

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 28 (1982)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** HARMONIZABLE PROCESSES: STRUCTURE THEORY  
**Autor:** Rao, M. M.  
**Kapitel:** 5. Domination problem for harmonizable fields  
**DOI:** <https://doi.org/10.5169/seals-52243>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 10.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$\begin{aligned}
0 &\leq \int_{\mathbf{R}} \int_{\mathbf{R}} f(s) \overline{f(t)} F_n(ds, dt) \leq \int_{\mathbf{R}} \int_{\mathbf{R}} |f(s) \overline{f(t)}| |F_n|(ds, dt) \\
&\leq \frac{1}{2} \left[ \int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)|^2 |F_n|(ds, dt) + \int_{\mathbf{R}} \int_{\mathbf{R}} |f(t)|^2 |F_n|(ds, dt) \right], \\
&\quad \text{since } |ab| \leq (|a|^2 + |b|^2)/2, \\
&= \int_{\mathbf{R}} |f(s)|^2 \beta_n(ds), \quad \text{by (54)}. \tag{55}
\end{aligned}$$

This and (53) yield

$$\begin{aligned}
\alpha_0^f &= \left\| \int_{\mathbf{R}} f(\lambda) Z(d\lambda) \right\|_2^2 = \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda) \overline{f(\lambda')} F_n(d\lambda, d\lambda') \\
&\leq \liminf_n \int_{\mathbf{R}} |f(\lambda)|^2 \beta_n(d\lambda), \quad f \in C_0(\mathbf{R}). \tag{56}
\end{aligned}$$

This completes the proof.

*Remark.* For a deeper analysis of the structure of these processes, it is desirable to replace the sequence  $\{\beta_n, n \geq 1\}$  by a single Borel measure. This is nontrivial. In the next section for a more general version, including harmonizable fields, such a result will be obtained.

## 5. DOMINATION PROBLEM FOR HARMONIZABLE FIELDS

The work of the preceding section indicates that the weakly harmonizable processes are included in the class of functions which are Fourier transformations of vector measures into Banach spaces. A characterization of such functions, based on the  $V$ -boundedness concept of [2], has been obtained first in [33]. For probabilistic applications (e.g., filtering theory) the domination problem, generalizing Theorem 4.5, should be solved. The following result illuminates the nature of the general problem under consideration.

**THEOREM 5.1.** *Let  $(\Omega, \Sigma)$  be a measurable space,  $\mathcal{X}$  a Banach space and  $\nu: \Sigma \rightarrow \mathcal{X}$  be a vector measure. Then there exists a (finite) measure  $\mu: \Sigma \rightarrow \mathbf{R}^+$ , a continuous convex function  $\varphi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  such that  $\frac{\varphi(x)}{x} \nearrow \infty$  as  $x \nearrow \infty$ ,  $\varphi(0) = 0$ , and  $\nu$  has  $\varphi$ -semivariation finite relative to  $\mu$  in the sense that*

$$\|\nu\|_{\varphi}(\Omega) = \sup \left\{ \left\| \int_{\Omega} f(\omega) \nu(d\omega) \right\|_{\mathcal{X}} : \|f\|_{\psi, \mu} \leq 1 \right\} < \infty, \tag{57}$$

where  $\|f\|_{\psi, \mu} = \inf \left\{ \alpha > 0 : \int_{\Omega} \psi \left( \frac{|f(\omega)|}{\alpha} \right) \mu(d\omega) \leq 1 \right\} < \infty$ , and the

integral relative to  $\nu$  in (57), is in the Dunford-Schwartz sense. Here  $\psi: \mathbf{R}^+ \rightarrow \mathbf{R}^+$  is a convex function given by  $\psi(x) = \sup \{ |x| y - \varphi(y) : y \geq 0 \}$ .

The proof of this result depends on some results of ([8], IV.10) and elementary properties of Orlicz spaces (cf. [47], p. 173). It will be omitted here, since the details are given in [38]. This is only motivational for what follows.

