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## 6. STATIONARY DILATIONS

The results of the last section play a key role in showing that each weakly harmonizable random field has a stationary dilation. It is a consequence of the preceding work that for any stationary field  $Y : G \rightarrow L_0^2(P)$  with  $G$  an LCA group, and each orthogonal projection  $Q : L_0^2(P) \rightarrow L_0^2(P)$ , the new random field  $X(g) = QY(g)$ ,  $g \in G$ , giving  $X : G \rightarrow L_0^2(P)$ , is shown to be weakly harmonizable. The dilation result yields the reverse implication. A "concrete" version of this is given by the following theorem and an operator version will be obtained later from it.

**THEOREM 6.1.** *Let  $G$  be an LCA group,  $X : G \rightarrow L_0^2(P) = \mathcal{H}$  a weakly harmonizable random field. Then there is a super (or extension) Hilbert space  $\mathcal{K} \supset \mathcal{H}$ , a probability measure space  $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$  with  $\mathcal{K} = L_0^2(\tilde{P})$ , and a stationary random field  $Y : G \rightarrow L_0^2(\tilde{P})$ , such that  $X(g) = QY(g)$ ,  $g \in G$ , where  $Q : L_0^2(\tilde{P}) \rightarrow L_0^2(\tilde{P})$  is the orthogonal projection with range  $L_0^2(P)$ . If moreover,  $\mathcal{H} = \overline{\text{sp}}\{X(g), g \in G\}$ , then  $Y$  determines  $\mathcal{K}$  in the sense that  $\mathcal{K} = \overline{\text{sp}}\{Y(g), g \in G\}$ . [Thus  $\mathcal{K}$  is the minimal super space for  $\mathcal{H}$ .]*

*Proof.* The "consequence" above is easily proved. In fact, if  $Y : G \rightarrow L_0^2(P)$  is stationary, then Theorem 3.3 implies

$$Y(g) = \int_{\hat{G}} \langle g, s \rangle Z(ds), \quad g \in G, \quad (63)$$

for a vector measure  $Z$  on  $\hat{G}$  into  $\mathcal{K} = L_0^2(P)$ , with orthogonal increments (also called orthogonally scattered) where  $\hat{G}$  is the dual group of the LCA group  $G$ , and  $\langle \cdot, s \rangle$  is a character of  $G$ . If  $Q : \mathcal{K} \rightarrow \mathcal{K}$  is any orthogonal projection, then  $\tilde{Z} = Q \circ Z$  is a stochastic measure on  $\hat{G}$  into  $\mathcal{K}$ . Indeed,

$$\|\tilde{Z}\|^2(\hat{G}) = \sup \left\{ \left\| \sum_{i=1}^n a_i \tilde{Z}(A_i) \right\|_2^2 : |a_i| \leq 1, A_i \subset \hat{G} \text{ disjoint Borel}, n \geq 1 \right\}$$

$$= \sup \left\{ \left\| Q \sum_{i=1}^n a_i Z(A_i) \right\|_2^2 : |a_i| \leq 1, A_i \subset \hat{G}, \text{ as above} \right\}$$

$$\leq \|Q\|^2 \sup \left\{ \left\| \sum_{i=1}^n a_i Z(A_i) \right\|_2^2 : |a_i| \leq 1, A_i \subset \hat{G}, \text{ as before} \right\}$$

$$= \|Q\|^2 \sup \left\{ \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} F(A_i \cap A_j) : |a_i| \leq 1, A_i \subset \hat{G} \text{ as before} \right\}$$

$$\text{where } F(A_i \cap A_j) = (Z(A_i), Z(A_j)) ,$$

$$= \|Q\|^2 |F|(\hat{G}) \leq F(\hat{G}) < \infty, \quad (64)$$

since  $F$  is the spectral measure of  $Z$  and so is finite and  $Q$  is a contraction. Hence  $\tilde{Z}$  has finite semivariation and is clearly  $\sigma$ -additive, so that it is a stochastic measure. By Theorem 3.3,  $X$  given by  $X(g) = QY(g) = \int_{\hat{G}} \langle g, s \rangle \tilde{Z}(ds)$ ,  $g \in G$ , is weakly harmonizable. (Note that the same conclusion holds if  $Q$  is replaced by any bounded linear operator on  $\mathcal{H}$ . If the range of the projection  $Q$  is not finite dimensional, then  $X$  need *not* be strongly harmonizable!)

