HARMONIZABLE PROCESSES: STRUCTURE THEORY [1]

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Dedicated to the memory of Prof. S. Bochner

CONTENTS

1. INTRODUCTION

If $\mathscr{H}$ is a complex Hilbert space and $X : \mathbf{R} \to \mathscr{H}$ is a mapping, then the curve $\{X(t), t \in \mathbf{R}\}$ is often called a *second order* (or Hilbertian) stochastic process, and if $\mathbf{R}$ is replaced by $\mathbf{R}^n$, $n \geqslant 2$, it is called a (Hilbertian) *random field*. Following Khintchine who developed the initial theory (1934), the process (or field) is called *weakly stationary* if $r : (s, t) \mapsto (X(s), X(t))$, termed the *covariance function* of the

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process, is continuous and depends only on $s - t$, where $(\cdot, \cdot)$ is the inner product in $\mathscr{H}$. Thus $r(s, t) = r(s-t)$. But then $r : \mathbf{R} \to \mathbf{C}$ is a continuous positive definite function and by the classical Bochner theorem (1932), r is expressible as:

$$r(t) = \int_{\mathbf{R}} e^{it\lambda} F(d\lambda), \qquad t \in \mathbf{R}, \tag{1}$$

for a unique positive bounded Borel measure F on $\mathbf{R}$. This F is called the *spectral measure* of the process. Because of the above connection with the Fourier transform theory, important advances have been made on the structural analysis of such stationary processes. For instance, according to a celebrated theorem of Cramér and Kolmogorov, each such stationary process admits an integral representation:

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R}, \tag{2}$$

where Z is an $\mathscr{H}$-valued "orthogonally scattered" measure on the Borel sets of $\mathbf{R}$ (i.e., Z is σ-additive and $(Z(A), Z(B)) = F(A \cap B)$), and the vector integral in (2) is suitably defined. Stationary processes find important applications in such areas as meteorology, communication and electrical engineering among others. The well developed theory and applications are now included in many monographs (cf. e.g. Doob [6, Ch. X-XII], Yaglom [44]), and especially for applications one may refer to Wiener's pioneering work [43].

While stationary processes (the qualification "weakly" will be dropped) admit a deep and beautiful mathematical theory, there are many problems for which stationarity is an unacceptable restriction. For instance, in econometrics and in the signal detection problems related to the navy, among others, it is quite desirable that the covariance function r be not so restricted as to be a function of a single variable. This necessitates a relaxation of stationarity and then (1) cannot obtain. To accommodate such problems while still retaining the methods of harmonic analysis, Loève has introduced in the middle 1940's the first weakening called "harmonizability". Thus a process $\{X(t), t \in \mathbf{R}\} \subset \mathscr{H}$ is *Loève* (to be called *strongly* hereafter) *harmonizable* if its covariance is expressible as (cf. [23], p. 474)

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}, \tag{3}$$

for a unique positive definite $F : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ of bounded variation (in the classical Vitali sense) in the plane. If F of (3) concentrates on the diagonal of $\mathbf{R} \times \mathbf{R}$, (3) reduces to (1). Loève also gave a representation of $X(t)$ analogous to (2), but now $Z(\cdot)$ will only satisfy $(Z(A), Z(B)) = F(A, B)$. Even though $r(\cdot, \cdot)$ of (3) is bounded and uniformly continuous, one does not have an elegant characterization of a harmonizable covariance analogous to (1). In fact

Loève raised this problem ([23], p. 477). A solution of it was presented in ([34], Thm. 5), but it is not effective in the sense that the conditions are not easily verifiable, although the characterization reduces to Bochner's theorem in the stationary case.

Other extensions of stationarity, of interest in applications, soon appeared. In 1947, Karhunen introduced a class of processes whose covariance r can be expressed as:

$$r(s, t) = \int_{\mathbf{R}} g(s, \lambda)\bar{g}(t, \lambda)F(d\lambda)\,, \qquad s, t \in \mathbf{R}\,, \tag{4}$$

where $\{g(t, \cdot), t \in \mathbf{R}\}$ is a family of Borel functions in $L^2(\mathbf{R}, F(d\lambda))$, with F as a bounded (or σ-finite) Borel measure on $\mathbf{R}$. If $g(t, \lambda) = e^{it\lambda}$, then for bounded F (4) reduces to (1). In 1951, Cramér has introduced in [3] a further generalization, to be called *class* (C) here, which contains both (3) and (4), by requiring only that r be representable as:

$$r(s, t) = \int_{\mathbf{R}}\int_{\mathbf{R}} g(s, \lambda)\bar{g}(t, \lambda')F(d\lambda, d\lambda')\,, \qquad s, t \in \mathbf{R}\,, \tag{5}$$

for a family $\{g(t, \cdot), t \in \mathbf{R}\}$ of Borel functions and a positive definite F of finite local (i.e., on each relatively compact rectangle) Vitali variation in $\mathbf{R}^2$, such that (5) holds. The corresponding stochastic integral representation of $X(t)$, generalizing (2), was also given. Both (4) and (5) have only a superficial contact with the methods of Fourier analysis. However, a very general concept which fully utilizes the advantages of Fourier analysis and which contains the Loève harmonizability was introduced by Bochner in 1953 under the name V-boundedness [2]. It turns out that (cf. Thm. 4.2 below) a second order process is V-bounded iff (= if and only if) it is the Fourier transform of a general vector measure on $\mathbf{R}$ into a Banach space $\mathcal{X}$. Independently of the work of [2], Rozanov [40], in 1959, considered a generalized concept again under the name "harmonizable", but which is different from Loève's definition. It will be called *weakly harmonizable* here. It turns out that, in this case, the covariance function r of the process is formally expressible in the form (3) relative to a positive definite F which is merely of Fréchet variation finite. The integral in (3) then cannot be defined in the Lebesgue sense, and a weaker Morse-Transue integral [26] appears in this work.

Even though each of these generalizations is inspired by the stationarity notion of Khintchine, each is different from one another, and their interrelations have not been fully established before. One of the main purposes of this paper is to present a detailed and unified structural analysis of these processes and obtain their characterization. This exposition utilizes some elementary aspects of vector measure theory which obviates a separate definition of the "stochastic integral" for each representation of the process under consideration in the form (2). From this analysis one finds that Loève's definition is more restrictive than Rozanov's and that Bochner's concept is mathematically the most elegant and general. Further in the Hilbert space context, it is shown that Bochner's and Rozanov's concepts coincide. It was already noted in [2] that Loève's definition is subsumed by V-boundedness. An interesting geometrical feature is that the Bochner class of second order processes is always a projection of a stationary family in a Hilbert space. Bochner's concept, as indicated above, is based on Fourier vector integration, and this abstract point of view yields different characterizations, one of which extends a scalar result of Helson [12] on characterizing Fourier transforms of signed measures, to separable reflexive Banach spaces. A further relation is that a process of the Bochner-Rozanov class in Hilbert space is a strong limit of a sequence of Loève harmonizable processes, uniformly on compact subsets of the line **R**.

A first comparative study of the Bochner and Loève classes in Hilbert space was given by Niemi in his thesis [29]. Then in [30] and [31] he essentially established that the V-boundedness in Hilbert space is the projection of a stationary family, extending a special case by Abreu [1]. The latter point was clarified and the same result was reestablished by a slightly different method in [25]. A further extension of the last work was announced in [39]. A key domination inequality, on which the projection results depend, is based on some work of Grothendieck. In particular, the methods of [25], [30] and [31] rest on Pietsch's form of this Grothendieck inequality. The work of the present paper utilizes some properties of the p-summing operators of [22]. I believe that the latter point of view yields a better understanding of the structure of the problem, with a more general solution and additional insight, not afforded by the earlier work. Thus the present paper is aimed at a comprehensive, unified and extended treatment of the structure of the Bochner-Rozanov class. It may be remarked that an essentially equivalent characterization of Bochner's Hilbert space version can be obtained using the results from an early paper due to Phillips [33], which seems to have been overlooked by almost all vector measure theorists and stochastic analysts. It is, in a sense, subsumed under a relatively recent paper by Kluvanek [21]. But most of all, Bochner's paper [2] has not been accorded the central place it deserves in probabilistic treatments on the subject. I hope that the present work will bring some of the many fundamental ideas of [2] to the forefront.

Finally, the concept of the spectral measure F of (1), so appropriate and natural in the stationary case (since it is *positive* and bounded) does not appear in a similar form for the harmonizable (or other nonstationary) processes, since F is usually complex valued as in (3) or (5). To overcome this problem, in the late 1950's, Kampé de Fériet and Frenkiel ([15], [16]) and independently Parzen [32] and Rozanov [40] have defined an "associated spectrum" for a class of second order processes $X : \mathbf{R} \to L_0^2(P)$. These are processes for which

$$\lim_{T \to \infty} \frac{1}{T} \int_0^{T-|h|} (X(s), X(s+|h|))ds = \tilde{r}(h), \quad h \in \mathbf{R}, \tag{6}$$

exists. Since $\tilde{r}(\cdot)$ is clearly positive definite, one can apply the Bochner representation theorem as in (1), in many cases. The resulting positive bounded measure F for this $\tilde{r}$ is called the *associated spectrum* of the process X. This class, to be termed *class* (KF), contains not only stationary processes but, among others, many almost periodic ones [35]. With the present methods it is shown in Section 8 that every weakly harmonizable process has an associated spectrum from which in fact several other properties can be obtained. A distinguishing feature of the weakly harmonizable case from the stationary, Cramér, Karhunen, or Loève definitions is that the theory of bimeasures and the consequent (nonabsolute) integration of Morse and Transue ([26], [27], [42], [45], [46]) play a vital role in their analyses. This difference has not been fully appreciated in the literature. (The most comprehensive characterizations of the harmonizable class are summarized in Theorems 7.3 and 7.4.) For vector valued processes, in both the weak and strong cases, some new technical problems have to be resolved. The same is true of random fields. All these aspects have important applications and some indications are given in Sections 8 and 9. A summary of some of these results is included in [37]. For greater accessibility and convenience, the next three sections are devoted to harmonizable processes and most of the remaining five consider the more general random fields with a natural transition. However, an essentially self-contained exposition (modulo some standard measure theory) is presented here.

Notation: The following notation is used: $\mathbf{R}$ for reals, $\mathbf{C}$ for complex numbers, $\mathbf{Z}$ for integers, $\mathbf{R}^n$ for the n-dimensional number space, and LCA for locally compact abelian. A *step function* is a mapping taking finitely many values on disjoint measurable sets, and a *simple function* on a measure space is a step function vanishing outside of a set of finite measure. Overbar denotes complex conjugation.

2. HARMONIZABILITY

For the work of this paper it is convenient to take the Hilbert space $\mathscr{H}$ as the standard function space. Namely, let (Ω, Σ, P) be a probability space and

$$L^2(P)\left(= L^2(\Omega, \Sigma, P)\right)$$

be the space of (equivalence classes of) scalar square integrable functions (= random variables) on Ω, and set

$$\mathscr{H} = L_0^2(P) = \{f \in L^2(P) : \int_\Omega f dP = 0\}.$$

This choice does not really restrict the generality since any abstract Hilbert space is known to be realizable isometrically as a subspace of $L^2(P)$ on some probability space (cf. e.g., [36], p. 414). From this point of view, a process $\{X(t), t \in \mathbf{R}\} \subset L_0^2(P)$ is stationary if its covariance r satisfies $r(s, t) = r(s-t)$, where

$$r(s, t) = E\left(X(s)\overline{X}(t)\right) = \int_\Omega X(s)\overline{X}(t) dP = \left(X(s), X(t)\right), \qquad s, t \in \mathbf{R},$$

and E is also called the "expectation" (= integral). Since $r(\cdot)$ is of positive type (= positive [semi-] definite), assuming it to be jointly measurable (this is implied by the measurability of the random function $\{X(t), t \in \mathbf{R}\}$), it follows that r admits the representation

$$r(t) = \int_{\mathbf{R}} e^{it\lambda} F(d\lambda), \qquad a.a.(t) \quad \text{(Leb.)}. \tag{1'}$$

It may be remarked that in the original (1932) version, Bochner assumed that $r(\cdot)$ is actually continuous, but soon afterward in (1933) F. Riesz showed that measurability itself yields this (slightly weaker) form (1′). This was also used in [33].

For a stationary process $\{X(t), t \in \mathbf{R}\}$, one easily verifies that it is mean continuous (i.e., $E(|X(s) - X(t)|^2) \to 0$ as $s \to t$) iff the covariance $r(\cdot, \cdot)$ is continuous on the diagonal of $\mathbf{R} \times \mathbf{R}$. Thus the measurability of r and the validity of (1′) everywhere implies already the mean continuity of the stationary process! So for certain applications of the type noted earlier, it is desirable to weaken the hypothesis of stationarity retaining some representative features. This was done by Loève, and it is restated in the following form:

Definition 2.1. A process $X : \mathbf{R} \to L_0^2(P)$ is *strongly harmonizable* if its covariance r is the Fourier transform of some covariance function ρ of bounded variation, so that one has

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} \rho(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}. \tag{3'}$$

It was noted in the Introduction that there is no efficient characterization of r given by (3′). There is however a more visible drawback of this concept. Since strong harmonizability is derived from stationarity, so that the latter class is included, consider a "truncated series" $\{\tilde{X}(n), n \in \mathbf{Z}\}$ of a stationary series $\{X(n), n \in \mathbf{Z}\}$ defined as: $\tilde{X}(n) = X(n)$ for finitely many n, and $\tilde{X}(n) = 0$ for all other $n \in \mathbf{Z}$. Then $\{\tilde{X}(n), n \in \mathbf{Z}\}$ is easily seen to be strongly harmonizable. But if $\tilde{X}(n) = X(n)$, for infinitely many n, and $= 0$ for all other n, then $\{\tilde{X}(n), n \in \mathbf{Z}\}$ need not be strongly harmonizable, as the following example illustrates.

Let (Ω, Σ, P) be separable and $\{f_n, n \in \mathbf{Z}\} \subset L_0^2(P)$ be a complete orthonormal set. Then $r(m, n) = \delta_{m-n} = r(m-n)$. So the sequence is stationary and (1′) becomes

$$r(m-n) = \int_{-\pi}^{\pi} e^{i(m-n)\lambda} \frac{d\lambda}{2\pi}, \qquad m, n \in \mathbf{Z}.$$

Now consider the truncated series, $\tilde{f}_n = f_n, n > 0$, and $= 0$ for $n \leqslant 0$. Then $\tilde{r}(m, n) = E(\tilde{f}_m \bar{\tilde{f}}_n) = 1$ if $m = n > 0$, $= 0$ otherwise. But $\tilde{r}$ does not admit the representation (3′). For, otherwise, $\tilde{r}(m, n)$ will be the Fourier coefficient of the representing ρ (of bounded variation) which is only nonvanishing on the ray $(m = n > 0)$ in $\mathbf{Z}^2$. It is a consequence of an important two dimensional extension by Bochner of the classical F. and M. Riesz theorem that ρ must then be absolutely continuous relative to the planar Lebesgue measure with density ρ'. But this implies $\tilde{r}(m, n) \to 0$ as $|m| + |n| \to \infty$ by the Riemann-Lebesgue lemma, and contradicts the fact that $\tilde{r}(m, n) = 1$, for all positive $m = n$ and $n \to \infty$. Hence $\tilde{r}$ cannot admit the representation (3′) so that $\{\tilde{f}_n, n \in \mathbf{Z}\}$ is *not* strongly harmonizable. This example is a slight modification of one due to Helson and Lowdenslager ([13], p. 183) who considered it for a similar purpose, and also appears in [1] for a related elucidation.