Note that (57) is a desired generalization of (51) if  $\{\beta_n, n \geq 1\}$  is replaced by  $\mu$  and  $\|\cdot\|_2$  is replaced by  $\|\cdot\|_\varphi$ . However  $\varphi$  may grow faster than a polynomial. What is useful here is a  $\varphi$  with  $\varphi(x) = |x|^p, 1 \leq p \leq 2$ . This can be proved for a special class of spaces  $\mathcal{X}$ , which is sufficient for our study of harmonizable fields.

It will be convenient to introduce a definition and to state a result (essentially) of Grothendieck and Pietsch, for the work below.

**Definition 5.2.** Let  $\mathcal{X}, \mathcal{Y}$  be a pair of Banach spaces and, as usual,  $B(\mathcal{X}, \mathcal{Y})$  be the space of bounded linear operators on  $\mathcal{X}$  into  $\mathcal{Y}$ . If  $1 \leq p \leq \infty$ ,  $T \in B(\mathcal{X}, \mathcal{Y})$ , then  $T$  is called *p-absolutely summing* if  $\alpha_p(T) < \infty$ , where

$$\alpha_p(T) = \inf \left\{ c > 0 : \left[ \sum_{i=1}^n \|Tx_i\|^p \right]^{1/p} \leq c \sup_{\|x^*\| \leq 1} \left( \sum_{i=1}^n |x^*(x_i)|^p \right)^{1/p}, x_i \in \mathcal{X}, \right. \\ \left. 1 \leq i \leq n, n \geq 1 \right\}, \quad (58)$$

with  $x^* \in \mathcal{X}^*$ , the adjoint space of  $\mathcal{X}$ .

The following result, which is alluded to above, with a short proof may be found in [22] together with some extensions and applications.

**PROPOSITION 5.3.** Let  $T \in B(\mathcal{X}, \mathcal{Y})$  be *p-absolutely summing*,  $1 \leq p < \infty$ . Let  $K^*$  be the weak-star closure of the set of extreme points of the unit ball  $U^*$  of  $\mathcal{X}^*$ . Then there is a regular Borel probability measure  $\mu$  on the compact space  $K^*$  such that

$$\|Tx\|_{\mathcal{Y}} \leq \alpha_p(T) \left[ \int_{K^*} |x^*(x)|^p \mu(dx^*) \right]^{1/p}, \quad x \in \mathcal{X}. \quad (59)$$

Conversely (and this is simple), if  $T$  satisfies (59) for some  $\mu$  on  $K^*$  with a constant  $\gamma_0$ , then  $T$  is *p-absolutely summing* and  $\alpha_p(T) \leq \gamma_0$ . Further any *p-absolutely summing operator* is weakly compact.

Let us specialize this result in the case that  $\mathcal{X} = C_r(S)[C(S)]$ , the space of real [complex] continuous functions on a compact set  $S$ . Let  $K$  be the set of all extreme points of the unit ball  $U^*$  of  $(C_r(S))^*$  and  $q: S \rightarrow (C_r(S))^*$  be the mapping defined by  $q(s) = l_s$  with  $l_s(f) = f(s)$ ,  $f \in C_r(S)$  so that  $l_s$  is the evaluation functional,  $\|l_s\| = 1$ , and  $l_s \in K$ ,  $s \in S$ . Some other known results needed from Linear Analysis, in the form used here, are as follows. (For details, see [4], Sec. V.3; [8], p. 441). In this case the spaces  $S$  and  $q(S)$  are homeomorphic and  $q(S)$  is closed since  $S$  is compact. By Mil'man's theorem  $U^*$  is the weak-star

closed convex hull of  $q(S) \cup (-q(S))$ , and (by the compactness of  $S$ ) the latter is equal to the extreme point-set of  $U^*$  and is closed. Further these are of the form  $\alpha l_s$ ,  $s \in S$ , and  $|\alpha| = 1$  (cf. [8], V.8.6). Consequently (59) becomes

$$\begin{aligned} \|Tf\|^p &\leq (\alpha_p(T))^p \cdot \int_{q(S) \cup (-q(S))} |l_s(f)|^p \mu(dl_s), \quad f \in C_r(S) \\ &\leq 2(\alpha_p(T))^p \cdot \int_{q(S)} |l_s(f)|^p \mu(dl_s), \\ &= 2(\alpha_p(T))^p \cdot \int_S |f(s)|^p \mu(ds), \end{aligned}$$

if  $S$  and  $q(S)$  are (as they can be) identified.

