde{Z} has finite semivariation and is clearly σ -additive, so that it is a stochastic measure. By Theorem 3.3, X given by $X(g) = QY(g) = \int_{\widehat{G}} \langle g, s \rangle \widetilde{Z}(ds), g \in G$, is weakly harmonizable. (Note that the same conclusion holds if Q is replaced by any bounded linear operator on \mathscr{K} . If the range of the projection Q is not finite dimensional, then X need *not* be strongly harmonizable!)

To go in the reverse direction, the (possibly) augmented space $\mathcal{K} \supset \mathcal{H}$ has to be constructed. Consider $X: G \to \mathcal{H} = L_0^2(P)$, the given weakly harmonizable random field. In order to get simultaneously the additional structure demanded in the last part, let $\mathcal{H} = \overline{sp}\{X(g), g \in G\}$ also. Then, as before, there is a stochastic measure on \hat{G} into \mathcal{H} such that

$$X(g) = \int_{\hat{G}} \langle g, s \rangle Z(ds) \in \mathcal{H}, \qquad g \in G.$$
 (65)

By Theorem 5.5, with $\mathcal{Y} = \mathcal{H}$, there exists a finite Radon (= regular Borel) measure μ on \hat{G} such that

$$\|\int_{\hat{G}} f(t)Z(dt)\|_{2}^{2} \leq \int_{\hat{G}} |f(t)|^{2} \mu(dt), \qquad f \in C_{0}(\hat{G}).$$
 (66)

Now define a mapping $v: \mathcal{B}(\hat{G} \times \hat{G}) \to \mathbf{R}^+$ by the equation

$$v(A, B) = \mu(A \cap B), A, B \in \mathcal{B}(\hat{G}), \tag{67}$$

where $\mathcal{B}(\hat{G})$ is the Borel σ -ring of \hat{G} and similarly $\mathcal{B}(\hat{G} \times \hat{G})$. Then v is a bimeasure of finite Vitali variation on $\mathcal{B}(\hat{G}) \times \mathcal{B}(\hat{G})$ and since this ring generates $\mathcal{B}(\hat{G} \times \hat{G})$, v extends to a Radon measure on the latter σ -ring. Morevoer, it is clear that v concentrates on the diagonal of the product space $\hat{G} \times \hat{G}$. If $C_b(\hat{G})$ denotes the Banach space of bounded continuous scalar functions on \hat{G} with uniform norm, then

$$\int_{\hat{G}} \int_{\hat{G}} f(s, t) v(ds, dt) = \int_{\hat{G}} f(s, s) \mu(ds), \qquad f \in C_b(\hat{G} \times \hat{G}).$$
 (68)

Let F(A, B) = (Z(A), Z(B)) so that $F : \mathcal{B}(\hat{G} \times \hat{G}) \to \mathbb{C}$ is a bimeasure of finite semivariation, from (65). Thus using the D-S and MT-integration techniques as before,

$$0 \le \| \int_{\hat{G}} f(s) Z(ds) \|_{2}^{2} = \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} F(ds, dt), \qquad f \in C_{b}(\hat{G}).$$
 (69)

Letting $f(s, t) = f(s) \cdot f(t)$ in (68), $\alpha = v - F$ one has from (66)-(69), $0 \le \int_{\hat{G}} |f(s)|^2 \mu(ds) - \|\int_{\hat{G}} f(s)Z(ds)\|_2^2$

$$= \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} \left[v(ds, dt) - F(ds, dt) \right]$$

$$= \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} \alpha(ds, dt), \qquad f \in C_b(\hat{G}). \tag{70}$$

So α is positive semi-definite and $\alpha = 0$ iff v = F, i.e., if F concentrates on the diagonal. This corresponds to X being stationary itself. Excluding this trivial case, $\alpha \neq 0$, and (70) is strictly positive, if f = 1. It follows from (70) that $[\cdot, \cdot]': C_b(\hat{G}) \times C_b(\hat{G}) \to \mathbb{C}$ defines a nontrivial semi-inner product, where

$$[f,g]' = \int_{\hat{G}} \int_{\hat{G}} f(s)\bar{g}(t)\alpha(ds,dt), \qquad f,g \in C_b(\hat{G}). \tag{71}$$