The above example and discussion lead us to look for a weakening of the conditions on the covariance function, since it is reasonable to expect *each* truncation of a stationary series to be included in a generalization, retaining the other properties as far as possible. Such an extension was successfully obtained in two different forms in the works of Bochner [2] and Rozanov [40]. The precise concept can be stated and its significance appreciated only after some preliminary considerations.

The measure function ρ of (3′) has the following properties:

(i) ρ is positive definite, i.e.

$$\rho(s, t) = \overline{\rho(t, s)}, \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j}\, \rho(s_i, s_j) \geqslant 0\,, \qquad a_i \in \mathbf{C}\,,$$

(ii) ρ is of bounded variation, i.e.

$$\sup \{ \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{A_i} \int_{B_j} | \rho(ds, dt) | : A_i, B_j \in \mathscr{B}, \text{ disjoint}\} < \infty\,,$$

where $\mathscr{B}$ is the Borel σ-algebra of $\mathbf{R}$. If $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ is defined by $F(A, B) = \int_A \int_B \rho(ds, dt)$, it follows from (i) and (ii) that there exists a complex Radon measure μ on $\mathbf{R}^2$ such that $F(A, B) = \mu(A \times B)$, where $A \times B \in \mathscr{B} \otimes \mathscr{B}$, and μ is positive definite. On the other hand, the defining equation of F implies that F is positive definite (so (i) holds with $\rho(s_i, s_j)$ replaced by $F(A_i, A_j)$) and (ii) becomes

$$V(F) = \sup \{ \sum_{i=1}^{n} \sum_{j=1}^{n} | F(A_i, B_j) | : A_i, B_j \in \mathscr{B}, \text{ disjoint}\} < \infty\,.$$

But (3′) is meaningful, if ρ is replaced by F under the following weaker conditions.

Let $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ be positive definite and be σ-additive in each variable separately. Equivalently, if $\mathscr{M}(\mathbf{R}, \mathscr{B})$ is the vector space of complex measures on $\mathscr{B}$, let $\nu(A) = F(A, \cdot)$, $A \in \mathscr{B}$ so that $\nu : \mathscr{B} \to \mathscr{M}(\mathbf{R}, \mathscr{B})$ is a vector measure. By symmetry, $\tilde{\nu} : B \to F(\cdot, B)$ is also a vector measure on $\mathscr{B} \to \mathscr{M}(\mathbf{R}, \mathscr{B})$. But $\mathscr{M}(\mathbf{R}, \mathscr{B}) = \mathscr{X}$ is a Banach space under the total variation norm, and hence ν (as well as $\tilde{\nu}$) has *finite semivariation* by a classical result (cf. [8], IV.10.4). This means,

$$\| \nu \| (\mathbf{R}) = \sup \{ \| \sum_{i=1}^{n} a_i \nu(A_i) \|_{\mathscr{X}} : | a_i | \leqslant 1, A_i \in \mathscr{B}, \text{ disjoint}\} < \infty\,.$$

Transferred to F, this translates to:

$$\| F \| (\mathbf{R} \times \mathbf{R}) = \sup \{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{a_j}\, F(A_i, A_j) : A_i \in \mathscr{B}, \text{ disjoint}, | a_i | \leqslant 1\} < \infty\,. \tag{7}$$

When (7) holds, $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ will be called a **C**-*bimeasure of finite semivariation.* [It should be noted that the σ-additivity of $F(\cdot, \cdot)$ in each of its components can be replaced by finite additivity and continuity of F from above at $\emptyset$ in that $| F(A_n, A_n) | \to 0$ as $A_n \downarrow \emptyset$.] The desired generalization follows from (7) if it is written in the following form. Let $\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and

$$\psi = \sum_{j=1}^{n} b_j \chi_{B_j},\ A_i \in \mathscr{B},\ B_j \in \mathscr{B}$$

and each collection is disjoint. Set

$$I(\varphi, \psi) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j). \tag{8}$$

Clearly I is well-defined, does not depend on the representation of φ or ψ, and $I(\varphi, \varphi) \geqslant 0$. So $(\varphi, \psi) = I(\varphi, \psi)$ is a semi-inner product on the space of $\mathscr{B}$-step functions. Hence by the generalized Schwarz's inequality one has:

$$| I(\varphi, \psi) |^2 \leqslant I(\varphi, \varphi) \cdot I(\psi, \psi). \tag{9}$$

Taking suprema on all such step functions φ, ψ such that

$$\| \varphi \|_u \leqslant 1, \| \psi \|_u \leqslant 1$$

($\| \cdot \|_u$ is the uniform norm), one deduces from (9) and (7) that

$$\| F \| (\mathbf{R} \times \mathbf{R}) \leqslant \sup \{ | \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j) | : | a_i | \leqslant 1,$$

$$| b_j | \leqslant 1, A_i, B_j \in \mathscr{B}, \text{disjoint} \} \leqslant \| F \| (\mathbf{R} \times \mathbf{R}), (\leqslant V(F)). \tag{10}$$

Thus $\| F \| (\mathbf{R} \times \mathbf{R})$ can be defined either by the middle term (as in [40]) or by (7). For a bimeasure, $\| F \| (\mathbf{R} \times \mathbf{R})$ is also called *Fréchet variation* of F (cf. [26], p. 292) and $V(F)$ the *Vitali variation*, (cf. [26], p. 298).

It should be emphasized that a set function F which is only a bimeasure (even positive definite), need *not* define a (complex) Radon measure on $\mathbf{R}^2$. In fact such bimeasures do not necessarily admit the Jordan decomposition, as counter examples show. Thus integrals relative to F (even if $\| F \| (\mathbf{R} \times \mathbf{R}) < \infty$) *cannot* generally be of Lebesgue-Stieltjes type. Treating $\nu: A \mapsto F(A, \cdot)$, $A \in \mathscr{B}$, as a vector measure into $\mathscr{M}(\mathbf{R}, \mathscr{B})$, one can employ the Dunford-Schwartz *(or D-S) integral* (cf. [8], IV.10), or alternately one can use the theory of bimeasures as developed in ([26], [27]) and [42]. This is the price paid to get the desired weakened concept, but it will be seen that a satisfactory solution of our problem is then obtained, and both these integrations will play key roles.

Let us therefore recall an appropriate integration concept to be used in the following. In ([40], p. 276) Rozanov has indicated a modification without detailing the consequences. (This resulted in a conjecture [40, p. 283] which will be resolved in Section 8 below.) Instead, a *different* route will be followed: namely the integration theory of Morse and Transue will be used from [27] together with a related result of Thomas ([42], p. 146). However, the Bourbaki set up of these papers is inconvenient here, and they will be converted to the set theoretical (or ensemble) versions and employed.

Let $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ be a bimeasure, i.e. $F(\cdot, B)$, $F(A, \cdot)$ are complex measures on $\mathscr{B}$. Hence one can define as usual ([8], III.6),

$$\tilde{I}_1(f, A) = \int_{\mathbf{R}} f(t)\, F(dt, A)\,. \tag{11}$$

for bounded Borel functions $f : \mathbf{R} \to \mathbf{C}$. Then $\tilde{I}_1(f, \cdot)$ is a complex measure. In fact $\tilde{I}_1 : \mathscr{B} \to (B(\mathbf{R}, \mathscr{B}, \mathbf{C}))^*$, the $B(\mathbf{R}, \mathscr{B}, \mathbf{C})$ being the Banach space of bounded complex Borel functions under the uniform norm, is a vector measure. So one can use the D-S integral (recalled at the beginning of the next section), defining

$$I_1(f, g) = \Big(\int_{\mathbf{R}} \bar{g}(t)\tilde{I}_1(dt)\Big)(f) \in \mathbf{C}\,, \qquad f, g \in B(\mathbf{R}, \mathscr{B}, \mathbf{C})\,. \tag{12}$$

Similarly starting with $F(A, \cdot)$ one can define $I_2(f, g)$. In general

$$I_1(f, g) \neq I_2(f, g)\,. \tag{13}$$

In fact the Fubini theorem does not hold in this context. For a counterexample, see ([27], §8). If there is equality in (13), then the pair (f, g) is said to be *integrable* relative to the bimeasure F, and the common value is denoted $I(f, g)$ and symbolically written as (f, g need not be bounded):

$$I(f, g) = \int_{\mathbf{R}}\int_{\mathbf{R}} f(s)\overline{g(t)}F(ds, dt)\,. \tag{14}$$

This is a *Morse-Transue* (or *MT-*) *integral.* While a characterization of MT-integrable functions is not easy, a good sufficient condition for this can be given as follows, (cf. [27], Thm. 7.1; [42], Théorème in §5.17). If f, g are step functions, so that $f = \sum_{i=1}^{n} a_i\chi_{A_i}$, $g = \sum_{j=1}^{n} b_j\chi_{B_j}$, then clearly $I(f, g)$ always exists and

$$I(f, g) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i\bar{b}_j\, F(A_i, B_j)\,. \tag{15}$$

Next define for any $\varphi \geqslant 0$, $\psi \geqslant 0$, Borel functions,

$$\tilde{I}(\varphi, \psi) = \sup\{|\, I(f, g)\,| : |\, f\,| \leqslant \varphi, |\, g\,| \leqslant \psi, f, g \text{ Borel step functions}\}\,,$$

and if u, v are any positive functions,

$$I^*(u, v) = \inf\{\tilde{I}(\varphi, \psi) : \varphi \geqslant u, \psi \geqslant v, \varphi, \psi \text{ are Borel}\}\,. \tag{16}$$

Now the desired result from the above papers is this: If (f, g) is a pair of complex Borel functions such that $I_1(f, g)$ and $I_2(f, g)$ exist in the sense of (12) and (13), and $I^*(|f|, |g|) < \infty$, then (f, g) is MT-integrable for the $\mathbf{C}$-bimeasure F. In the case that the bimeasure F is also positive definite and has *finite* semivariation, then each pair (f, g) of *bounded* complex Borel functions is MT-integrable relative to F. Moreover, using the notations of (7), one has

$$|\, I(f, g)\,| \leqslant \|\, F\,\| \cdot \|\, f\,\|_u \cdot \|\, g\,\|_u\,, \tag{17}$$

where $\| F \| = \| F \| (\mathbf{R} \times \mathbf{R})$. It should be noted, however, that the integrability of (f, g) generally need not imply that of $(|f|, |g|)$, and the MT-integral is *not* an absolutely continuous functional in contrast to the Lebesgue-Stieltjes theory, as already shown by counterexamples in [26] and [27]. Fortunately a certain dominated convergence theorem ([27], Thm. 3.3) is valid and this implies some density properties which can and will be utilized in our treatment below. Also f is termed F-integrable if (f, f) is MT-integrable. Our definition above is somewhat more restrictive than that of [27], but it suffices for this work. For the theory of [27], the space $B(\mathbf{R}, \mathscr{B}, \mathbf{C})$ in (12) and (13) is replaced by $C_{00}(\mathbf{R})$, its subset of continuous functions with compact supports, with the locally convex (inductive limit) topology. Note that, thus far, no special properties of $\mathbf{R}$ were used in the definition of the MT- integral, and the *definition and properties are valid if* $\mathbf{R}$ *is replaced by an arbitrary locally compact space (group in the present context).* This remark will be utilized later on.

With this necessary detour, the second concept can be given as follows:

Definition 2.2. A process $X : \mathbf{R} \to L_0^2(P)$, with $r(\cdot, \cdot)$ as its covariance function, is called *weakly harmonizable* if

$$r(s, t) = I(e^{is(\cdot)}, e^{it(\cdot)}) = \int_{\mathbf{R}}\int_{\mathbf{R}} e^{is\lambda - it\lambda'} F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}\,, \tag{18}$$

relative to some positive definite bimeasure F of finite semivariation where the right side is the MT-integral.

In particular r is bounded and continuous (by (17) and Thm. 3.2 below). Moreover, if F is of bounded variation, then the MT-integral reduces to the Lebesgue-Stieltjes integral and (18) goes over to (3). The following work shows that the process of the counterexample following Definition 2.1 is weakly harmonizable. The same counterexample also shows that harmonizable processes generally do *not* admit shift operators on them, in that there need not be a continuous linear operator

$$\tau_s : X(t) \mapsto X(t+s) \in L_0^2(P)\,, \qquad t \in \mathbf{R}$$

on $L_0^2(P)$. This is in distinction to certain other nonstationary processes of Karhunen type (cf. [9]).

3. Integral representation of a class of second order processes

In order to introduce and utilize the "V-boundedness" concept of Bochner's, it will be useful to have an integral representation of weakly harmonizable processes. This is done by presenting a comprehensive result for a more general class including the (weakly) harmonizable ones. It is based on a method of Cramér's [3], and the resulting representation yields by specializations both the harmonizable, stationary, Cramér class of [3], as well as the Karhunen class (restated below). This is detailed as follows.

Recall that if $(\Omega_0, \mathscr{A})$ is a measurable space (i.e., $\mathscr{A}$ is a σ-algebra of sets of Ω_0) and $\mathscr{X}$ a Banach space, then a mapping $Z : \mathscr{A} \to \mathscr{X}$ is called a *vector measure* if Z is σ-additive, or

$$Z\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} Z(A_i), \qquad A_i \in \mathscr{A},$$

disjoint, the series converging unconditionally in the norm of $\mathscr{X}$. If $\mathscr{X} = L_0^2(P)$ where (Ω, Σ, P) is a probability space, then a vector measure is sometimes termed a *stochastic measure*. The integration of scalar functions relative to a vector measure Z is needed, and it will be in the sense of Dunford-Schwartz ([8], IV.10). This may be briefly outlined here. If $f = \sum_{i=1}^{n} a_i \chi_{A_i}$, $A_i \in \mathscr{A}$, disjoint, define as usual

$$\int_A f(s)Z(ds) = \sum_{i=1}^{n} a_i Z(A \cap A_i) \in \mathscr{X}, \qquad A \in \mathscr{A}. \tag{19}$$

Now if $g : \Omega_0 \to \mathbf{C}$ is $\mathscr{A}$-measurable, and g_n are $\mathscr{A}$-step functions such that $g_n \to g$ pointwise, one says that g is *D-S integrable* whenever for each $A \in \mathscr{A}$,

$$\Big\{\int_A g_n(s)Z(ds),\ n \geqslant 1\Big\} \subset \mathscr{X}$$

is a Cauchy sequence. Then the limit, denoted g_A, of this sequence is called the integral of g on A, and is denoted as

$$g_A = \int_A g(s)Z(ds) = \lim_{n\to\infty} \int_A g_n(s)Z(ds), \qquad A \in \mathscr{A}. \tag{20}$$

It is a standard (but non-obvious) matter to show that the integral is well-defined, independent of the sequence used, and the mapping $A \mapsto \int_A g(s)Z(ds)$ is σ-additive on $\mathscr{A}$, and $g \mapsto \int_A g(s)Z(ds)$ is linear. Also

$$\Big\| \int_A g(s)Z(ds) \Big\| \leqslant \| g \|_u \, \| Z \| (A), \qquad g \in B(\Omega, \mathscr{A}, \mathbf{C}), \tag{21}$$

where $\| Z \| (\cdot)$ is the semivariation of Z (cf. (7)) which is always finite on the σ-algebra $\mathscr{A}$. [If $\mathscr{A}$ is only a δ-ring and $\Omega_0 \notin \mathscr{A}$, then Z need not have finite semivariation on $\mathscr{A}$.] The dominated convergence theorem is true for the D-S integral. (See [8], IV.10, for proofs and related results. The latter exposition is very readable and nice.)