For the complex case,  $C(S) = C_r(S) + iC_r(S)$ , and so the same holds if the constants are doubled. Thus

$$\|Tf\| \leq C_p \left[ \int_S |f(s)|^p \mu(ds) \right]^{\frac{1}{p}} = C_p \|f\|_{p, \mu}, \quad f \in C(S), \quad (60)$$

where  $C_p^p = 4[\alpha_p(T)]^p$ . This form of (59) will be utilized below.

**Definition 5.4.** Let  $\mathcal{X}$  be a Banach space,  $1 \leq p \leq \infty$  and  $1 \leq \lambda < \infty$ . Then  $\mathcal{X}$  is called an  $\mathcal{L}_{p, \lambda}$ -space if for each  $n$ -dimensional space  $E \subset \mathcal{X}$ ,  $1 \leq n < \infty$ , there is a finite dimensional  $F \subset \mathcal{X}$ ,  $E \subset F$ , such that  $d(F, l_p^n) \leq \lambda$  where  $l_p^n$  is the  $n$ -dimensional sequence space with  $p$ -th power norm and where

$$d(E_1, E_2) = \inf \{ \|T\| \|T^{-1}\| : T \in B(E_1, E_2) \}$$

for any pair of normed linear spaces  $E_1, E_2$ . A Banach space  $\mathcal{X}$  is an  $\mathcal{L}_p$ -space if it is an  $\mathcal{L}_{p, \lambda}$ -space for some  $\lambda \geq 1$ .

It is known (and easy to verify) that each  $L^p(\mu)$ ,  $p \geq 1$ , is an  $\mathcal{L}_{p, \lambda}$ -space for every  $\lambda > 1$ , and  $C(S)$  [indeed each abstract  $(M)$ -space] is an  $\mathcal{L}_{\infty, \lambda}$ -space for every  $\lambda > 1$ . The class of  $\mathcal{L}_2$ -spaces coincides with the class of Banach spaces isomorphic to a Hilbert space. For proofs and more on these ideas the reader is referred to the article of Lindenstrauss and Pełczyński [22].

With this set up the following general result can be established at this time on the domination problem for vector measures.

**THEOREM 5.5.** Let  $S$  be a locally compact space and  $C_0(S)$  be the Banach space of continuous scalar functions on  $S$  vanishing at " $\infty$ ". If  $\mathcal{Y}$  is an  $\mathcal{L}_p$ -space  $1 \leq p \leq 2$ , and  $T \in B(C_0(S), \mathcal{Y})$ , then there exist a finite positive Borel measure  $\mu$  on  $S$ , and a vector measure  $Z$  on  $S$  into  $\mathcal{Y}$ , such that

$$\left\| \int_S f(s) Z(ds) \right\|_{\mathcal{Y}} = \|Tf\|_{\mathcal{Y}} \leq \|f\|_{2, \mu}, \quad f \in C_0(S). \quad (61)$$