If $\mathcal{N}_0 = \{f : [f, f]' = 0, f \in C_b(\widehat{G})\}$, and $\mathcal{H}_1 = C_b(\widehat{G})/\mathcal{N}_0$ is the factor space, let $[\cdot, \cdot] : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ be defined by

$$[(f), (g)] = [f, g]', \qquad f \in (f) \in \mathcal{H}_1, g \in (g) \in \mathcal{H}_1. \tag{72}$$

Then $[\cdot, \cdot]$ is an inner product on \mathcal{H}_1 and define \mathcal{H}_0 as its completion in $[\cdot, \cdot]$. Let $\pi_0: C_b(\hat{G}) \to \mathcal{H}_0$ be the canonical projection. Thus \mathcal{H}_0 is nontrivial and need not be separable. Now let us replace \mathcal{H}_0 by $L_0^2(P')$ on a probability space (Ω', Σ', P') . This can be done based on the Fubini-Jessen theorem where P' can even be taken to be a Gaussian measure (for the real \mathcal{H} , see [36], pp. 414-415). The complex case is similar. A quick outline is as follows: Let $\{h_i, i \in I\} \subset \mathcal{H}_0$ be a CON set. If $(\Omega_i, \Sigma_i, P_i)$ is a probability space determined by a complex Gaussian variable, so that one can take $\Omega_i = \mathbb{C}$, $\Sigma_i = \text{Borel } \sigma\text{-algebra of } \mathbb{C}$, and

$$P_{i}(A) = (2\pi)^{-1} \int_{A} \exp\left(-\frac{|t|^{2}}{2}\right) dt_{1} dt_{2}, A \in \Sigma_{i}, (t = t_{1} + \sqrt{-1} t_{2}),$$

let $(\Omega', \Sigma', P') = \bigotimes_{i \in I} (\Omega_i, \Sigma_i, P_i)$ the product space given by the Fubini-Jessen theorem. If $X_i(\omega) = \omega(i)$, $\omega \in \Omega' = \mathbb{C}^I$, the coordinate function, then $E(X_i) = 0$ and $E(|X_i|^2) = 1$. Also $\{X_i, i \in I\}$ forms a CON basis of $\mathcal{L} = \sup\{X_i, i \in I\}$ $\subset L_0^2(P')$. The correspondence $\tau: h_i \to X_i$, extended linearly, sets up an isomorphism of \mathcal{H}_0 onto \mathcal{L} , and

$$\| \tau(h_i) \|_2^2 = E(|X_i|^2) = 1 = [h_i, h_i], \quad i \in I.$$

Then by polarization one has $[h_i, h_j] = E(\tau(h_i)\overline{\tau(h_j)})$, so that τ is an isometric isomorphism of \mathcal{H}_0 onto $\mathcal{L} \subset L_0^2(P')$, as desired.

If $\pi = \tau \circ \pi_0$: $f \mapsto \tau(\pi_0(f)) \in \mathcal{H}' \subset L_0^2(P')$, $f \in C_b(\widehat{G})$, is the composite (canonical) mapping, let $X_1(t) = \pi(e_t(\cdot)) \in \mathcal{H}'$ where $e_t : s \mapsto (t, s)$, is a character of G at $t \in G$. Note that $e_0 = 1 \notin \mathcal{N}_0$, so $\pi_0(1)$ can be identified with the constant $1 \in C_b(\widehat{G})$. Thus