To go in the reverse direction, the (possibly) augmented space  $\mathcal{K} \supset \mathcal{H}$  has to be constructed. Consider  $X : G \rightarrow \mathcal{K} = L_0^2(P)$ , the given weakly harmonizable random field. In order to get simultaneously the additional structure demanded in the last part, let  $\mathcal{K} = \overline{\text{sp}}\{X(g), g \in G\}$  also. Then, as before, there is a stochastic measure on  $\hat{G}$  into  $\mathcal{K}$  such that

$$X(g) = \int_{\hat{G}} \langle g, s \rangle Z(ds) \in \mathcal{K}, \quad g \in G. \quad (65)$$

By Theorem 5.5, with  $\mathcal{Y} = \mathcal{K}$ , there exists a finite Radon (= regular Borel) measure  $\mu$  on  $\hat{G}$  such that

$$\left\| \int_{\hat{G}} f(t) Z(dt) \right\|_2^2 \leq \int_{\hat{G}} |f(t)|^2 \mu(dt), \quad f \in C_0(\hat{G}). \quad (66)$$

Now define a mapping  $v : \mathcal{B}(\hat{G} \times \hat{G}) \rightarrow \mathbf{R}^+$  by the equation

$$v(A, B) = \mu(A \cap B), \quad A, B \in \mathcal{B}(\hat{G}), \quad (67)$$

where  $\mathcal{B}(\hat{G})$  is the Borel  $\sigma$ -ring of  $\hat{G}$  and similarly  $\mathcal{B}(\hat{G} \times \hat{G})$ . Then  $v$  is a bimeasure of finite Vitali variation on  $\mathcal{B}(\hat{G}) \times \mathcal{B}(\hat{G})$  and since this ring generates  $\mathcal{B}(\hat{G} \times \hat{G})$ ,  $v$  extends to a Radon measure on the latter  $\sigma$ -ring. Moreover, it is clear that  $v$  concentrates on the diagonal of the product space  $\hat{G} \times \hat{G}$ . If  $C_b(\hat{G})$  denotes the Banach space of bounded continuous scalar functions on  $\hat{G}$  with uniform norm, then

$$\int_{\hat{G}} \int_{\hat{G}} f(s, t) v(ds, dt) = \int_{\hat{G}} f(s, s) \mu(ds), \quad f \in C_b(\hat{G} \times \hat{G}). \quad (68)$$

Let  $F(A, B) = (Z(A), Z(B))$  so that  $F : \mathcal{B}(\hat{G} \times \hat{G}) \rightarrow \mathbf{C}$  is a bimeasure of finite semivariation, from (65). Thus using the D-S and MT-integration techniques as before,

$$0 \leq \left\| \int_{\hat{G}} f(s) Z(ds) \right\|_2^2 = \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} F(ds, dt), \quad f \in C_b(\hat{G}). \quad (69)$$

Letting  $f(s, t) = f(s) \cdot \overline{f(t)}$  in (68),  $\alpha = v - F$  one has from (66)-(69),

$$\begin{aligned} 0 &\leq \int_{\hat{G}} |f(s)|^2 \mu(ds) - \left\| \int_{\hat{G}} f(s) Z(ds) \right\|_2^2 \\ &= \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} [v(ds, dt) - F(ds, dt)] \\ &= \int_{\hat{G}} \int_{\hat{G}} f(s) \overline{f(t)} \alpha(ds, dt), \quad f \in C_b(\hat{G}). \end{aligned} \quad (70)$$

So  $\alpha$  is positive semi-definite and  $\alpha = 0$  iff  $v = F$ , i.e., if  $F$  concentrates on the diagonal. This corresponds to  $X$  being stationary itself. Excluding this trivial case,  $\alpha \neq 0$ , and (70) is strictly positive, if  $f = 1$ . It follows from (70) that  $[\cdot, \cdot]': C_b(\hat{G}) \times C_b(\hat{G}) \rightarrow \mathbf{C}$  defines a nontrivial semi-inner product, where

$$[f, g]' = \int_{\hat{G}} \int_{\hat{G}} f(s) \bar{g}(t) \alpha(ds, dt), \quad f, g \in C_b(\hat{G}). \quad (71)$$