The general class noted above is the following:

Definition 3.1. A process $X : \mathbf{R} \to L_0^2(P)$, with covariance $r(\cdot, \cdot)$, is said to be *weakly of class* (C) (C for Cramér) if (i) there exists a covariance bimeasure F on $\mathbf{R} \times \mathbf{R}$ of locally bounded semivariation in the sense that

$$F(A, B) = \bar{F}(B, A), \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{a}_j F(A_i, A_j) \geqslant 0, \qquad a_i \in \mathbf{C}.$$

Here $A_i \in \mathscr{B}$, $1 \leqslant i \leqslant n$, bounded, and for each bounded Borel $A \subset \mathbf{R}$, if $\mathscr{B}(A) = \{A \cap B : B \in \mathscr{B}\}$, then

$$\| F \| (A \times A) = \sup \{ | \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \bar{b}_j F(A_i, B_j) | : | a_i | \leqslant 1, | b_j | \leqslant 1,$$
$$A_i, B_j \in \mathscr{B}(A), \text{ disjoint}\} < \infty ;$$

(ii) there exists an MT-integrable (for F) family $g_t : \mathbf{R} \to \mathbf{C}$ of Borel functions, $t \in \mathbf{R}$, such that $I(|g_s|, |g_s|) < \infty$, $s \in \mathbf{R}$, where I denotes the MT-integral relative to F, in terms of which one has ($g_t(\lambda)$ is also written as $g(t, \lambda)$):

$$r(s, t) = I(g_s, \bar{g}_t) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda) \bar{g}_t(\lambda') F(d\lambda, d\lambda'), \qquad s, t \in \mathbf{R}. \tag{22}$$

Remark. Note that in this definition F can be given by a covariance function ρ as in (3′) since, for $A = [a, b)$ and $B = [c, d)$ one defines $(\Delta^2 F)(A, B)$ as the increment $\rho(b, d) - \rho(a, d) - \rho(b, c) + \rho(a, c)$ and extend it to $\mathscr{B} \times \mathscr{B}$. Also in (22) it is possible that $\| F \| (\mathbf{R} \times \mathbf{R}) = \infty$. If F has finite variation on each compact rectangle of $\mathbf{R}^2$, then F determines a locally bounded complex Radon measure, and the above class reduces to the family defined by Cramér in [3], and called class (C) and analyzed in [35]. If $\| F \| (\mathbf{R} \times \mathbf{R}) < \infty$, then one can take $g_t(\lambda) = g(t, \lambda) = e^{it\lambda}$ so that the weakly harmonizable class is included. Again it may be noted that $\mathbf{R}$ can be replaced by a locally compact space or an abelian group in (22) so that $\mathbf{R}^n$ or the n-torus $\mathbf{T}^n$ is included.

To present the general representation, it is necessary also to note the validity of the D-S integration embodied in (20), (21) when the set functions are defined on arbitrary δ-rings instead of σ-algebras, assumed in [8]. Further our measure $Z : \tilde{\mathscr{B}} \to \mathscr{X}$ has the property that it is Baire regular in the sense that for each $A \in \tilde{\mathscr{B}}$ and $\varepsilon > 0$, there exist a compact $C \in \tilde{\mathscr{B}}$, open $U \in \tilde{\mathscr{B}}$ such that $C \subset A \subset U$ and $\| Z(D) \| < \varepsilon$ for each $D \in \tilde{\mathscr{B}}$, $D \subset U - C$, where $\tilde{\mathscr{B}}$ is the Baire (= Borel here) σ-ring of $\mathbf{R}$. Even if $\mathbf{R}$ is replaced by a general locally compact space S, with $\tilde{\mathscr{B}}$ as its Baire σ-ring and $Z : \tilde{\mathscr{B}} \to \mathscr{X}$ σ-additive, one has Z to be Baire regular having a unique regular extension to the Borel σ-ring $\mathscr{B}$ of S.

Actually Z concentrates on a σ-compact Baire set $S_0 \subset S$. Moreover if Z is weakly regular in that $x^* \circ Z$ is a scalar regular signed measure, $x^* \in \mathscr{X}^*$, then Z is itself regular. (See [21], pp. 262-263 for proofs with only simple modifications of the arguments given in [8], IV.10.) In each case the measure Z has finite semivariation on bounded sets in $\tilde{\mathscr{B}}$ (cf. (7) where $\mathscr{B}$ is replaced by the ring generated by all bounded Baire sets for S). If $\mathscr{B}_0 \subset \mathscr{B}$ is the class of all bounded sets (a set is bounded if it is contained in a compact set), then it is a δ-ring, and the D-S integration of a scalar function relative to $Z : \mathscr{B}_0 \to \mathscr{X}$ holds as noted above. With this understanding the following is the desired general result.

THEOREM 3.2. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a process which is weakly of class* (C) *in the sense of Definition 3.1, relative to a positive definite bimeasure* F *of locally finite semivariation, and a family* $\{g_s, s \in \mathbf{R}\}$ *of Borel functions such that each* $|g_s|$ *is MT-integrable for* F. *Then there exists a stochastic measure* $Z : \mathscr{B}_0 \to L_0^2(\tilde{P})$ *where* $\mathscr{B}_0$ *is the* δ*-ring of bounded Borel sets of* $\mathbf{R}$, *and* $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ *is an enlargement of* (Ω, Σ, P) *so* $L_0^2(\tilde{P}) \supset L_0^2(P)$, *such that*

$$\text{(i) } E\big(Z(A) \cdot \overline{Z}(B)\big) = \big(Z(A), Z(B)\big) = F(A, B),\ A, B \in \mathscr{B}_0\,,$$

$$\text{(ii) } X(t) = \int_{\mathbf{R}} g(t, \lambda) Z(d\lambda),\ t \in \mathbf{R}\,, \tag{23}$$

where the integral is in the D-S *sense for the* δ*-ring* $\mathscr{B}_0$.

Conversely, if $\{X(t), t \in \mathbf{R}\}$ *is a process defined by* (23) *relative to a stochastic measure* $Z : \mathscr{B}_0 \to L_0^2(P)$ *and a Borel family* $\{g_t, t \in \mathbf{R}\}$, D-S *integrable for* Z *and* $\mathscr{B}_0$, *then it is weakly of class* (C) *relative to* F *defined by*

$$F(A, B) = E\big(Z(A) \cdot \overline{Z}(B)\big), \qquad A, B \in \mathscr{B}_0\,,$$

and each $|g_t|, t \in \mathbf{R}$, *is MT-integrable for* F. *Moreover, if*

$$\mathscr{H}_X = \overline{sp}\{X(t), t \in \mathbf{R}\}$$

and

$$\mathscr{H}_Z = \overline{sp}\{Z(A), A \in \mathscr{B}_0\}$$

in $L_0^2(P)$, *then* $\mathscr{H}_X = \mathscr{H}_Z$ *when and only when the* $\{g_t, t \in \mathbf{R}\}$ *has the property that*

$$\int_{\mathbf{R}}\int_{\mathbf{R}} f(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda') = 0, \qquad \text{all } t \in \mathbf{R}\,,$$

implies $\int_{\mathbf{R}}\int_{\mathbf{R}} f(\lambda)\bar{f}(\lambda')F(d\lambda, d\lambda') = 0$ *both being* MT*-integrals.*

Proof: The basic layout is that of [3]. The integrals used there will have to be replaced by the D-S and MT-integrals appropriately. Since the changes are not immediately obvious, the essential details are spelled out so that in subsequent discussions, such arguments can be compressed.

For the direct part, let the process be weakly of class (C). Then its covariance r admits a representation (with the MT-integration) as:

$$r(s, t) = E(X(s)\bar{X}(t)) = \int_{\mathbf{R}}\int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda'). \tag{24}$$

Since F is a positive definite bimeasure, if

$$L_F^2 = \{f: \int_{\mathbf{R}}\int_{\mathbf{R}} f(\lambda)\bar{f}(\lambda')F(d\lambda, d\lambda') = (f, f)_F < \infty,\ f \text{ is MT-integrable for } F\},$$

and since $I_F(f, f) = (f, f)_F \geqslant 0$, the earlier discussion implies $\{L_F^2, (\cdot, \cdot)_F\}$ is a semi-inner product space, and $g_t \in L_F^2$, $t \in \mathbf{R}$. Let $T: L_F^2 \to \mathscr{H}_X$ be defined by $T: g_s \mapsto X(s)$, extending it linearly. Then (24) implies

$$(Tg_s, Tg_t)_{\mathscr{H}_X} = (g_s, g_t)_F, \qquad s, t \in \mathbf{R}. \tag{25}$$

Thus T is an isometric mapping of $\Lambda_F^2 = sp\{g_t, t \in \mathbf{R}\} \subset L_F^2$ onto $\mathscr{H}_X$ where $\mathscr{H}_X$ is the space given in the statement of the theorem.

Suppose first that Λ_F^2 is dense in L_F^2. By ([27], Thm. 11.1) every Borel function with $I^*(|f|, |f|) < \infty$ is in L_F^2, so that, in particular $\chi_A \in L_F^2$ for each $A \in \mathscr{B}_0$ since F is locally of finite semivariation. By the density of Λ_F^2 in L_F^2 and the isometry, there is a $Z_A \in \mathscr{H}_X$ such that $T\chi_A = Z_A$. If $A, B \in \mathscr{B}_0$, then

$$E(Z_A \cdot \bar{Z}_B) = (T\chi_A, T\chi_B)_{\mathscr{H}_X} = (\chi_A, \chi_B)_F = F(A, B),$$

and if $A \cap B = \emptyset$ also holds, then

$$E(|Z_{A\cup B} - Z_A - Z_B|^2) = (\chi_{A\cup B} - \chi_A - \chi_B, \chi_{A\cup B} - \chi_A - \chi_B)_F = 0$$

since F is additive in both components. Thus $Z_{(\cdot)}: \mathscr{B}_0 \to \mathscr{H}_X \subset L_0^2(P)$ is additive. If $\{A_n\}_1^\infty \subset \mathscr{B}_0$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathscr{B}_0$, then

$$E(|Z_A - \sum_{i=1}^{n} Z_{A_i}|^2) = E(|Z_{\bigcup_{i=1}^{n} A_i} + Z_{\bigcup_{i>n} A_i} - \sum_{i=1}^{n} Z_{A_i}|^2)$$

$$= E(|Z_{\bigcup_{i>n} A_i}|^2) = F(\bigcup_{i>n} A_i, \bigcup_{i>n} A_i) \to 0$$

as $n \to \infty$, since F is continuous at $\emptyset$ from above (cf. discussion after (7)). This Z is σ-additive on $\mathscr{B}_0$ and hence is a stochastic measure of finite semivariation on each compact set there. Clearly $\mathscr{H}_Z \subset \mathscr{H}_X$. Since $\{g_t, t \in \mathbf{R}\}$ is dense in L_F^2, $\chi_A \in L_F^2$, and each g_t is assumed MT-integrable for F, there is a sequence $\tilde{g}_n = \sum_{i=1}^{n} a_i g_{t_i} \to \chi_A$ in L_F^2 so that $(\tilde{g}_n - \chi_A, \tilde{g}_n - \chi_A)_F \to 0$. Hence by the isometry $E(|\sum_{i=1}^{n} a_i X(t_i) - Z_A|^2) \to 0$, as $n \to \infty$. It now follows easily that $\{Z_A, A \in \mathscr{B}_0\}$ is dense in $\mathscr{H}_X$. Thus $\mathscr{H}_X = \mathscr{H}_Z$, and each element in $\mathscr{H}_Z$ corresponds uniquely to an element of $\bar{L}_F^2$, the completion of L_F^2 and where elements $h \in \bar{L}_F^2$ with $(h, h)_F = 0$ and 0 are identified. Let $Y(t)$ be defined as:

$$Y(t) = \int_{\mathbf{R}} g_t(\lambda) Z(d\lambda) \qquad \in \mathscr{H}_Z = \mathscr{H}_X . \tag{26}$$

Here the right side is the D-S integral on the δ-ring $\mathscr{B}_0$, which can be defined by a slight modification of the work of ([8], IV.10), as noted in [21]. Thus,

$$\begin{aligned}(Y(s), Y(t)) &= \left(\int_{\mathbf{R}} g_s(\lambda) Z(d\lambda), \int_{\mathbf{R}} g_t(\lambda') Z(d\lambda')\right) \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda) \bar{g}_t(\lambda') F(d\lambda, d\lambda')\end{aligned}$$

which holds if g_s is a $\mathscr{B}_0$-measurable step function and then the general case follows by ([27], Thm. 3.3 or [46], p. 126), since $|g_s|$ is MT-integrable in our sense. Now by definition (l·i·m denoting $L^2(P)$-mean):

$$\begin{aligned}Z(A) &= T(\chi_A) = T(\lim_n \tilde{g}_n), \quad \text{where} \quad \tilde{g}_n \to \chi_A \quad \text{in} \quad L_F^2 \\ &= \operatorname{l\cdot i\cdot m}_n T(\tilde{g}_n) = \operatorname{l\cdot i\cdot m}_n \sum_{i=1}^{n} a_i T(g_{t_i}) \\ &= \operatorname{l\cdot i\cdot m}_n \sum_{i=1}^{n} a_i X(t_i) = \operatorname{l\cdot i\cdot m}_n \tilde{X}_n \quad \text{(say)}.\end{aligned}$$

The $L^2(P)$-limits imply

$$\begin{aligned}E(X(s)\bar{Z}(A)) &= \lim_n E(X(s)\overline{\tilde{X}_n}) \\ &= \lim_n \sum_{i=1}^{n} a_i E(X(s)\bar{X}(t_i)) = \lim_n \sum_{i=1}^{n} a_i r(s, t_i) \\ &= \lim_n \sum_{i=1}^{n} a_i \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda) \bar{g}_{t_i}(\lambda') F(d\lambda, d\lambda') \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda) \chi_A(\lambda') F(d\lambda, d\lambda') .\end{aligned}$$

By isometry, if $\zeta_n = \sum_{j=1}^{n} b_j \cdot Z(A_j)$, one gets $\tilde{h}_n \leftrightarrow \zeta_n$ where $\tilde{h}_n = \sum_{j=1}^{n} b_j \chi_{A_j} \in L_F^2$,

$$E(X(s)\bar{\zeta}_n) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_n(\lambda')F(d\lambda, d\lambda') .$$

So again by the MT-integrability of $g_s(\cdot)$, the preceding result yields

$$E(X(s)\bar{Y}(t)) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda') .$$

It follows from this that

$$E(|X(s) - Y(s)|^2) = E(|X(s)|^2) + E(|Y(s)|^2) - E(X(s)\overline{Y(s)}) - E(Y(s)\overline{X(s)}) = 0 .$$

Hence $X(s) = Y(s)$ *a.e.*, $s \in \mathbf{R}$. So (26) implies (23) in the event that Λ_F^2 is dense in L_F^2.