$$X_1(0) = \tau(1), E(|\tau(1)|^2) = 1.$$

Let $\mathscr{H}'' = \operatorname{sp}\{X_1(t), t \in G\} \subset \mathscr{H}'$. Then there exists a probability space $(\Omega'', \Sigma'', P'')$, as above, such that $\mathscr{H}'' \subset L^2(P'')$. Finally set $\mathscr{H} = \mathscr{H} \oplus \mathscr{H}''$, in the

direct sum of Hilbert spaces $L_0^2(P)$ and $L_0^2(P'')$. If $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ = $(\Omega, \Sigma, P) \otimes (\Omega'', \Sigma'', P'')$ then one can identify, in a natural way, $\mathcal{K} \subset L_0^2(\tilde{P})$. Define $Y(t) = X(t) + X_1(t)$, $t \in G$, so that $(X(t), X_1(t)) = 0$ since $\mathcal{H} \perp \mathcal{H}''$ in \mathcal{K} . Then $\{Y(t), t \in G\} \subset \mathcal{K} \subset L_0^2(\tilde{P})$, and if $Q: \mathcal{K} \to \mathcal{H} = \{\mathcal{H} \oplus \{0\}\}$ is the orthogonal projection, one has X(t) = QY(t), $t \in G$. It remains to show that $Y: G \to L_0^2(\tilde{P})$ is stationary. By construction $Y(0) = X(0) + X_1(0)$ and this is X(0) only when $X_1(0) = 0$ which can happen iff $\mathcal{H}'' = \{0\}$, i.e., when no enlargement is needed.

To verify stationarity, consider

$$r(s, t) = (Y(s), Y(t)) = (X(s), X(t)) + (X_1(s), X_1(t)) \text{ since } X \perp X_1,$$

$$= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) (\overline{t}, \lambda') F(d\lambda, d\lambda') + \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) (\overline{t}, \lambda') \alpha(d\lambda, d\lambda'),$$
by (69) and (72) and these are MT-integrals,
$$= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) (\overline{t}, \lambda') v(d\lambda, d\lambda'), \text{ since } \alpha = v - F$$

$$= \int_{\hat{G}} (s, \lambda) (\overline{t}, \lambda) \mu(d\lambda), \text{ by (68)},$$

 $= \int_{\hat{G}} (s - t, \lambda) \mu(d\lambda), \text{ by the composition of characters.}$ (73)

Since μ is a finite positive measure, (73) implies

$$r(s+h, t+h) = r(s, t) = \tilde{r}(s-t),$$

and so the $Y: G \to L_0^2(\tilde{P})$ is stationary. The construction also implies that $\overline{\sup}\{Y(t), t \in G\} = \mathcal{K}$ in the case that $\mathcal{H} = \overline{\sup}\{X(t), t \in G\}$. This completes the proof.

The following is a useful deduction:

COROLLARY 6.2. Every vector measure $v: \mathcal{B}(G) \to \mathcal{H}$ where G is an LCA group, $\mathcal{B}(G)$ being its Borel algebra, and \mathcal{H} is a Hilbert space, has an orthogonally scattered dilation.

Proof. Since $G = \widehat{G}$ consider the mapping $X : \widehat{G} \to \mathcal{H}$ defined as the D-S integral $X(\widehat{g}) = \int_G \langle \widehat{g}, \lambda \rangle \nu(d\lambda)$. Then X is V-bounded; so it is weakly harmonizable. By the above theorem there are an extension Hilbert space $\mathcal{H} \to \mathcal{H}$, an orthogonal projection $Q : \mathcal{H} \to \mathcal{H}$, with range \mathcal{H} , and a stationary field $Y : \widehat{G} \to \mathcal{H}$ such that $X(\widehat{g}) = QY(\widehat{g})$. Let Z be the stochastic measure representing Y, (cf. Theorem 3.3). Hence for each $h \in \mathcal{H}$ one has $(Z : \mathcal{B}(\widehat{G}) \to \mathcal{H})$

$$\int_{G} (\hat{g}, \lambda) (v(d\lambda), h) = (X(\hat{g}), h) = (QY(\hat{g}), h) = \int_{\hat{G}} (\hat{g}, \lambda) (Q \circ Z(d\lambda), h).$$

These are now scalar (Lebesgue-Stieltjes) integrals. By the classical uniqueness theorem of Fourier analysis for such integrals, one has

$$(v(A) - Q \circ Z(A), h) = 0, A \in \mathcal{B}(G), h \in \mathcal{H}$$
.