If  $\mathcal{N}_0 = \{f: [f, f]' = 0, f \in C_b(\hat{G})\}$ , and  $\mathcal{H}_1 = C_b(\hat{G})/\mathcal{N}_0$  is the factor space, let  $[\cdot, \cdot]: \mathcal{H}_1 \times \mathcal{H}_1 \rightarrow \mathbf{C}$  be defined by

$$[(f), (g)] = [f, g]', \quad f \in (f) \in \mathcal{H}_1, g \in (g) \in \mathcal{H}_1. \quad (72)$$

Then  $[\cdot, \cdot]$  is an inner product on  $\mathcal{H}_1$  and define  $\mathcal{H}_0$  as its completion in  $[\cdot, \cdot]$ . Let  $\pi_0: C_b(\hat{G}) \rightarrow \mathcal{H}_0$  be the canonical projection. Thus  $\mathcal{H}_0$  is nontrivial and need not be separable. Now let us replace  $\mathcal{H}_0$  by  $L_0^2(P')$  on a probability space  $(\Omega', \Sigma', P')$ . This can be done based on the Fubini-Jessen theorem where  $P'$  can even be taken to be a Gaussian measure (for the real  $\mathcal{H}$ , see [36], pp. 414-415). The complex case is similar. A quick outline is as follows: Let  $\{h_i, i \in I\} \subset \mathcal{H}_0$  be a CON set. If  $(\Omega_i, \Sigma_i, P_i)$  is a probability space determined by a complex Gaussian variable, so that one can take  $\Omega_i = \mathbf{C}$ ,  $\Sigma_i =$  Borel  $\sigma$ -algebra of  $\mathbf{C}$ , and

$$P_i(A) = (2\pi)^{-1} \int_A \exp\left(-\frac{|t|^2}{2}\right) dt_1 dt_2, \quad A \in \Sigma_i, (t = t_1 + \sqrt{-1} t_2),$$

let  $(\Omega', \Sigma', P') = \bigotimes_{i \in I} (\Omega_i, \Sigma_i, P_i)$  the product space given by the Fubini-Jessen theorem. If  $X_i(\omega) = \omega(i)$ ,  $\omega \in \Omega' = \mathbf{C}^I$ , the coordinate function, then  $E(X_i) = 0$  and  $E(|X_i|^2) = 1$ . Also  $\{X_i, i \in I\}$  forms a CON basis of  $\mathcal{L} = \overline{\text{sp}}\{X_i, i \in I\} \subset L_0^2(P')$ . The correspondence  $\tau: h_i \rightarrow X_i$ , extended linearly, sets up an isomorphism of  $\mathcal{H}_0$  onto  $\mathcal{L}$ , and

$$\|\tau(h_i)\|_2^2 = E(|X_i|^2) = 1 = [h_i, h_i], \quad i \in I.$$

Then by polarization one has  $[h_i, h_j] = E(\tau(h_i) \overline{\tau(h_j)})$ , so that  $\tau$  is an isometric isomorphism of  $\mathcal{H}_0$  onto  $\mathcal{L} \subset L_0^2(P')$ , as desired.

If  $\pi = \tau \circ \pi_0: f \mapsto \tau(\pi_0(f)) \in \mathcal{H}' \subset L_0^2(P')$ ,  $f \in C_b(\hat{G})$ , is the composite (canonical) mapping, let  $X_1(t) = \pi(e_t(\cdot)) \in \mathcal{H}'$  where  $e_t: s \mapsto (t, s)$ , is a character of  $G$  at  $t \in G$ . Note that  $e_0 = 1 \notin \mathcal{N}_0$ , so  $\pi_0(1)$  can be identified with the constant  $1 \in C_b(\hat{G})$ . Thus

$$X_1(0) = \tau(1), E(|\tau(1)|^2) = 1.$$

Let  $\mathcal{H}'' = \overline{\text{sp}}\{X_1(t), t \in G\} \subset \mathcal{H}'$ . Then there exists a probability space  $(\Omega'', \Sigma'', P'')$ , as above, such that  $\mathcal{H}'' \subset L^2(P'')$ . Finally set  $\mathcal{H} = \mathcal{H} \oplus \mathcal{H}''$ , in the