For the general case, where $\tilde{\Lambda}_F^2 = \bar{L}_F^2 \ominus \bar{\Lambda}_F^2$ is nontrivial and where the "bar" again denotes completion, let $\{h_t, t \in \tilde{R}\}$ be a basis of $\tilde{\Lambda}_F^2$. If $\tilde{\tilde{R}} = \mathbf{R} \dot{+} \tilde{R}$ is a disjoint sum to give a new index set, let $\tilde{g}_s = g_s$ for $s \in \mathbf{R}$, and $= h_s$ for $s \in \tilde{R}$, then $\{\tilde{g}_s, s \in \tilde{\tilde{R}}\}$ is dense in $\bar{L}_F^2$. So by the preceding case, on extending T to τ from $L_F^2 \to L_0^2(\tilde{P})$, where $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ is possibly an enlargement of (Ω, Σ, P) by adjunction (cf., e.g., [36], p. 82), with $\tau\chi_A = Z_A \in L_0^2(\tilde{P})$, one has

$$\tilde{Y}(s) = \int_{\mathbf{R}} \tilde{g}_s(\lambda)Z(d\lambda) \qquad \in L_0^2(P) . \tag{27}$$

Observe that all $\tilde{g}_s$ are Borel and MT-integrable in this procedure. Hence, as before, $\tilde{Y}(s) = X(s)$ for $s \in \mathbf{R}$, and (23) holds again. In this case $\mathscr{H}_Z \supset \mathscr{H}_X$, and the inclusion is proper.

Conversely, let $\{X(t), t \in \mathbf{R}\}$ be a process defined by (23). Let $F(A, B) = (Z(A), Z(B))$ and $g_n = \sum_{i=1}^{n} a_i\chi_{A_i}, A_i, A, B$ in $\mathscr{B}_0$. Then for the D-S integral (23) one has

$$\begin{aligned} \| F \|(A, A) &= \sup \{\sum_{i=1}^{n} \sum_{j=1}^{n} a_i\bar{a}_j F(A_i, A_j) : A_i \in \mathscr{B}(A), | a_i | \leqslant 1\} \\ &= \sup \{\| \sum_{i=1}^{n} a_i Z(A_i) \|_2^2 : | a_i | \leqslant 1, A_i \in \mathscr{B}(A)\} \\ &\leqslant \| Z \|^2(A) < \infty, A \in \mathscr{B}_0 . \end{aligned}$$

Thus if $X_{g_n} = \int_{\mathbf{R}} g_n(\lambda)Z(d\lambda)$, one has with h_n another such step function,

$$E(X_{g_n}\bar{X}_{h_n}) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_n(\lambda)\bar{h}_n(\lambda')F(d\lambda, d\lambda') . \tag{28}$$

Now given $g_s \in L_F^2$ which is MT-integrable in our (restricted) sense (this is analogous to a definition of [46]) and for which (23) holds, the g_s can be approximated by suitable Borel step functions $\{g_n\}_1^\infty \subset L_F^2$ such that $g_n \to g_s$ pointwise $|g_n| \leq |g_s|$ and similarly with $\tilde{g}_n \to g_t$ such that

$$I(g_n, \tilde{g}_n) \to I(g_s, g_t), I(|g_s|, |g_s|) < \infty .$$

Applying this to (28), one obtains

$$\begin{aligned}\int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda') &= \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} g_n(\lambda)\bar{\tilde{g}}_n(\lambda)F(d\lambda, d\lambda') \\ &= \lim_n (X_{g_n}, X_{\tilde{g}_n}) \\ &= \lim_n \left(\int_{\mathbf{R}} g_n(\lambda)Z(d\lambda), \int_{\mathbf{R}} \tilde{g}_n(\lambda')Z(d\lambda')\right) \\ &= \left(\int_{\mathbf{R}} g_s(\lambda)Z(d\lambda), \int_{\mathbf{R}} g_t(\lambda)Z(d\lambda)\right),\end{aligned}$$

since for the D-S integral the dominated convergence holds,

$$= \big(X(s), X(t)\big) = r(s, t) . \tag{29}$$

This shows $\{X(t), t \in \mathbf{R}\}$ is of weakly class (C).

Regarding the last assertion, it is evident that $\{g_s, s \in \mathbf{R}\}$ is a basis in L_F^2 iff $I(f, g_t) = 0, t \in \mathbf{R}$ implies $I(f, f) = 0$. This is clearly necessary and sufficient for $\mathscr{H}_Z = \mathscr{H}_X$ since otherwise, (with possibly an enlargement of the underlying probability space) $\mathscr{H}_Z \underset{\neq}{\supset} \mathscr{H}_X$ and $\mathscr{H}_Z = \mathscr{H}_{\tilde{Y}}$ in the notation of (27). Thus the proof is complete.

Remarks. 1. If F is of locally finite variation, then it defines a locally finite (i.e., finite on compact sets) complex Borel (= Radon) measure in the plane $\mathbf{R}^2$, and then the MT-integrals for F reduce to the Lebesgue-Stieltjes integrals. Thus $I(g_s, g_s) < \infty$ is equivalent to the classical theory, and the above result specializes to Cramér's theorem of [3]. However, for the general case of bimeasures (as here), the MT-theory (or a form of it) appears essential.

2. The above theorem is true if $\mathbf{R}$ is replaced by a *locally compact space*, since no special property of $\mathbf{R}$ is used. Only the concept of boundedness is needed.

When $\| F \|(\mathbf{R} \times \mathbf{R}) < \infty$, so that F is of finite semivariation on $\mathbf{R}^2$, then by ([27], Thm. 11.1) each bounded Borel function g is MT-integrable for F. Taking $g_t(\lambda) = e^{it\lambda}$ in the above theorem, one deduces from this result the important representation given by Rozanov ([40], p. 279). The last statement is not too hard to establish. [A separate proof of it is also found in ([29], p. 36).]

THEOREM 3.3. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a process such that* $\| X(t) \|_2 \leq M < \infty, t \in \mathbf{R}$, *and be weakly continuous. Then the process is weakly harmonizable relative to some covariance bimeasure* F *of finite semivariation* (cf. *Definition* 2.2) *iff there is a stochastic measure* $Z : \mathscr{B} \to L_0^2(P)$ *such that for each* A, B *in* $\mathscr{B}$, $F(A, B) = \big(Z(A), Z(B)\big)$ *and*

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R}, \tag{30}$$

the right side symbol being the D-S integral and $\| Z \|(\mathbf{R}) < \infty$. *Moreover,* X *is strongly harmonizable iff the covariance bimeasure* F *of* Z *in* (30) *is of bounded variation in* $\mathbf{R}^2$ (cf. *Definition* 2.1). *In either case the harmonizable process* X *is uniformly continuous, and is represented as in* (30).

Suppose that in the representation (23) the Z-process is *orthogonally scattered* implying $(Z(A), Z(B)) = 0$ whenever $A \cap B = \emptyset$. Then

$$F(A, B) = (Z(A), Z(B)) = \tilde{F}(A \cap B),$$

where F is the covariance bimeasure and $\tilde{F}$ is a positive locally finite measure on $\mathscr{B}$ so that it is σ-finite there. Then

$$r(s, t) = E(X_s \bar{X}_t) = \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda)\tilde{F}(d\lambda). \tag{31}$$

A process whose covariance function **R** satisfies this condition is termed a *Karhunen process.* Moreover, if $\tilde{F}$ is a finite measure and $g_s(\lambda) = e^{is\lambda}$, the resulting one is the classical (Khintchine) stationary process. In both these cases there are no weak type extensions.

Let us introduce a further generalization of the (weak) Cramér class to illuminate the above Definition 3.1, and for a future analysis. Let (Ω, Σ, μ) be a measure space and $M(\mu)$ be the space of scalar μ-measurable functions on Ω. Let $N(\cdot) : M(\mu) \to \mathbf{R}^+$ be a *function norm* in that for f, f_n in $M(\mu)$, (i) $N(f) = N(|f|) \geqslant 0$, (ii) $0 \leqslant f_n \uparrow \Rightarrow N(f_n) \uparrow$, (iii) $N(af) = |a| N(f)$, $a \in \mathbf{C}$ and (iv) $N(f+g) \leqslant N(f) + N(g)$. The functional N has the weak Fatou property if

$$0 \leqslant f_n \uparrow f, \lim_n N(f_n) < \infty \Rightarrow N(f) < \infty,$$

and has the Fatou property if instead $N(f_n) \uparrow N(f)$ $(\leqslant \infty)$. The associate norm N' of N is defined by:

$$N'(f) = \sup \{ | \int_{\Omega} (fg)(\omega)\mu(d\omega) | : N(g) \leqslant 1 \}. \tag{32}$$

One sees that N' is a function norm with the Fatou property. If

$$N(\cdot) = \| \cdot \|_p, 1 \leqslant p \leqslant \infty,$$

then

$$N'(\cdot) = \| \cdot \|_q, p^{-1} + q^{-1} = 1.$$

The general concept alluded to above is as follows:

Definition 3.4. (a) If $r : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ is a covariance function, it is said to be of *class*$_N$ (C) relative to a function norm N, if there is a covariance bimeasure

$F : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ of locally finite N-variation (let N' be the associate norm of N), and there exists a family $\{g_t, t \in \mathbf{R}\}$ of Borel functions which are MT-integrable relative to F, such that

$$r(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} g_s(\lambda)\bar{g}_t(\lambda')F(d\lambda, d\lambda'), \; s, t \in \mathbf{R}\,, \tag{33}$$

and where locally finite N-variation is meant the following:

$$\infty > \| F \|_N(A \times A) = \sup \{ | I(f, g) | : N'(f) \leqslant 1, N'(g) \leqslant 1 \}\,. \tag{34}$$

Here f, g are Borel step functions, with $\mathrm{supp}(f) \subset A$, $\mathrm{supp}(g) \subset A$, $A \in \mathscr{B}_0$, the δ-ring of bounded Borel sets of $\mathbf{R}$.

(b) A process $X : \mathbf{R} \to L_0^2(P)$ is of $\mathrm{class}_N(\mathrm{C})$ if its covariance function r is of class_N (C) so that it is representable as (33).

It is clear that if $N(\cdot) = \| \cdot \|_1$ so that $N'(\cdot) = \| \cdot \|_\infty$, the N-variation is simply the 1-semivariation of Definition 3.1 and that

$$\| F \|_N = \| F \|_1 \, (= \| F \|)\,.$$

Remark. Without further restrictions, class_N (C) need not contain the weak or strong harmonizable processes. However if N is restricted so that, letting

$$L^N(P) = \{f \in M(P) : N(f) < \infty\}, L^\infty(P) \subset L^N(P) \subset L^1(P)\,,$$

where $\mu = P$ is a probability, then every class_N (C) will contain both the weak and strong harmonizable families, as an easy computation shows. If $N(\cdot) = \| \cdot \|_1$, then class_1 (C) is the class which corresponds to the covariance bimeasure of *finite semivariation.* This includes the classical Loève and Rozanov definitions. Again this definition holds, with only a notational change, if $\mathbf{R}$ is replaced by a locally compact group G. A brief discussion on some analysis of these classes which extend the present work is included at the end of the paper.

4. V-BOUNDEDNESS, WEAK AND STRONG HARMONIZABILITY

The definition of weak harmonizability is of interest only when an effective characterization of it is found and when its relations with strong harmonizability are made concrete. These points will be clarified and answered here. Now Theorem 3.3 shows that a weakly harmonizable process is the Fourier transform of a stochastic measure and this leads to a fundamental concept called V-boundedness ('V' for "variation"), introduced much earlier by Bochner [2], which is valid in a more general context. This notion plays a central role in the theory and applications of weakly harmonizable processes (and fields) which are shown to be V-bounded in the context of $L_0^2(P)$. Further this characterization facilitates a use of the powerful tools of Fourier analysis of vector measures. The desired concept is as follows (cf. [2], and also [33]):

Definition 4.1. A process $X : \mathbf{R} \to \mathscr{X}$, a Banach space, is *V-bounded* if $X(\mathbf{R})$ lies in a ball of $\mathscr{X}$, X as an $\mathscr{X}$-valued function is strongly measurable (i.e., range of X is separable and $X^{-1}(B) \in \mathscr{B}$ for each Borel set $B \subset \mathscr{X}$), and if the set C is relatively weakly compact in $\mathscr{X}$, where

$$C = \{\int_{\mathbf{R}} f(t)X(t)dt : \| \hat{f} \|_u \leqslant 1, f \in L^1(\mathbf{R})\} \subset \mathscr{X}, \tag{35}$$

and where $\hat{f}(t) = \int_{\mathbf{R}} f(\lambda)e^{it\lambda}d\lambda$, $\int_{\mathbf{R}} f(t)X(t)dt$ being the Bochner integral. If $\mathscr{X}$ is reflexive then the condition on C may be replaced by its boundedness. (Here if the measurability of X is strengthened to weak continuity, then it actually implies the strong [and even uniform] continuity.)

Let us establish the following basic fact when $\mathscr{X} = L_0^2(P)$:

THEOREM 4.2. *A process $X : \mathbf{R} \to L_0^2(P)$ is weakly harmonizable iff X is V-bounded (i.e., $\| X(t) \|_2 \leqslant M_0 < \infty, t \in \mathbf{R}$, and the set in (35) is bounded) and weakly continuous.*

Proof: For the direct part, let X be weakly continuous and V-bounded. Then

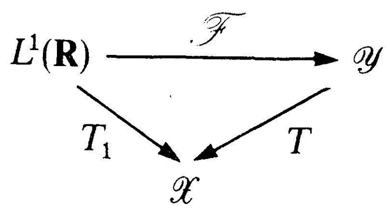
$$\| \int_{\mathbf{R}} f(t)X(t)dt \|_2 \leqslant c \| \hat{f} \|_u, f \in L^1(\mathbf{R}), \tag{36}$$

by Definition 4.1. Let $\mathscr{Y} = \{\hat{f} : f \in L^1(\mathbf{R})\} \subset C_0(\mathbf{R})$, the space of complex continuous functions vanishing at "∞"; the inclusion holds by the Riemann-Lebesgue lemma. Moreover, $\mathscr{Y}$ is uniformly dense in $C_0(\mathbf{R})$, since $\mathscr{Y}$ is a real algebra in $C_0(\mathbf{R})$ and separates points of $\mathbf{R}$ so that the Stone-Weierstrass theorem applies (cf. [24], §26.B). Let $\mathscr{F} : f \mapsto \int_{\mathbf{R}} f(\lambda)\bar{e}_t(\lambda)d\lambda, t \in \mathbf{R}$, where $e_t(\lambda) = e^{it\lambda}$. Then $\mathscr{F} : L^1(\mathbf{R}) \to C_0(\mathbf{R})$ is a one-to-one contractive operator. Consider the mapping

$$T : \mathscr{Y} \to \mathscr{X} = L_0^2(P), \quad \text{by} \quad T(\hat{f}) = \int_{\mathbf{R}} f(t)X(t)dt \in \mathscr{X}.$$

This is well-defined, and the following diagram is commutative:

$$T_1(f) = \int_{\mathbf{R}} f(t)X(t)dt \in \mathscr{X}.$$



By hypothesis T is bounded and by the density of $\mathscr{Y}$ in $C_0(\mathbf{R})$, it has a norm preserving extension $\tilde{T}$ to $C_0(\mathbf{R})$. Now $\tilde{T}$ will be given an integral representation using a classical theorem due to Dunford-Schwartz ([8], VI.7.3) since $\tilde{T}$ is a weakly compact operator because $\mathscr{X}$ is reflexive.