Hence $v = Q \circ Z$. Since Z is orthogonally scattered by virtue of the fact that Y is stationary, the result follows.

With the last theorem, a more perspicuous version of the dilation problem for a weakly harmonizable random field can be given. This, however, depends also on an interesting theorem of Sz.-Nagy [41] and will be presented. Recall from the classical theory of stationary processes ([6], p. 512 and p. 638) every such process $\{Y_t, t \in \mathbf{R}\} \subset L_0^2(P)$, can be expressed as $Y_t = U_t Y_0$, where $\{U_t, t \in \mathbf{R}\}$ is a group of unitary operators acting on $L_0^2(P)$ (first on $\operatorname{sp}\{Y_t, t \in \mathbf{R}\}$ and then, for instance, define each U_t as an identity on the orthogonal complement of this subspace). The spectral theory of U_t then yields immediately the corresponding integral representation of Y_t 's. The same result holds if \mathbf{R} is replaced by an LCA group G. The corresponding operator representation for harmonizable processes (or fields) is not so simple. Its solution will be presented in the following theorem. Recall that a family $T: G \to B(\mathcal{X})$, \mathcal{X} a Hilbert space, is of positive type if $T(-g) = T(g)^*$ (adjoint operator) and for each finite set $\{x_{s_1}, ..., x_{s_n}\}$ of \mathcal{X} indexed by $J = \{s_1, s_2, ..., s_n\} \subset G$, one has

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(T(s_j^{-1} s_i) x_{s_i}, x_{s_j} \right) \geqslant 0.$$
 (74)

Theorem 6.3. Let G be an ICA group and $X:G\to L_0^2(P)=\mathscr{X}$, a Hilbert space, be weakly harmonizable. Then there exists a super Hilbert space $\mathscr{K}=L_0^2(\tilde{P})\supset \mathscr{X}$ on an enlarged probability space $(\tilde{\Omega},\tilde{\Sigma},\tilde{P})$, a random variable $Y_0\in \mathscr{K}$ a weakly continuous family $\{T(g),g\in G\}$ of contractive linear operators from \mathscr{K} to \mathscr{X} with T(0) as the identity on \mathscr{X} (0 being the neutral element of G), such that, when its domain is restricted to \mathscr{X} , it is of positive type, in terms of which $X(g)=T(g)Y_0,g\in G$. Conversely every weakly continuous contractive family $\{T(g),g\in G\}$ of the above type from any super Hilbert space $\mathscr{K}\supseteq \mathscr{X}$ into \mathscr{X} which, when restricted to \mathscr{X} is of positive type, defines a weakly harmonizable process $X:G\to \mathscr{X}$, by the equation $X(g)=T(g)Y_0$ for any $Y_0\in \mathscr{X}$, T(0) being identity on \mathscr{X} .

Proof. The direct part is an operator-theoretic reformulation of Theorem 6.1. Briefly, let $X: G \to L_0^2(P) = \mathcal{X}$ be weakly harmonizable. Then there exist a $\mathcal{K} = L_0^2(\tilde{P}) \supset \mathcal{X}$ and a stationary $Y: G \to \mathcal{K}$ such that $X(g) = QY(g), g \in G$, by Theorem 6.1 with Q as the orthogonal projection on \mathcal{K} and range \mathcal{X} . But Y(g) = U(g)Y(0) where $\{U(g), g \in G\}$ is a (strongly) continuous group of unitary operators on \mathcal{K} . Let $T(g) = QU(g), g \in G$. It is asserted that $\{T(g), g \in G\}$ is the desired family.