direct sum of Hilbert spaces  $L_0^2(P)$  and  $L_0^2(P'')$ . If  $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) = (\Omega, \Sigma, P) \otimes (\Omega'', \Sigma'', P'')$  then one can identify, in a natural way,  $\mathcal{H} \subset L_0^2(\tilde{P})$ . Define  $Y(t) = X(t) + X_1(t)$ ,  $t \in G$ , so that  $(X(t), X_1(t)) = 0$  since  $\mathcal{H} \perp \mathcal{H}''$  in  $\mathcal{H}$ . Then  $\{Y(t), t \in G\} \subset \mathcal{H} \subset L_0^2(\tilde{P})$ , and if  $Q: \mathcal{H} \rightarrow \mathcal{H} = \{\mathcal{H} \oplus \{0\}\}$  is the orthogonal projection, one has  $X(t) = QY(t)$ ,  $t \in G$ . It remains to show that  $Y: G \rightarrow L_0^2(\tilde{P})$  is stationary. By construction  $Y(0) = X(0) + X_1(0)$  and this is  $X(0)$  only when  $X_1(0) = 0$  which can happen iff  $\mathcal{H}'' = \{0\}$ , i.e., when no enlargement is needed.

To verify stationarity, consider

$$\begin{aligned} r(s, t) &= (Y(s), Y(t)) = (X(s), X(t)) + (X_1(s), X_1(t)) \text{ since } X \perp X_1, \\ &= \int_{\tilde{G}} \int_{\tilde{G}} (s, \lambda) \overline{(t, \lambda')} F(d\lambda, d\lambda') + \int_{\tilde{G}} \int_{\tilde{G}} (s, \lambda) \overline{(t, \lambda')} \alpha(d\lambda, d\lambda'), \\ &\quad \text{by (69) and (72) and these are MT-integrals,} \\ &= \int_{\tilde{G}} \int_{\tilde{G}} (s, \lambda) \overline{(t, \lambda')} v(d\lambda, d\lambda'), \text{ since } \alpha = v - F \\ &= \int_{\tilde{G}} (s, \lambda) \overline{(t, \lambda)} \mu(d\lambda), \text{ by (68),} \\ &= \int_{\tilde{G}} (s-t, \lambda) \mu(d\lambda), \text{ by the composition of characters.} \end{aligned} \tag{73}$$

Since  $\mu$  is a finite positive measure, (73) implies

$$r(s+h, t+h) = r(s, t) = \tilde{r}(s-t),$$

and so the  $Y: G \rightarrow L_0^2(\tilde{P})$  is stationary. The construction also implies that  $\overline{\text{sp}\{Y(t), t \in G\}} = \mathcal{H}$  in the case that  $\mathcal{H} = \overline{\text{sp}\{X(t), t \in G\}}$ . This completes the proof.

The following is a useful deduction:

**COROLLARY 6.2.** *Every vector measure  $v: \mathcal{B}(G) \rightarrow \mathcal{H}$  where  $G$  is an LCA group,  $\mathcal{B}(G)$  being its Borel algebra, and  $\mathcal{H}$  is a Hilbert space, has an orthogonally scattered dilation.*

*Proof.* Since  $G = \hat{\hat{G}}$  consider the mapping  $X: \hat{\hat{G}} \rightarrow \mathcal{H}$  defined as the D-S integral  $X(\hat{g}) = \int_{\hat{G}} \langle \hat{g}, \lambda \rangle v(d\lambda)$ . Then  $X$  is  $V$ -bounded; so it is weakly harmonizable. By the above theorem there are an extension Hilbert space  $\mathcal{K} \supset \mathcal{H}$ , an orthogonal projection  $Q: \mathcal{K} \rightarrow \mathcal{H}$ , with range  $\mathcal{H}$ , and a stationary field  $Y: \hat{\hat{G}} \rightarrow \mathcal{K}$  such that  $X(\hat{g}) = QY(\hat{g})$ . Let  $Z$  be the stochastic measure representing  $Y$ , (cf. Theorem 3.3). Hence for each  $h \in \mathcal{H}$  one has  $(Z: \mathcal{B}(\hat{\hat{G}}) \rightarrow \mathcal{K})$

$$\int_{\hat{G}} (\hat{g}, \lambda) (v(d\lambda), h) = (X(\hat{g}), h) = (QY(\hat{g}), h) = \int_{\hat{G}} (\hat{g}, \lambda) (Q \circ Z(d\lambda), h).$$

These are now scalar (Lebesgue-Stieltjes) integrals. By the classical uniqueness theorem of Fourier analysis for such integrals, one has

$$(v(A) - Q \circ Z(A), h) = 0, A \in \mathcal{B}(G), h \in \mathcal{H}.$$

Hence  $v = Q \circ Z$ . Since  $Z$  is orthogonally scattered by virtue of the fact that  $Y$  is stationary, the result follows.