To invoke the above cited theorem, however, it should first be observed that the result holds even if the space $C(S)$ of continuous (scalar) functions on a compact space S (for which it is proved) is replaced by $C_0(\mathscr{S})$ with a locally compact space $\mathscr{S}$. Here $\mathscr{S} = \mathbf{R}$. Indeed, let $\bar{\mathscr{S}}$ be the one-point (at "∞") compactification of $\mathscr{S}$ and consider the space $C(\bar{\mathscr{S}})$. Now $C_0(\mathscr{S})$ can be identified with the subspace $\{f \in C(\bar{\mathscr{S}}) : f(\infty) = 0\}$. Since $\tilde{T} : C_0(\mathscr{S}) \to \mathscr{X}$ is continuous and $C_0(\mathscr{S})$ is an "abstract M-space", there is a continuous operator $\bar{T} : C(\bar{\mathscr{S}}) \to \mathscr{X}$ such that $\bar{T} \mid C_0(\mathscr{S}) = \tilde{T}$. This follows from the fact that for any Banach space $\mathscr{Z}$ containing a subspace which is an abstract M-space, there is a projection of norm one on $\mathscr{Z}$ onto that subspace, by the well-known Kelley-Nachbin-Goodner theorem (cf. e.g., [8], p. 398), and $\bar{T} = \tilde{T} \circ Q$. Hence by the Dunford-Schwartz theorem noted above, there is a vector measure $\tilde{Z}$ on $\bar{\mathscr{S}}$ into $\mathscr{X}$ such that

$$\bar{T}(f) = \int_{\bar{\mathscr{S}}} f(t)\tilde{Z}(dt),\ f \in C(\bar{\mathscr{S}}), \tag{37}$$

and $\| \bar{T} \| = \| \tilde{Z} \|(\bar{\mathscr{S}})$, the integral on the right being in the D-S sense. Define $Z : \mathscr{B}(\mathscr{S}) \to \mathscr{X}$ as $Z(A) = \tilde{Z}(\mathscr{S} \cap A)$, $A \in \mathscr{B}(\mathscr{S})$. Then Z is a vector measure and $\| Z \| \leqslant \| \tilde{Z} \|$. Moreover, if $f_0 = f |_{\mathscr{S}}$, then

$$\begin{aligned} \bar{T}(f) &= \int_{\mathscr{S}} f_0(t)Z(dt) + \int_{``\{\infty\}"} f(\infty)\tilde{Z}(dt),\ f \in C(\bar{\mathscr{S}}) \\ &= \tilde{T}(f_0), \qquad \text{since} \quad f(\infty) = 0 . \end{aligned}$$

Hence $\bar{T}(f) = \tilde{T}(f)$, $f \in C_0(\mathscr{S})$ with $\| \tilde{T} \| \leqslant \| \bar{T} \| = \| \tilde{T}Q \| \leqslant \| \tilde{T} \|$, and

$$\tilde{T}(f) = \int_{\mathscr{S}} f(t)Z(dt),\ f \in C_0(\mathscr{S}) . \tag{38}$$

Thus writing $\mathbf{R}$ for $\mathscr{S}$ from now on (the above general case is needed later), it follows that

$$\begin{aligned} \| \tilde{T} \| = \sup \{ \| \textstyle\int_{\mathbf{R}} f(t)Z(dt) \| : f \in C_0(\mathbf{R}), \| f \|_u \leqslant 1 \} &= \| Z \|(\mathbf{R}) \\ &= \| \tilde{Z} \|(\bar{\mathbf{R}}) , \end{aligned}$$

and T and Z correspond to each other uniquely. Since $\tilde{T} \mid \mathscr{Y} = T$, this implies

$$T(\hat{f}) = \int_{\mathbf{R}} \hat{f}(t)Z(dt) = \int_{\mathbf{R}} f(t)X(t)dt,\ f \in L^1(\mathbf{R}) , \tag{39}$$

and $\| T \| = \| Z \|(\mathbf{R})$.

Let $l \in \mathscr{X}^*$. Then (39) becomes (since a continuous operator commutes with the D-S integral, cf. [8], p. 324 and p. 153, and $\mathscr{X}^*$ is the adjoint space of $\mathscr{X}$),

$$\int_{\mathbf{R}} \hat{f}(t)\, l \circ Z(dt) = \int_{\mathbf{R}} f(t)\, l \circ X(t)dt . \tag{40}$$

In (40) now both are ordinary Lebesgue integrals, and hence using the Fubini theorem (for signed measures) on the left one has:

$$\int_{\mathbf{R}} f(t)dt \int_{\mathbf{R}} e_t(\lambda) l \circ Z(d\lambda) = \int_{\mathbf{R}} f(t) l \circ X(t)dt .$$

Subtracting and using the same theorem of ([8], p. 324),

$$\int_{\mathbf{R}} f(t) l\left(\int_{\mathbf{R}} e_t(\lambda) Z(d\lambda) - X(t)\right)dt = 0, l \in \mathscr{X}^*, f \in L^1(\mathbf{R}) . \tag{41}$$

It follows that the coefficient of f vanishes *a.e.*, (everywhere as it is continuous). Since $l \in \mathscr{X}^*$ is arbitrary it finally results that the quantity inside l is zero, for each $t \in \mathbf{R}$. Thus

$$X(t) = \int_{\mathbf{R}} e_t(\lambda) Z(d\lambda) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), t \in \mathbf{R} . \tag{42}$$

Hence X is weakly harmonizable by Theorem 3.3.

For the converse, let $X : \mathbf{R} \to L_0^2(P)$ be weakly harmonizable. Then X admits a representation of (42) by Theorem 3.3. Since $\| Z \| (\mathbf{R}) < \infty$, (21) implies $\| X(t) \|_2 \leqslant M_0 < \infty$ for all $t \in \mathbf{R}$, and as $l \circ X(\cdot)$ is the Fourier transform of $l \circ Z, l \in \mathscr{X}^*$, X is weakly continuous. Consider the Bochner integral for $(fX)(\cdot)$ as

$$l\left(\int_{\mathbf{R}} f(t)X(t)dt\right) = \int_{\mathbf{R}} f(t) l \circ X(t)dt = \int_{\mathbf{R}} f(t) \cdot \int_{\mathbf{R}} e_t(\lambda)(l \circ Z)(d\lambda)dt , \tag{43}$$

since $l \circ X$ is the Fourier transform of a signed measure

$$= \int_{\mathbf{R}}\int_{\mathbf{R}} f(t)e_t(\lambda) l \circ Z(d\lambda)dt, \text{ by Fubini's theorem,}$$

$$= \int_{\mathbf{R}} \hat{f}(\lambda) l \circ Z(d\lambda)$$

$$= l\left(\int_{\mathbf{R}} \hat{f}(\lambda)Z(d\lambda)\right), \text{ by ([8], p. 324) again.} \tag{44}$$

Since $l \in \mathscr{X}^*$ is arbitrary, (44) implies

$$\int_{\mathbf{R}} f(t)X(t)dt = \int_{\mathbf{R}} \hat{f}(\lambda)Z(d\lambda) \in \mathscr{X} . \tag{45}$$

Hence, using (21), one has

$$\left\| \int_{\mathbf{R}} f(t)X(t)dt \right\|_2 \leqslant \| \hat{f} \|_u \| Z \|(\mathbf{R}) = c \| \hat{f} \|_u , \quad f \in L^1(\mathbf{R}) , \tag{46}$$

where $c = \| Z \| (\mathbf{R}) < \infty$. It therefore follows that the set

$$\left\{\int_{\mathbf{R}} f(t)X(t)dt : \| \hat{f} \|_u \leqslant 1, f \in L^1(\mathbf{R})\right\} \subset L_0^2(P) ,$$

and is bounded. Since $\mathscr{X}$ is reflexive, X is V-bounded. This completes the proof.

Remarks. 1. Since V-boundedness concept is defined for general Banach spaces (for a treatment of this case, cf. [33]), and its Hilbert space version is equivalent to weak harmonizability, by the above theorem, *the latter term will be used in the Hilbert space context.* (Using the general definition of V-boundedness, a characterization of a process $X : \mathbf{R} \to \mathscr{X}$, a reflexive space, which is a Fourier transform of a vector measure is given in Theorem 7.2 below. It extends a result of [12].)

2. The preceding proof is arranged so that if $\mathbf{R}$ is replaced by a locally compact abelian (LCA) group G, the result and proof hold with essentially no change. The functions $\{e_t(\cdot), t \in G\}$ will then be group characters. Thus the result takes care of $G = \mathbf{R}^n$; so the (weakly) harmonizable random fields are included. Precise statements and further results in the general case will be given later.

If $\mathscr{W}$ is the set of all weakly harmonizable processes on $\mathbf{R} \to L_0^2(P) = \mathscr{X}$, and $T \in B(\mathscr{X})$, the algebra of bounded linear operators on $\mathscr{X}$, then $Y(t) = TX(t)$, $t \in \mathbf{R}$ defines a process which can be written as:

$$Y(t) = T\left(\int_{\mathbf{R}} e^{it\lambda} Z(d\lambda)\right) = \int_{\mathbf{R}} e^{it\lambda} (T \circ Z)(d\lambda), \tag{47}$$

by ([8], p. 324), and it can be seen that $\tilde{Z} = T \circ Z : \mathscr{B} \to \mathscr{X}$ is a stochastic measure, $\| \tilde{Z} \| (\mathbf{R}) \leqslant \| T \| \, \| Z \| (\mathbf{R}) < \infty$. Hence $Y \in \mathscr{W}$. Thus one has:

COROLLARY 4.3. $B(\mathscr{X}) \cdot \mathscr{W} = \mathscr{W}$, *or in words, the linear space of weakly harmonizable processes is a module over the class of all bounded linear transformations on* $\mathscr{X} = L_0^2(P)$.

Since each stationary process X is trivially strongly (hence weakly) harmonizable, if $P : \mathscr{X} \to \mathscr{X}$ is any orthogonal projection, then $Y = PX \in \mathscr{W}$, i.e. weakly harmonizable by Corollary 4.3. In particular if $\{X_n, n \in Z\} \subset \mathscr{X}$ is an orthonormal sequence, $\mathscr{X}_0 = \overline{sp}(X_n, n>0)$, let $Q(\mathscr{X}) = \mathscr{X}_0$ be the orthogonal projection and $Y_n = QX_n = X_n$ if $n > 0$, $= 0$ if $n \leqslant 0$. The process $\{Y_n, n \in \mathbf{Z}\} \in \mathscr{W}$, but it is *not* strongly harmonizable. Thus the class of weakly harmonizable processes is strictly larger than the strongly harmonizable class. (The latter is not a module over $B(\mathscr{X})$.)

In spite of the above comment, each weakly harmonizable process can be approximated "pointwise" by a sequence of strongly harmonizable ones. This observation is essentially due to Niemi [29]. The precise result is as follows:

THEOREM 4.4. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a weakly harmonizable process. Then there exists a sequence of strongly harmonizable processes* $X_n : \mathbf{R} \to L_0^2(P)$ *such that* $X_n(t) \to X(t)$, *as* $n \to \infty$, *in* $L_0^2(P)$ *uniformly (in* t*) on compact subsets of* $\mathbf{R}$. *If* $\mathbf{R}$ *is replaced by an LCA group* G *the same result holds with* $\{X_n, n \in I\}$ *being a net of such process. (The convergence is here in* $L^2(P)$*-mean.)*

Proof. By hypothesis, there is a stochastic measure $Z : \mathscr{B} \to \mathscr{X} = L_0^2(P)$, such that

$$X(t) = \int_{\mathbf{R}} e_t(\lambda) Z(d\lambda), \qquad t \in \mathbf{R}.$$

Thus $X : \mathbf{R} \to \mathscr{X}$ is a continuous mapping. If $\mathscr{H}_X = \overline{sp}\{X(t), t \in \mathbf{R}\} \subset \mathscr{X}$, then the continuity of X (and the separability of $\mathbf{R}$) implies $\mathscr{H}_X$ is separable. Hence there exists a sequence $\{\varphi_n, n \geqslant 1\} \subset \mathscr{X}_X$ which is a complete orthonormal (CON) basis for $\mathscr{X}_X$, so that

$$X(t) = \sum_{n=1}^{\infty} \varphi_n(X(t), \varphi_n), \qquad t \in \mathbf{R}, \tag{48}$$

the series converging in the (norm) topology of $\mathscr{H}_X$ for each t. Define

$$X_n(t) = \sum_{k=1}^{n} \varphi_k(X(t), \varphi_k), \qquad t \in \mathbf{R}. \tag{49}$$

Claim: $\{X_n(t), t \in \mathbf{R}\}$, $n \geqslant 1$, is the desired sequence. [In the general LCA group case $\{\varphi_n, n \in I\}$ is a net of CON elements of $\mathscr{H}_X$, since G, hence $\mathscr{H}_X$, need not be separable. Otherwise the same argument works with trivial modifications.]

To verify the claim, it is clear that $X_n(t) \to X(t)$ in $\mathscr{H}_X$ for each $t \in \mathbf{R}$. To see that X_n is strongly harmonizable, let

$$l_k : X \mapsto (X, \varphi_k), \qquad X \in \mathscr{H}_X.$$

Then $l_k \in \mathscr{H}_X^*$ for each k. Hence using the standard properties of the D-S integral, one has

$$X_n(t) = \sum_{k=1}^{n} \varphi_k l_k(X(t)) = \sum_{k=1}^{n} \varphi_k \cdot l_k\left(\textstyle\int_{\mathbf{R}} e_t(\lambda) Z(d\lambda)\right),$$

since X is weakly harmonizable,

$$= \sum_{k=1}^{n} \varphi_k \int_{\mathbf{R}} e_t(\lambda) l_k \circ Z(d\lambda) = \int_{\mathbf{R}} e_t(\lambda) \zeta_n(d\lambda), \tag{50}$$

where $\zeta_n(\cdot) = \sum_{k=1}^{n} \varphi_k l_k \circ Z(\cdot)$. Let $G_n(A, B) = (\zeta_n(A), \zeta_n(B))$. Then G_n is of finite total variation. Indeed, if $\mu_k = l_k \circ Z$, which is a signed measure (hence has finite variation) on $\mathbf{R}$, let

$$\eta_k(A, B) = (\varphi_k \mu_k(A), \varphi_k \mu_k(B)) = \mu_k(A)\overline{\mu_k(B)}.$$

So $G_n(A, B) = \sum_{k=1}^{n} \mu_k(A)\overline{\mu_k(B)}$. Since

$$|\mu_k(A)|\,|\mu_k(B)| \leqslant (|\mu_k|(\mathbf{R}))^2 < \infty$$

for each k, it follows that each η_k and hence G_n for each n has finite variation so that each X_n is strongly harmonizable.