*Proof.* Since  $\mathcal{X} = C_0(S)$  is an abstract  $(M)$ -space, it is an  $\mathcal{L}_{\infty}$ -space by the preceding remarks. But  $\mathcal{Y}$  is an  $\mathcal{L}_p$ -space  $1 \leq p \leq 2$ , and so  $T \in B(\mathcal{X}, \mathcal{Y})$  is 2-absolutely summing by ([22], Thm. 4.3), and therefore (cf. Prop. 5.3 above) it is also weakly compact. By the argument presented for (37), (38) above, one can use

the theorem ([8], VI.7.3) even when  $S$  is locally compact (and noncompact) to conclude that there is a vector measure  $Z$  on the Borel  $\sigma$ -ring of  $S$  into  $\mathcal{Y}$  such that

$$Tf = \int_S f(s)Z(ds), \quad (\text{D-S integral}).$$

Using the argument of (37), if  $\tilde{S}$  is the one point compactification of  $S$ , and  $\tilde{T} \in B(C(\tilde{S}), \mathcal{Y})$  is the norm preserving extension, then  $\tilde{T}$  is 2-absolutely summing (since  $C(\tilde{S})$  is an abstract  $(M)$ -space), and weakly compact. So by (60) there exists a finite Borel measure  $\tilde{\mu}$  on  $\tilde{S}$  such that

$$\|\tilde{T}f\|_{\mathcal{Y}} \leq c_p \|f\|_{2, \tilde{\mu}}, \quad f \in C(\tilde{S}).$$

Letting  $\bar{\mu} = c_p^p \tilde{\mu}$ , one has  $\|\tilde{T}f\|_{\mathcal{Y}} \leq \|f\|_{2, \bar{\mu}}$ ,  $f \in C(\tilde{S})$ . So (61) holds on  $\tilde{S}$ . Let  $\mu(\cdot) = \bar{\mu}(S \cap \cdot)$  so that  $\mu$  is a finite Borel measure on  $S$ . If now one restricts to  $C_0(S)$  identified as a subset of  $C(\tilde{S})$ , so that  $T = \tilde{T}|_{C_0(S)}$ , it follows from the preceding analysis that  $\|Tf\|_{\mathcal{Y}} \leq \|f\|_{2, \mu}$  for all  $f \in C_0(S)$ . Since the integral representation of  $T$  is evidently true, this establishes (61), and completes the proof of the theorem.

If  $\mathcal{Y}$  is a Hilbert space, it is an  $\mathcal{L}_2$ -space so that the above theorem considerably strengthens Theorem 4.5, since the sequence there is now replaceable by a single measure.

The following statement is actually a consequence of the above result, and it will be invoked in the last section.

**PROPOSITION 5.6.** *Let  $(\Omega, \Sigma)$  be any measurable space, and  $\mathcal{X} = B(\Omega, \Sigma)$  be the Banach space (under uniform norm) of scalar measurable functions. If  $\mathcal{Y}$  is an  $\mathcal{L}_p$ -space,  $1 \leq p \leq 2$ , as above,  $T \in B(\mathcal{X}, \mathcal{Y})$  is such that for each  $f_n \in \mathcal{X}$ ,  $f_n \rightarrow f$  pointwise boundedly implies  $\|Tf_n\| \rightarrow \|Tf\|$ , then there exist  $\sigma$ -additive functions  $Z: \Sigma \rightarrow \mathcal{Y}$ ,  $\mu: \Sigma \rightarrow \mathbf{R}^+$ , such that*

$$\left\| \int_{\Omega} f(\omega)Z(d\omega) \right\|_{\mathcal{Y}} = \|Tf\|_{\mathcal{Y}} \leq \|f\|_{2, \mu}, \quad f \in \mathcal{X}. \quad (62)$$

The proof uses the fact that  $B(\Omega, \Sigma)$  is isometrically isomorphic to  $C(S)$ , for a compact (extremely disconnected) Hausdorff space (cf. [8], IV.6.18), and reduces to the preceding result. The computations, using the standard Carathéodory measure theory, will be omitted here. The details, however, may be found in [38].

*Remark.* The preceding results show that the domination problem for vector measures in  $L^p$ -spaces,  $1 \leq p \leq 2$ , is solved and hence also for harmonizable fields since only the  $\mathcal{L}_2$ -type spaces are involved in the latter. But, for  $p > 2$ , such a satisfactory solution of the problem is not available.