With the last theorem, a more perspicuous version of the dilation problem for a weakly harmonizable random field can be given. This, however, depends also on an interesting theorem of Sz.-Nagy [41] and will be presented. Recall from the classical theory of stationary processes ([6], p. 512 and p. 638) every such process  $\{Y_t, t \in \mathbf{R}\} \subset L_0^2(P)$ , can be expressed as  $Y_t = U_t Y_0$ , where  $\{U_t, t \in \mathbf{R}\}$  is a group of unitary operators acting on  $L_0^2(P)$  (first on  $\overline{\text{sp}}\{Y_t, t \in \mathbf{R}\}$  and then, for instance, define each  $U_t$  as an identity on the orthogonal complement of this subspace). The spectral theory of  $U_t$  then yields immediately the corresponding integral representation of  $Y_t$ 's. The same result holds if  $\mathbf{R}$  is replaced by an LCA group  $G$ . The corresponding operator representation for harmonizable processes (or fields) is not so simple. Its solution will be presented in the following theorem. Recall that a family  $T : G \rightarrow B(\mathcal{X})$ ,  $\mathcal{X}$  a Hilbert space, is of positive type if  $T(-g) = T(g)^*$  (adjoint operator) and for each finite set  $\{x_{s_1}, \dots, x_{s_n}\}$  of  $\mathcal{X}$  indexed by  $J = \{s_1, s_2, \dots, s_n\} \subset G$ , one has

$$\sum_{i=1}^n \sum_{j=1}^n (T(s_j^{-1} s_i) x_{s_i}, x_{s_j}) \geq 0. \quad (74)$$

**THEOREM 6.3.** *Let  $G$  be an LCA group and  $X : G \rightarrow L_0^2(P) = \mathcal{X}$ , a Hilbert space, be weakly harmonizable. Then there exists a super Hilbert space  $\mathcal{K} = L_0^2(\tilde{P}) \supset \mathcal{X}$  on an enlarged probability space  $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ , a random variable  $Y_0 \in \mathcal{K}$  a weakly continuous family  $\{T(g), g \in G\}$  of contractive linear operators from  $\mathcal{K}$  to  $\mathcal{X}$  with  $T(0)$  as the identity on  $\mathcal{X}$  ( $0$  being the neutral element of  $G$ ), such that, when its domain is restricted to  $\mathcal{X}$ , it is of positive type, in terms of which  $X(g) = T(g)Y_0, g \in G$ . Conversely every weakly continuous contractive family  $\{T(g), g \in G\}$  of the above type from any super Hilbert space  $\mathcal{K} \supseteq \mathcal{X}$  into  $\mathcal{X}$  which, when restricted to  $\mathcal{X}$  is of positive type, defines a weakly harmonizable process  $X : G \rightarrow \mathcal{X}$ , by the equation  $X(g) = T(g)Y_0$  for any  $Y_0 \in \mathcal{X}$ ,  $T(0)$  being identity on  $\mathcal{X}$ .*

*Proof.* The direct part is an operator-theoretic reformulation of Theorem 6.1. Briefly, let  $X : G \rightarrow L_0^2(P) = \mathcal{X}$  be weakly harmonizable. Then there exist a  $\mathcal{K} = L_0^2(\tilde{P}) \supset \mathcal{X}$  and a stationary  $Y : G \rightarrow \mathcal{K}$  such that  $X(g) = QY(g), g \in G$ , by Theorem 6.1 with  $Q$  as the orthogonal projection on  $\mathcal{X}$  and range  $\mathcal{X}$ . But  $Y(g) = U(g)Y(0)$  where  $\{U(g), g \in G\}$  is a (strongly) continuous group of unitary operators on  $\mathcal{K}$ . Let  $T(g) = QU(g), g \in G$ . It is asserted that  $\{T(g), g \in G\}$  is the desired family.