It was already noted that X being weakly harmonizable, it is strongly continuous. [This is true even if $\mathbf{R}$ is replaced by an LCA group G (cf. [21], p. 270).] So if $K \subset \mathbf{R}$ is a compact set, then its image $X(K) \subset \mathcal{H}_X \subset L_0^2(P)$ is also (norm) compact. But $\mathcal{H}_X$ being a Hilbert space it has the (metric) approximation property. [This means the identity on $\mathcal{H}_X$ can be uniformly approximated by a sequence (net) of (contractive) degenerate, or finite rank, operators on each compact subset of $\mathcal{H}_X$.] Then $X_n(t) \to X(t)$ in $\mathcal{X}$ for each $t \in \mathbf{R}$ implies, by a result in Abstract Analysis in the presence of the approximation property, that the convergence holds in $\mathcal{X}$ uniformly on compact subsets of $\mathcal{X}$. This and the fact that $X(K)$ is compact implies that $X_n(t) \to X(t)$ in $L_0^2(P)$, uniformly for $t \in K \subset \mathbf{R}$. In the general LCA case, the same holds with nets replacing sequences. This completes the proof.

Remark. Even though the weakly harmonizable process is bounded and weakly (hence strongly here) continuous with some nice closure properties demonstrated above, it does not exhaust the class of all bounded continuous processes in $L_0^2(P)$. This can be seen from Theorem 3.2 by a suitable choice of a vector measure of finite local semivariation but which is not of finite semivariation. The following example demonstrates this point. Let $L^1(\mathbf{R})$ be identified with $\mathcal{M}(\mathbf{R})$ of regular signed measures on $\mathbf{R}$ by the Radon-Nikodým theorem (i.e. $f \in L^1(\mathbf{R}) \leftrightarrow \int_{(\cdot)} f(t)dt \in \mathcal{M}(R)$). Now it is known that there are nontrivial functions in $C_0(\mathbf{R}) - \mathcal{Y}_1$ where $\mathcal{Y}_1 = \{\hat{\mu} : \mu \in \mathcal{M}(\mathbf{R})\}$. Let $f \in C_0(\mathbf{R}) - \mathcal{Y}$. For instance

$$f(x) = \operatorname{sgn}(x)\left((\log|x|)^{-1}\chi_{[|x| \geq e]} + \frac{|x|}{e}\chi_{[|x|<e]}\right), \qquad x \in \mathbf{R},$$

is known to be such an f. Let $\varphi \in L_0^2(P)$, $\|\varphi\|_2 = 1$. Let $l \in \left(L_0^2(P)\right)^*$ such that $l(\varphi) = 1$. Consider the trivial process $X_0 : t \mapsto f(t)\varphi$. Then $X_0 : \mathbf{R} \to L_0^2(P)$ is bounded and continuous but not weakly harmonizable, since otherwise there exists a stochastic measure Z such that (by Theorem 3.3)

$$X_0(t) = \int_{\mathbf{R}} e_t(\lambda)Z(d\lambda), \qquad \text{and}$$

$$f(t) = l(X_0(t)) = \int_{\mathbf{R}} e_t(\lambda)\,(l \circ Z)\,(d\lambda).$$

Since $l \circ Z \in \mathcal{M}(\mathbf{R})$, this would contradict the choice of f.

Here is an interesting consequence of the preceding theorem.

THEOREM 4.5. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a weakly harmonizable process and let* $Z : \mathscr{B} \to L_0^2(P)$ *be its representing measure by (30). Then there is (nonuniquely) a fixed sequence of finite regular Borel measures* $\beta_n : \mathscr{B} \to \mathbf{R}^+$ *such that for each* $f \in C_0(\mathbf{R})$,

$$\left\| \int_{\mathbf{R}} f(t)Z(dt) \right\|_2 \leq \liminf_n \| f \|_{2, \beta_n}$$

$$\left(= \liminf_n \left[\int_{\mathbf{R}} | f(t) |^2 \beta_n(dt)\right]^{1/2}\right). \tag{51}$$

Remark. Even though this result is deducible from the general Theorem 5.5 below, the present proof is elementary and has some interest and will be given here. It leads to the general case.

Proof: By hypothesis, $X(\cdot)$ is represented by a stochastic measure Z (cf. (30)), and by the preceding theorem there are strongly harmonizable $X_n \to X$, uniformly on compact subsets of $\mathbf{R}$. Let ζ_n be the representing measure of X_n, so that $\zeta_n, Z : \mathscr{B} \to L_0^2(P)$, and

$$\int_{\mathbf{R}} f(\lambda)Z(d\lambda) = \lim_{n\to\infty} \int_{\mathbf{R}} f(\lambda)\zeta_n(d\lambda), \tag{52}$$

the limit existing in $L_0^2(P)$ when f is a trigonometric polynomial. Since such polynomials separate points of $\mathbf{R}$ and so are uniformly dense in $C_0(\mathbf{R})$, and the integrals in (52) define bounded operators from $C_0(\mathbf{R})$ into $L_0^2(P)$, it follows that (52) holds for all $f \in C_0(\mathbf{R})$, by standard reasoning (cf. [8], II.3.6). Hence

$$\alpha_0^f = \left\| \int_{\mathbf{R}} f(\lambda)Z(d\lambda) \right\|_2^2 = \lim_{n\to\infty} \left\| \int_{\mathbf{R}} f(\lambda)\zeta_n(d\lambda) \right\|_2^2, \quad f \in C_0(\mathbf{R})$$

$$= \lim_{n\to\infty} \int_{\mathbf{R}}\int_{\mathbf{R}} f(\lambda)\overline{f}(\lambda')F_n(d\lambda, d\lambda'), \tag{53}$$

where $F_n(s, t) = \big(\zeta_n(-\infty, s), \zeta_n(-\infty, t)\big)$ is a covariance function of bounded variation for each n. Let $| F_n | (\cdot, \cdot)$ be the (Vitali) variation measure of the bimeasure F_n. Then the hermitian property of F_n implies, in an obvious notation, $| F_n | (A, B) = | F_n | (B, A)$. Now define a mapping $\beta_n : \mathscr{B} \to \mathbf{R}^+$ by the equation:

$$\beta_n(A) = | F_n | (A, \mathbf{R}) = \frac{1}{2}\{| F_n | (A, \mathbf{R}) + | F_n | (\mathbf{R}, A)\}, \qquad A \in \mathscr{B},$$

so that β_n is a finite Borel measure, and

$$\int_{\mathbf{R}} f(\lambda)\beta_n(d\lambda) = \frac{1}{2}\left[\int_{\mathbf{R}}\int_{\mathbf{R}} f(s) \;| F_n | (ds, dt) + \int_{\mathbf{R}}\int_{\mathbf{R}} f(t) | F_n | (ds, dt)\right]. \tag{54}$$

Since F_n is positive (semi-) definite,

$$0 \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} f(s)\overline{f(t)} F_n(ds, dt) \leqslant \int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)\overline{f(t)}| \, |F_n|(ds, dt)$$

$$\leqslant \frac{1}{2} \left[\int_{\mathbf{R}} \int_{\mathbf{R}} |f(s)|^2 \, |F_n|(ds, dt) + \int_{\mathbf{R}} \int_{\mathbf{R}} |f(t)|^2 \, |F_n|(ds, dt)\right],$$

$$\text{since} \quad |ab| \leqslant (|a|^2 + |b|^2)/2,$$

$$= \int_{\mathbf{R}} |f(s)|^2 \beta_n(ds), \qquad \text{by (54)}. \tag{55}$$

This and (53) yield

$$\alpha_0^f = \| \int_{\mathbf{R}} f(\lambda) Z(d\lambda) \|_2^2 = \lim_n \int_{\mathbf{R}} \int_{\mathbf{R}} f(\lambda)\overline{f}(\lambda') F_n(d\lambda, d\lambda')$$

$$\leqslant \liminf_n \int_{\mathbf{R}} |f(\lambda)|^2 \beta_n(d\lambda), \qquad f \in C_0(\mathbf{R}). \tag{56}$$

This completes the proof.

Remark. For a deeper analysis of the structure of these processes, it is desirable to replace the sequence $\{\beta_n, n \geqslant 1\}$ by a single Borel measure. This is nontrivial. In the next section for a more general version, including harmonizable fields, such a result will be obtained.

5. Domination problem for harmonizable fields

The work of the preceding section indicates that the weakly harmonizable processes are included in the class of functions which are Fourier transformations of vector measures into Banach spaces. A characterization of such functions, based on the V-boundedness concept of [2], has been obtained first in [33]. For probabilistic applications (e.g., filtering theory) the domination problem, generalizing Theorem 4.5, should be solved. The following result illuminates the nature of the general problem under consideration.

THEOREM 5.1. *Let* (Ω, Σ) *be a measurable space,* $\mathscr{X}$ *a Banach space and* $\nu : \Sigma \to \mathscr{X}$ *be a vector measure. Then there exists a (finite) measure* $\mu : \Sigma \to \mathbf{R}^+$, *a continuous convex function* $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $\frac{\varphi(x)}{x} \nearrow \infty$ *as* $x \nearrow \infty$, $\varphi(0) = 0$, *and* ν *has* φ*-semivariation finite relative to* μ *in the sense that*

$$\| \nu \|_{\varphi}(\Omega) = \sup \{ \| \int_{\Omega} f(\omega) \nu(d\omega) \|_{\mathscr{X}} : \| f \|_{\psi, \mu} \leqslant 1 \} < \infty, \tag{57}$$

where $\| f \|_{\psi,\mu} = \inf \{ \alpha > 0 : \int_{\Omega} \psi\left(\frac{|f(\omega)|}{\alpha}\right) \mu(d\omega) \leqslant 1 \} < \infty,$ *and the* *integral relative to* ν *in (57), is in the Dunford-Schwartz sense. Here* $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ *is a convex function given by* $\psi(x) = \sup\{|x|y - \varphi(y) : y \geq 0\}$.

The proof of this result depends on some results of ([8], IV.10) and elementary properties of Orlicz spaces (cf. [47], p. 173). It will be omitted here, since the details are given in [38]. This is only motivational for what follows.

Note that (57) is a desired generalization of (51) if $\{\beta_n, n \geq 1\}$ is replaced by μ and $\|\cdot\|_2$ is replaced by $\|\cdot\|_\varphi$. However φ may grow faster than a polynomial. What is useful here is a φ with $\varphi(x) = |x|^p$, $1 \leq p \leq 2$. This can be proved for a special class of spaces $\mathcal{X}$, which is sufficient for our study of harmonizable fields.

It will be convenient to introduce a definition and to state a result (essentially) of Grothendieck and Pietsch, for the work below.

Definition 5.2. Let $\mathcal{X}, \mathcal{Y}$ be a pair of Banach spaces and, as usual, $B(\mathcal{X}, \mathcal{Y})$ be the space of bounded linear operators on $\mathcal{X}$ into $\mathcal{Y}$. If $1 \leq p \leq \infty$, $T \in B(\mathcal{X}, \mathcal{Y})$, then T is called *p-absolutely summing* if $\alpha_p(T) < \infty$, where

$$\alpha_p(T) = \inf\{c > 0 : [\sum_{i=1}^{n} \|Tx_i\|^p]^{\frac{1}{p}} \leq c \sup_{\|x^*\| \leq 1} (\sum_{i=1}^{n} |x^*(x_i)|^p)^{\frac{1}{p}}, x_i \in \mathcal{X}, \quad 1 \leq i \leq n, n \geq 1\}, \tag{58}$$

with $x^* \in \mathcal{X}^*$, the adjoint space of $\mathcal{X}$.

The following result, which is alluded to above, with a short proof may be found in [22] together with some extensions and applications.

PROPOSITION 5.3. *Let* $T \in B(\mathcal{X}, \mathcal{Y})$ *be p-absolutely summing,* $1 \leq p < \infty$. *Let* K^* *be the weak-star closure of the set of extreme points of the unit ball* U^* *of* $\mathcal{X}^*$. *Then there is a regular Borel probability measure* μ *on the compact space* K^* *such that*

$$\|Tx\|_{\mathcal{Y}} \leq \alpha_p(T)\,[\int_{K^*} |x^*(x)|^p \,\mu(dx^*)]^{1/p}, \qquad x \in \mathcal{X}. \tag{59}$$

Conversely (and this is simple), if T *satisfies (59) for some* μ *on* K^* *with a constant* γ_0, *then* T *is p-absolutely summing and* $\alpha_p(T) \leq \gamma_0$. *Further any p-absolutely summing operator is weakly compact.*

Let us specialize this result in the case that $\mathcal{X} = C_r(S)[C(S)]$, the space of real [complex] continuous functions on a compact set S. Let K be the set of all extreme points of the unit ball U^* of $(C_r(S))^*$ and $q : S \to (C_r(S))^*$ be the mapping defined by $q(s) = l_s$ with $l_s(f) = f(s)$, $f \in C_r(S)$ so that l_s is the evaluation functional, $\|l_s\| = 1$, and $l_s \in K$, $s \in S$. Some other known results needed from Linear Analysis, in the form used here, are as follows. (For details, see [4], Sec. V.3; [8], p. 441). In this case the spaces S and $q(S)$ are homeomorphic and $q(S)$ is closed since S is compact. By Mil'man's theorem U^* is the weak-star closed convex hull of $q(S) \cup (-q(S))$, and (by the compactness of S) the latter is equal to the extreme point-set of U^* and is closed. Further these are of the form αl_s, $s \in S$, and $|\alpha| = 1$ (cf. [8], V.8.6). Consequently (59) becomes

$$\begin{aligned} \| Tf \|^p &\leqslant (\alpha_p(T))^p \cdot \int_{q(S) \cup (-q(S))} | l_s(f) |^p \mu(dl_s), \qquad f \in C_r(S) \\ &\leqslant 2(\alpha_p(T))^p \cdot \int_{q(S)} | l_s(f) |^p \mu(dl_s), \\ &= 2(\alpha_p(T))^p \cdot \int_S | f(s) |^p \mu(ds), \end{aligned}$$

if S and $q(S)$ are (as they can be) identified. For the complex case, $C(S) = C_r(S) + iC_r(S)$, and so the same holds if the constants are doubled. Thus

$$\| Tf \| \leqslant C_p [\int_S | f(s) |^p \mu(ds)]^{\frac{1}{p}} = C_p \| f \|_{p, \mu}, \qquad f \in C(S), \tag{60}$$

where $C_p^p = 4[\alpha_p(T)]^p$. This form of (59) will be utilized below.

Definition 5.4. Let $\mathscr{X}$ be a Banach space, $1 \leqslant p \leqslant \infty$ and $1 \leqslant \lambda < \infty$. Then $\mathscr{X}$ is called *an* $\mathscr{L}_{p,\lambda}$*-space* if for each n-dimensional space $E \subset \mathscr{X}$, $1 \leqslant n < \infty$. there is a finite dimensional $F \subset \mathscr{X}, E \subset F$, such that $d(F, l_p^n) \leqslant \lambda$ where l_p^n is the n-dimensional sequence space with p-th power norm and where

$$d(E_1, E_2) = \inf \{\| T \| \| T^{-1} \| : T \in B(E_1, E_2)\}$$

for any pair of normed linear spaces E_1, E_2. A Banach space $\mathscr{X}$ is *an* $\mathscr{L}_p$*-space* if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqslant 1$.