Indeed, T(0) = Q (= identity on \mathscr{X}), and $||T(g)|| \le ||Q|| ||U(g)|| \le 1$. The continuity of U(g) on G clearly implies the weak continuity of T(g)'s. To verify the positive definiteness on \mathscr{X} , let h_{s_1} , ..., h_{s_n} be a finite set in \mathscr{X} . Then letting $\tilde{T}(g) = T(g)|_{\mathscr{X}}$ one has $\tilde{T}(-g) = (\tilde{T}(g))^*$ since

$$(\tilde{T}(-g)h_{s_1}, h_{s_2}) = (QU(-g)h_{s_1}, h_{s_2}) = (U^*(g)h_{s_1}, Qh_{s_2})$$

$$= (h_{s_1}, U(g)h_{s_2}), \text{ since } Qh_{s_i} = h_{s_i} \text{ and } U^{**}(g) = U(g),$$

$$= (Qh_{s_1}, U(g)h_{s_2}) = (h_{s_1}, QU(g)h_{s_2})$$

$$= (h_{s_1}, \tilde{T}(g)h_{s_2}) = (\tilde{T}(g)^*h_{s_1}, h_{s_2}), h_{s_i} \in \mathcal{X}, i = 1, 2.$$
(75)

Similarly,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\widetilde{T}(s_{j}^{-1} s_{i}) h_{s_{i}}, h_{s_{j}} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(QU(-s_{j})U(s_{i}) h_{s_{i}}, h_{s_{j}} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(U(s_{j}) * U(s_{i}) h_{s_{i}}, h_{s_{j}} \right)$$

$$= \| \sum_{i=1}^{n} U(s_{i}) h_{s_{i}} \|^{2} \ge 0.$$
(76)

The converse depends explicitly on an important theorem of Sz.-Nagy ([41], Thm. III; this is an extension of a classical result of Naĭmark). According to this result if $\tilde{T}(\cdot) = T(\cdot)|_{\mathcal{X}}$, then there is a super Hilbert space $\mathcal{K}_1 \supset \mathcal{X}(\mathcal{K}_1 \text{ may be quite different from } \mathcal{K})$ and a weakly (hence strongly) continuous group $\{V(g), g \in G\}$ of unitary operators on \mathcal{K}_1 such that $\tilde{T}(g) = Q_1 V(g)|_{\mathcal{X}}, Q_1$ being the orthogonal projection of \mathcal{K}_1 onto \mathcal{X} . Here \mathcal{K}_1 can be chosen as $\mathcal{K}_1 = \overline{\sup}\{V(g)\mathcal{X}, g \in G\}$. If $x_0 \in \mathcal{X}$ is arbitrary, then $x_0 \in \mathcal{K}_1 \cap \mathcal{K}$, and

$$T(g)x_0 = \tilde{T}(g)x_0 = Q_1V(g)x_0 = X(g),$$
 (say), $g \in G$.

But $\{Y(g) = V(g)x_0, g \in G\} \subset \mathcal{K}_1$ is a stationary process so that by the first paragraph of the proof of Theorem 6.1, $\{X_0(g), g \in G\} \subset \mathcal{X}$ is weakly harmonizable. Thus for each $x_0 \in \mathcal{X}$, $\{T(g)x_0, g \in G\}$ is weakly harmonizable, and this completes the proof.

Remark. In the converse direction one can take $\mathcal{K} = \mathcal{X}$ However in the forward direction, it is not always possible to take Y_0 in \mathcal{X} , so that $X(0) = Y_0$, as the example following Definition 2.1 shows. Thus there is an inherent asymmetry in the statement of this theorem, and the mention of the super Hilbert space \mathcal{K} in the enunciation cannot be avoided. It should also be noted that the above quoted theorem of Sz.-Nagy [41] can be deduced also from Naĭmark's theorem and Theorem 6.1. See [38] for a further discussion on this point.