Indeed,  $T(0) = Q$  (= identity on  $\mathcal{X}$ ), and  $\|T(g)\| \leq \|Q\| \|U(g)\| \leq 1$ . The continuity of  $U(g)$  on  $G$  clearly implies the weak continuity of  $T(g)$ 's. To verify the positive definiteness on  $\mathcal{X}$ , let  $h_{s_1}, \dots, h_{s_n}$  be a finite set in  $\mathcal{X}$ . Then letting  $\tilde{T}(g) = T(g)|_{\mathcal{X}}$  one has  $\tilde{T}(-g) = (\tilde{T}(g))^*$  since

$$\begin{aligned} (\tilde{T}(-g)h_{s_1}, h_{s_2}) &= (QU(-g)h_{s_1}, h_{s_2}) = (U^*(g)h_{s_1}, Qh_{s_2}) \\ &= (h_{s_1}, U(g)h_{s_2}), \text{ since } Qh_{s_i} = h_{s_i} \text{ and } U^{**}(g) = U(g), \\ &= (Qh_{s_1}, U(g)h_{s_2}) = (h_{s_1}, QU(g)h_{s_2}) \\ &= (h_{s_1}, \tilde{T}(g)h_{s_2}) = (\tilde{T}(g)^*h_{s_1}, h_{s_2}), h_{s_i} \in \mathcal{X}, i = 1, 2. \end{aligned} \quad (75)$$

Similarly,

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n (\tilde{T}(s_j^{-1}s_i)h_{s_i}, h_{s_j}) &= \sum_{i=1}^n \sum_{j=1}^n (QU(-s_j)U(s_i)h_{s_i}, h_{s_j}) \\ &= \sum_{i=1}^n \sum_{j=1}^n (U(s_j)^*U(s_i)h_{s_i}, h_{s_j}) \\ &= \left\| \sum_{i=1}^n U(s_i)h_{s_i} \right\|^2 \geq 0. \end{aligned} \quad (76)$$

The converse depends explicitly on an important theorem of Sz.-Nagy ([41], Thm. III; this is an extension of a classical result of Naïmark). According to this result if  $\tilde{T}(\cdot) = T(\cdot)|_{\mathcal{X}}$ , then there is a super Hilbert space  $\mathcal{K}_1 \supset \mathcal{X}$  ( $\mathcal{K}_1$  may be quite different from  $\mathcal{K}$ ) and a weakly (hence strongly) continuous group  $\{V(g), g \in G\}$  of unitary operators on  $\mathcal{K}_1$  such that  $\tilde{T}(g) = Q_1V(g)|_{\mathcal{X}}$ ,  $Q_1$  being the orthogonal projection of  $\mathcal{K}_1$  onto  $\mathcal{X}$ . Here  $\mathcal{K}_1$  can be chosen as  $\mathcal{K}_1 = \overline{\text{sp}\{V(g)\mathcal{X}, g \in G\}}$ . If  $x_0 \in \mathcal{X}$  is arbitrary, then  $x_0 \in \mathcal{K}_1 \cap \mathcal{K}$ , and

$$T(g)x_0 = \tilde{T}(g)x_0 = Q_1V(g)x_0 = X(g), \quad (\text{say}), g \in G.$$

But  $\{Y(g) = V(g)x_0, g \in G\} \subset \mathcal{K}_1$  is a stationary process so that by the first paragraph of the proof of Theorem 6.1,  $\{X_0(g), g \in G\} \subset \mathcal{X}$  is weakly harmonizable. Thus for each  $x_0 \in \mathcal{X}$ ,  $\{T(g)x_0, g \in G\}$  is weakly harmonizable, and this completes the proof.

*Remark.* In the converse direction one can take  $\mathcal{K} = \mathcal{X}$ . However in the forward direction, it is not always possible to take  $Y_0$  in  $\mathcal{X}$ , so that  $X(0) = Y_0$ , as the example following Definition 2.1 shows. Thus there is an inherent asymmetry in the statement of this theorem, and the mention of the super Hilbert space  $\mathcal{K}$  in the enunciation cannot be avoided. It should also be noted that the above quoted theorem of Sz.-Nagy [41] can be deduced also from Naïmark's theorem and Theorem 6.1. See [38] for a further discussion on this point.