It is known (and easy to verify) that each $L^p(\mu)$, $p \geqslant 1$, is an $\mathscr{L}_{p,\lambda}$-space for every $\lambda > 1$, and $C(S)$ [indeed each abstract (M)-space] is an $\mathscr{L}_{\infty,\lambda}$-space for every $\lambda > 1$. The class of $\mathscr{L}_2$-spaces coincides with the class of Banach spaces isomorphic to a Hilbert space. For proofs and more on these ideas the reader is referred to the article of Lindenstrauss and Pelczyński [22].

With this set up the following general result can be established at this time on the domination problem for vector measures.

THEOREM 5.5. *Let S be a locally compact space and $C_0(S)$ be the Banach space of continuous scalar functions on S vanishing at "∞". If $\mathscr{Y}$ is an $\mathscr{L}_p$-space $1 \leqslant p \leqslant 2$, and $T \in B(C_0(S), \mathscr{Y})$, then there exist a finite positive Borel measure μ on S, and a vector measure Z on S into $\mathscr{Y}$, such that*

$$\| \int_S f(s) Z(ds) \|_{\mathscr{Y}} = \| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2,\mu}, \qquad f \in C_0(S). \tag{61}$$

Proof. Since $\mathscr{X} = C_0(S)$ is an abstract (M)-space, it is an $\mathscr{L}_\infty$-space by the preceding remarks. But $\mathscr{Y}$ is an $\mathscr{L}_p$-space $1 \leqslant p \leqslant 2$, and so $T \in B(\mathscr{X}, \mathscr{Y})$ is 2-absolutely summing by ([22], Thm. 4.3), and therefore (cf. Prop. 5.3 above) it is also weakly compact. By the argument presented for (37), (38) above, one can use the theorem ([8], VI.7.3) even when S is locally compact (and noncompact) to conclude that there is a vector measure Z on the Borel σ-ring of S into $\mathscr{Y}$ such that

$$Tf = \int_S f(s)Z(ds), \qquad \text{(D-S integral)}.$$

Using the argument of (37), if $\tilde{S}$ is the one point compactification of S, and $\tilde{T} \in B(C(\tilde{S}), \mathscr{Y})$ is the norm preserving extension, then $\tilde{T}$ is 2-absolutely summing (since $C(\tilde{S})$ is an abstract (M)-space), and weakly compact. So by (60) there exists a finite Borel measure $\tilde{\mu}$ on $\tilde{S}$ such that

$$\| \tilde{T}f \|_{\mathscr{Y}} \leqslant c_p \| f \|_{2, \tilde{\mu}}, \qquad f \in C(\tilde{S}).$$

Letting $\bar{\mu} = c_p^p \tilde{\mu}$, one has $\| \tilde{T}f \| \leqslant \| f \|_{2, \bar{\mu}}$, $f \in C(\tilde{S})$. So (61) holds on $\tilde{S}$. Let $\mu(\cdot) = \bar{\mu}(S \cap \cdot)$ so that μ is a finite Borel measure on S. If now one restricts to $C_0(S)$ identified as a subset of $C(\tilde{S})$, so that $T = \tilde{T} \mid C_0(S)$, it follows from the preceding analysis that $\| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2, \mu}$ for all $f \in C_0(S)$. Since the integral representation of T is evidently true, this establishes (61), and completes the proof of the theorem.

If $\mathscr{Y}$ is a Hilbert space, it is an $\mathscr{L}_2$-space so that the above theorem considerably strengthens Theorem 4.5, since the sequence there is now replaceable by a single measure.

The following statement is actually a consequence of the above result, and it will be invoked in the last section.

PROPOSITION 5.6. *Let (Ω, Σ) be any measurable space, and $\mathscr{X} = B(\Omega, \Sigma)$ be the Banach space (under uniform norm) of scalar measurable functions. If $\mathscr{Y}$ is an $\mathscr{L}_p$-space, $1 \leqslant p \leqslant 2$, as above, $T \in B(\mathscr{X}, \mathscr{Y})$ is such that for each $f_n \in \mathscr{X}$, $f_n \to f$ pointwise boundedly implies $\| Tf_n \| \to \| Tf \|$, then there exist σ-additive functions $Z: \Sigma \to \mathscr{Y}$, $\mu: \Sigma \to \mathbf{R}^+$, such that*

$$\left\| \int_\Omega f(\omega)Z(d\omega) \right\|_{\mathscr{Y}} = \| Tf \|_{\mathscr{Y}} \leqslant \| f \|_{2, \mu}, \qquad f \in \mathscr{X}. \tag{62}$$

The proof uses the fact that $B(\Omega, \Sigma)$ is isometrically isomorphic to $C(S)$, for a compact (extremelly disconnected) Hausdorff space (cf. [8], IV.6.18), and reduces to the preceding result. The computations, using the standard Carathéodory measure theory, will be omitted here. The details, however, may be found in [38].

Remark. The preceding results show that the domination problem for vector measures in L^p-spaces, $1 \leqslant p \leqslant 2$, is solved and hence also for harmonizable fields since only the $\mathscr{L}_2$-type spaces are involved in the latter. But, for $p > 2$, such a satisfactory solution of the problem is not available.

6. STATIONARY DILATIONS

The results of the last section play a key role in showing that each weakly harmonizable random field has a stationary dilation. It is a consequence of the preceding work that for any stationary field $Y : G \to L_0^2(P)$ with G an LCA group, and each orthogonal projection $Q : L_0^2(P) \to L_0^2(P)$, the new random field $X(g) = QY(g), g \in G$, giving $X : G \to L_0^2(P)$, is shown to be weakly harmonizable. The dilation result yields the reverse implication. A "concrete" version of this is given by the following theorem and an operator version will be obtained later from it.

THEOREM 6.1. *Let G be an LCA group, $X : G \to L_0^2(P) = \mathscr{H}$ a weakly harmonizable random field. Then there is a super (or extension) Hilbert space $\mathscr{K} \supset \mathscr{H}$, a probability measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ with $\mathscr{K} = L_0^2(\tilde{P})$, and a stationary random field $Y : G \to L_0^2(\tilde{P})$, such that $X(g) = QY(g), g \in G$, where $Q : L_0^2(\tilde{P}) \to L_0^2(\tilde{P})$ is the orthogonal projection with range $L_0^2(P)$. If moreover, $\mathscr{H} = \overline{sp}\{X(g), g \in G\}$, then Y determines $\mathscr{K}$ in the sense that $\mathscr{K} = \overline{sp}\{Y(g), g \in G\}$. [Thus $\mathscr{K}$ is the minimal super space for $\mathscr{H}$.]*

Proof. The "consequence" above is easily proved. In fact, if $Y : G \to L_0^2(P)$ is stationary, then Theorem 3.3 implies

$$Y(g) = \int_{\hat{G}} \langle g, s\rangle \, Z(ds), \qquad g \in G, \tag{63}$$

for a vector measure Z on $\hat{G}$ into $\mathscr{K} = L_0^2(P)$, with orthogonal increments (also called orthogonally scattered) where $\hat{G}$ is the dual group of the LCA group G, and $\langle \cdot, s\rangle$ is a character of G. If $Q : \mathscr{K} \to \mathscr{K}$ is any orthogonal projection, then $\tilde{Z} = Q \circ Z$ is a stochastic measure on $\tilde{G}$ into $\mathscr{K}$. Indeed,

$$\begin{aligned}
\|\tilde{Z}\|^2(\hat{G}) &= \sup\{\|\sum_{i=1}^{n} a_i\tilde{Z}(A_i)\|_2^2 : |a_i| \leqslant 1, A_i \subset \hat{G} \text{ disjoint Borel}, n \geqslant 1\} \\
&= \sup\{\|Q\sum_{i=1}^{n} a_i Z(A_i)\|_2^2 : |a_i| \leqslant 1, A_i \subset \hat{G}, \text{ as above}\} \\
&\leqslant \|Q\|^2 \sup\{\|\sum_{i=1}^{n} a_i Z(A_i)\|_2^2 : |a_i| \leqslant 1, A_i \subset \hat{G}, \text{ as before}\} \\
&= \|Q\|^2 \sup\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i\overline{a_j}\, F(A_i \cap A_j) : |a_i| \leqslant 1, A_i \subset \hat{G} \text{ as before}\} \\
&\qquad\qquad \text{where } F(A_i \cap A_j) = (Z(A_i), Z(A_j)), \\
&= \|Q\|^2 |F|(\hat{G}) \leqslant F(\hat{G}) < \infty,
\end{aligned} \tag{64}$$

since F is the spectral measure of Z and so is finite and Q is a contraction. Hence $\tilde{Z}$ has finite semivariation and is clearly σ-additive, so that it is a stochastic measure. By Theorem 3.3, X given by $X(g) = QY(g) = \int_{\hat{G}} \langle g, s\rangle \tilde{Z}(ds), g \in G$, is weakly harmonizable. (Note that the same conclusion holds if Q is replaced by any bounded linear operator on $\mathscr{K}$. If the range of the projection Q is not finite dimensional, then X need *not* be strongly harmonizable!)

To go in the reverse direction, the (possibly) augmented space $\mathscr{K} \supset \mathscr{H}$ has to be constructed. Consider $X : G \to \mathscr{H} = L_0^2(P)$, the given weakly harmonizable random field. In order to get simultaneously the additional structure demanded in the last part, let $\mathscr{H} = \overline{sp}\{X(g), g \in G\}$ also. Then, as before, there is a stochastic measure on $\hat{G}$ into $\mathscr{H}$ such that

$$X(g) = \int_{\hat{G}} \langle g, s\rangle Z(ds) \in \mathscr{H}, \qquad g \in G. \tag{65}$$

By Theorem 5.5, with $\mathscr{Y} = \mathscr{H}$, there exists a finite Radon (= regular Borel) measure μ on $\hat{G}$ such that

$$\| \int_{\hat{G}} f(t)Z(dt) \|_2^2 \leqslant \int_{\hat{G}} | f(t) |^2 \mu(dt), \qquad f \in C_0(\hat{G}). \tag{66}$$

Now define a mapping $v : \mathscr{B}(\hat{G} \times \hat{G}) \to \mathbf{R}^+$ by the equation

$$v(A, B) = \mu(A \cap B), A, B \in \mathscr{B}(\hat{G}), \tag{67}$$

where $\mathscr{B}(\hat{G})$ is the Borel σ-ring of $\hat{G}$ and similarly $\mathscr{B}(\hat{G} \times \hat{G})$. Then v is a bimeasure of finite Vitali variation on $\mathscr{B}(\hat{G}) \times \mathscr{B}(\hat{G})$ and since this ring generates $\mathscr{B}(\hat{G} \times \hat{G})$, v extends to a Radon measure on the latter σ-ring. Morevoer, it is clear that v concentrates on the diagonal of the product space $\hat{G} \times \hat{G}$. If $C_b(\hat{G})$ denotes the Banach space of bounded continuous scalar functions on $\hat{G}$ with uniform norm, then

$$\int_{\hat{G}} \int_{\hat{G}} f(s, t)v(ds, dt) = \int_{\hat{G}} f(s, s)\mu(ds), \qquad f \in C_b(\hat{G} \times \hat{G}). \tag{68}$$

Let $F(A, B) = \big(Z(A), Z(B)\big)$ so that $F : \mathscr{B}(\hat{G} \times \hat{G}) \to \mathbf{C}$ is a bimeasure of finite semivariation, from (65). Thus using the D-S and MT-integration techniques as before,

$$0 \leqslant \| \int_{\hat{G}} f(s)Z(ds) \|_2^2 = \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)}F(ds, dt), \qquad f \in C_b(\hat{G}). \tag{69}$$

Letting $f(s, t) = f(s) \cdot f(t)$ in (68), $\alpha = v - F$ one has from (66)-(69),

$$\begin{aligned} 0 &\leqslant \int_{\hat{G}} | f(s) |^2 \mu(ds) - \| \int_{\hat{G}} f(s)Z(ds) \|_2^2 \\ &= \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)}\,[v(ds, dt) - F(ds, dt)] \\ &= \int_{\hat{G}} \int_{\hat{G}} f(s)\overline{f(t)}\alpha(ds, dt), \qquad f \in C_b(\hat{G}). \end{aligned} \tag{70}$$

So α is positive semi-definite and $\alpha = 0$ iff $v = F$, i.e., if F concentrates on the diagonal. This corresponds to X being stationary itself. Excluding this trivial case, $\alpha \not\equiv 0$, and (70) is strictly positive, if $f = 1$. It follows from (70) that $[\cdot, \cdot]' : C_b(\hat{G}) \times C_b(\hat{G}) \to \mathbf{C}$ defines a nontrivial semi-inner product, where

$$[f, g]' = \int_{\hat{G}} \int_{\hat{G}} f(s)\bar{g}(t)\alpha(ds, dt), \qquad f, g \in C_b(\hat{G}). \tag{71}$$

If $\mathcal{N}_0 = \{f : [f,f]' = 0, f \in C_b(\hat{G})\}$, and $\mathcal{H}_1 = C_b(\hat{G})/\mathcal{N}_0$ is the factor space, let $[\cdot, \cdot] : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbf{C}$ be defined by

$$[(f), (g)] = [f, g]', \qquad f \in (f) \in \mathcal{H}_1, g \in (g) \in \mathcal{H}_1. \tag{72}$$

Then $[\cdot, \cdot]$ is an inner product on $\mathcal{H}_1$ and define $\mathcal{H}_0$ as its completion in $[\cdot, \cdot]$. Let $\pi_0 : C_b(\hat{G}) \to \mathcal{H}_0$ be the canonical projection. Thus $\mathcal{H}_0$ is nontrivial and need not be separable. Now let us replace $\mathcal{H}_0$ by $L_0^2(P')$ on a probability space (Ω', Σ', P'). This can be done based on the Fubini-Jessen theorem where P' can even be taken to be a Gaussian measure (for the real $\mathcal{H}$, see [36], pp. 414-415). The complex case is similar. A quick outline is as follows: Let $\{h_i, i \in I\} \subset \mathcal{H}_0$ be a CON set. If $(\Omega_i, \Sigma_i, P_i)$ is a probability space determined by a complex Gaussian variable, so that one can take $\Omega_i = \mathbf{C}$, $\Sigma_i =$ Borel σ-algebra of $\mathbf{C}$, and

$$P_i(A) = (2\pi)^{-1} \int_A \exp\left(-\frac{|t|^2}{2}\right) dt_1 dt_2, A \in \Sigma_i, (t = t_1 + \sqrt{-1}\, t_2),$$

let $(\Omega', \Sigma', P') = \bigotimes_{i \in I} (\Omega_i, \Sigma_i, P_i)$ the product space given by the Fubini-Jessen theorem. If $X_i(\omega) = \omega(i), \omega \in \Omega' = \mathbf{C}^I$, the coordinate function, then $E(X_i) = 0$ and $E(|X_i|^2) = 1$. Also $\{X_i, i \in I\}$ forms a CON basis of $\mathcal{L} = \overline{\operatorname{sp}}\{X_i, i \in I\} \subset L_0^2(P')$. The correspondence $\tau : h_i \to X_i$, extended linearly, sets up an isomorphism of $\mathcal{H}_0$ onto $\mathcal{L}$, and

$$\| \tau(h_i) \|_2^2 = E(|X_i|^2) = 1 = [h_i, h_i], \qquad i \in I.$$

Then by polarization one has $[h_i, h_j] = E(\tau(h_i)\overline{\tau(h_j)})$, so that τ is an isometric isomorphism of $\mathcal{H}_0$ onto $\mathcal{L} \subset L_0^2(P')$, as desired.

If $\pi = \tau \circ \pi_0 : f \mapsto \tau(\pi_0(f)) \in \mathcal{H}' \subset L_0^2(P')$, $f \in C_b(\hat{G})$, is the composite (canonical) mapping, let $X_1(t) = \pi(e_t(\cdot)) \in \mathcal{H}'$ where $e_t : s \mapsto (t, s)$, is a character of G at $t \in G$. Note that $e_0 = 1 \notin \mathcal{N}_0$, so $\pi_0(1)$ can be identified with the constant $1 \in C_b(\hat{G})$. Thus

$$X_1(0) = \tau(1), E(|\tau(1)|^2) = 1.$$

Let $\mathcal{H}'' = \overline{\operatorname{sp}}\{X_1(t), t \in G\} \subset \mathcal{H}'$. Then there exists a probability space $(\Omega'', \Sigma'', P'')$, as above, such that $\mathcal{H}'' \subset L^2(P'')$. Finally set $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}''$, in the direct sum of Hilbert spaces $L_0^2(P)$ and $L_0^2(P'')$. If $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) = (\Omega, \Sigma, P) \otimes (\Omega'', \Sigma'', P'')$ then one can identify, in a natural way, $\mathscr{K} \subset L_0^2(\tilde{P})$. Define $Y(t) = X(t) + X_1(t)$, $t \in G$, so that $(X(t), X_1(t)) = 0$ since $\mathscr{H} \perp \mathscr{H}''$ in $\mathscr{K}$. Then $\{Y(t), t \in G\} \subset \mathscr{K} \subset L_0^2(\tilde{P})$, and if $Q : \mathscr{K} \to \mathscr{H} = \{\mathscr{H} \oplus \{0\}\}$ is the orthogonal projection, one has $X(t) = QY(t)$, $t \in G$. It remains to show that $Y : G \to L_0^2(\tilde{P})$ is stationary. By construction $Y(0) = X(0) + X_1(0)$ and this is $X(0)$ only when $X_1(0) = 0$ which can happen iff $\mathscr{H}'' = \{0\}$, i.e., when no enlargement is needed.

To verify stationarity, consider

$$\begin{aligned} r(s, t) &= (Y(s), Y(t)) = (X(s), X(t)) + (X_1(s), X_1(t)) \text{ since } X \perp X_1, \\ &= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} F(d\lambda, d\lambda') + \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} \alpha(d\lambda, d\lambda'), \\ &\qquad \text{by (69) and (72) and these are MT-integrals,} \\ &= \int_{\hat{G}} \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda')} v(d\lambda, d\lambda'), \text{ since } \alpha = v - F \\ &= \int_{\hat{G}} (s, \lambda) \overline{(t, \lambda)} \mu(d\lambda), \text{ by (68),} \\ &= \int_{\hat{G}} (s-t, \lambda) \mu(d\lambda), \text{ by the composition of characters.} \end{aligned} \tag{73}$$

Since μ is a finite positive measure, (73) implies

$$r(s+h, t+h) = r(s, t) = \tilde{r}(s-t),$$

and so the $Y : G \to L_0^2(\tilde{P})$ is stationary. The construction also implies that $\overline{\text{sp}}\{Y(t), t \in G\} = \mathscr{K}$ in the case that $\mathscr{H} = \overline{\text{sp}}\{X(t), t \in G\}$. This completes the proof.

The following is a useful deduction:

COROLLARY 6.2. *Every vector measure* $\nu : \mathscr{B}(G) \to \mathscr{H}$ *where* G *is an LCA group,* $\mathscr{B}(G)$ *being its Borel algebra, and* $\mathscr{H}$ *is a Hilbert space, has an orthogonally scattered dilation.*

Proof. Since $G = \hat{\hat{G}}$ consider the mapping $X : \hat{G} \to \mathscr{H}$ defined as the D-S integral $X(\hat{g}) = \int_G \langle \hat{g}, \lambda \rangle \nu(d\lambda)$. Then X is V-bounded; so it is weakly harmonizable. By the above theorem there are an extension Hilbert space $\mathscr{K} \supset \mathscr{H}$, an orthogonal projection $Q : \mathscr{K} \to \mathscr{K}$, with range $\mathscr{H}$, and a stationary field $Y : \hat{G} \to \mathscr{K}$ such that $X(\hat{g}) = QY(\hat{g})$. Let Z be the stochastic measure representing Y, (cf. Theorem 3.3). Hence for each $h \in \mathscr{H}$ one has $(Z : \mathscr{B}(\hat{G}) \to \mathscr{K})$

$$\int_G (\hat{g}, \lambda) (\nu(d\lambda), h) = (X(\hat{g}), h) = (QY(\hat{g}), h) = \int_{\hat{G}} (\hat{g}, \lambda) (Q \circ Z(d\lambda), h).$$

These are now scalar (Lebesgue-Stieltjes) integrals. By the classical uniqueness theorem of Fourier analysis for such integrals, one has

$$(\nu(A) - Q \circ Z(A), h) = 0, A \in \mathscr{B}(G), h \in \mathscr{K} .$$

Hence $\nu = Q \circ Z$. Since Z is orthogonally scattered by virtue of the fact that Y is stationary, the result follows.

With the last theorem, a more perspicuous version of the dilation problem for a weakly harmonizable random field can be given. This, however, depends also on an interesting theorem of Sz.-Nagy [41] and will be presented. Recall from the classical theory of stationary processes ([6], p. 512 and p. 638) every such process $\{Y_t, t \in \mathbf{R}\} \subset L_0^2(P)$, can be expressed as $Y_t = U_t Y_0$, where $\{U_t, t \in \mathbf{R}\}$ is a group of unitary operators acting on $L_0^2(P)$ (first on $\overline{\text{sp}}\{Y_t, t \in \mathbf{R}\}$ and then, for instance, define each U_t as an identity on the orthogonal complement of this subspace). The spectral theory of U_t then yields immediately the corresponding integral representation of Y_t's. The same result holds if $\mathbf{R}$ is replaced by an LCA group G. The corresponding operator representation for harmonizable processes (or fields) is not so simple. Its solution will be presented in the following theorem. Recall that a family $T : G \to B(\mathscr{X})$, $\mathscr{X}$ a Hilbert space, is of positive type if $T(-g) = T(g)^*$ (adjoint operator) and for each finite set $\{x_{s_1}, ..., x_{s_n}\}$ of $\mathscr{X}$ indexed by $J = \{s_1, s_2, ..., s_n\} \subset G$, one has

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(T(s_j^{-1} s_i) x_{s_i}, x_{s_j}\right) \geq 0 . \tag{74}$$

THEOREM 6.3. *Let G be an LCA group and $X : G \to L_0^2(P) = \mathscr{X}$, a Hilbert space, be weakly harmonizable. Then there exists a super Hilbert space $\mathscr{K} = L_0^2(\tilde{P}) \supset \mathscr{X}$ on an enlarged probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, a random variable $Y_0 \in \mathscr{K}$ a weakly continuous family $\{T(g), g \in G\}$ of contractive linear operators from $\mathscr{K}$ to $\mathscr{X}$ with $T(0)$ as the identity on $\mathscr{X}$ (0 being the neutral element of G), such that, when its domain is restricted to $\mathscr{X}$, it is of positive type, in terms of which $X(g) = T(g) Y_0, g \in G$. Conversely every weakly continuous contractive family $\{T(g), g \in G\}$ of the above type from any super Hilbert space $\mathscr{K} \supseteq \mathscr{X}$ into $\mathscr{X}$ which, when restricted to $\mathscr{X}$ is of positive type, defines a weakly harmonizable process $X : G \to \mathscr{X}$, by the equation $X(g) = T(g) Y_0$ for any $Y_0 \in \mathscr{X}$, $T(0)$ being identity on $\mathscr{X}$.*

Proof. The direct part is an operator-theoretic reformulation of Theorem 6.1. Briefly, let $X : G \to L_0^2(P) = \mathscr{X}$ be weakly harmonizable. Then there exist a $\mathscr{K} = L_0^2(\tilde{P}) \supset \mathscr{X}$ and a stationary $Y: G \to \mathscr{K}$ such that $X(g) = QY(g), g \in G$, by Theorem 6.1 with Q as the orthogonal projection on $\mathscr{K}$ and range $\mathscr{X}$. But $Y(g) = U(g) Y(0)$ where $\{U(g), g \in G\}$ is a (strongly) continuous group of unitary operators on $\mathscr{K}$. Let $T(g) = QU(g), g \in G$. It is asserted that $\{T(g), g \in G\}$ is the desired family.

Indeed, $T(0) = Q$ (= identity on $\mathscr{X}$), and $\| T(g) \| \leqslant \| Q \| \| U(g) \| \leqslant 1$. The continuity of $U(g)$ on G clearly implies the weak continuity of $T(g)$'s. To verify the positive definiteness on $\mathscr{X}$, let $h_{s_1}, \ldots, h_{s_n}$ be a finite set in $\mathscr{X}$. Then letting $\tilde{T}(g) = T(g)|_{\mathscr{X}}$ one has $\tilde{T}(-g) = (\tilde{T}(g))^*$ since

$$\begin{aligned}(\tilde{T}(-g)h_{s_1}, h_{s_2}) &= (QU(-g)h_{s_1}, h_{s_2}) = (U^*(g)h_{s_1}, Qh_{s_2}) \\ &= (h_{s_1}, U(g)h_{s_2}), \text{ since } Qh_{s_i} = h_{s_i} \text{ and } U^{**}(g) = U(g), \\ &= (Qh_{s_1}, U(g)h_{s_2}) = (h_{s_1}, QU(g)h_{s_2}) \\ &= (h_{s_1}, \tilde{T}(g)h_{s_2}) = (\tilde{T}(g)^*h_{s_1}, h_{s_2}),\ h_{s_i} \in \mathscr{X},\ i = 1, 2\,. \end{aligned} \tag{75}$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n} (\tilde{T}(s_j^{-1}s_i)h_{s_i}, h_{s_j}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} (QU(-s_j)U(s_i)h_{s_i}, h_{s_j}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} (U(s_j)^*U(s_i)h_{s_i}, h_{s_j}) \\ &= \| \sum_{i=1}^{n} U(s_i)h_{s_i} \|^2 \geqslant 0\,. \end{aligned} \tag{76}$$

The converse depends explicitly on an important theorem of Sz.-Nagy ([41], Thm. III; this is an extension of a classical result of Naĭmark). According to this result if $\tilde{T}(\cdot) = T(\cdot)|_{\mathscr{X}}$, then there is a super Hilbert space $\mathscr{K}_1 \supset \mathscr{X}$ ($\mathscr{K}_1$ may be quite different from $\mathscr{K}$) and a weakly (hence strongly) continuous group $\{V(g), g \in G\}$ of unitary operators on $\mathscr{K}_1$ such that $\tilde{T}(g) = Q_1 V(g)|_{\mathscr{X}}$, Q_1 being the orthogonal projection of $\mathscr{K}_1$ onto $\mathscr{X}$. Here $\mathscr{K}_1$ can be chosen as $\mathscr{K}_1 = \overline{\text{sp}}\{V(g)\mathscr{X}, g \in G\}$. If $x_0 \in \mathscr{X}$ is arbitrary, then $x_0 \in \mathscr{K}_1 \cap \mathscr{K}$, and

$$T(g)x_0 = \tilde{T}(g)x_0 = Q_1 V(g)x_0 = X(g)\,, \qquad \text{(say)}, g \in G\,.$$

But $\{Y(g) = V(g)x_0, g \in G\} \subset \mathscr{K}_1$ is a stationary process so that by the first paragraph of the proof of Theorem 6.1, $\{X_0(g), g \in G\} \subset \mathscr{X}$ is weakly harmonizable. Thus for each $x_0 \in \mathscr{X}$, $\{T(g)x_0, g \in G\}$ is weakly harmonizable, and this completes the proof.

Remark. In the converse direction one can take $\mathscr{K} = \mathscr{X}$ However in the forward direction, it is not always possible to take Y_0 in $\mathscr{X}$, so that $X(0) = Y_0$, as the example following Definition 2.1 shows. Thus there is an inherent asymmetry in the statement of this theorem, and the mention of the super Hilbert space $\mathscr{K}$ in the enunciation cannot be avoided. It should also be noted that the above quoted theorem of Sz.-Nagy [41] can be deduced also from Naĭmark's theorem *and* Theorem 6.1. See [38] for a further discussion on this point.

7. Characterizations of weak harmonizability

In this section a different type of characterization, based on the V-boundedness concept crucially, of weak harmonizability as well as a comprehensive statement embodying all the other equivalences of this concept are given. The comparison will illuminate the structure of this general class of processes. However, it is interesting and useful to obtain a characterization of V-boundedness for a general Banach space, and then specialize the result for the harmonizable case.

In this context let us say that $X : G \to \mathscr{X}$, a Banach space, is a *generalized (or vector) Fourier transform* if G is an LCA group, and if there is a vector measure $\nu : \mathscr{B}(\hat{G}) \to \mathscr{X}$ such that $X(g) = \int_{\hat{G}} \langle g, s \rangle \, \nu(ds), g \in G$. In [33], Phillips has extended the fundamental scalar result of Bochner's V-boundedness to certain Banach spaces with $G = \mathbf{R}$. Later but apparently independently, the LCA group case was given by Kluvanek in ([21], p. 269). In the present terminology this can be stated as follows:

Proposition 7.1. *Let G be an LCA group and $\mathscr{X}$ a Banach space. Then a mapping $X : G \to \mathscr{X}$ is a generalized Fourier transform of a regular vector measure $\nu : \mathscr{B}(\hat{G}) \to \mathscr{X}$ (i.e., for given $\varepsilon > 0$ and $E \in \mathscr{B}(\hat{G})$, there exist an open set O and a compact set C with $O \supset E \supset C$ such that for each $F \subset O - C$, $F \in \mathscr{B}(\hat{G})$ one has $\| \nu(F) \| < \varepsilon$) iff X is weakly continuous and V-bounded (in the sense of Definition 4.1).*

On the other hand, when $\mathscr{X} = \mathbf{C}$, a different kind of characterization was given by Helson [12]. A vector extension of this is used for the weak harmonizability problem, and will be presented here. Let $L^k(G)$ be the Lebesgue space, $k \geqslant 1$, on G relative to a Haar measure, denoted dg. Similarly $L^k(\hat{G})$ is defined on the dual group $\hat{G}$, and $L^k_{\mathscr{X}}(\hat{G})$ for $\mathscr{X}$-valued function space. Let

$$\hat{L}^1(G) = \{\hat{f} : \hat{f}(t) = \int_G \langle t, s \rangle f(s) ds, \qquad f \in L^1(G)\} \subset C_0(\hat{G}),$$

a similar definition for $\hat{L}^1_{\mathscr{X}}(G)$, the integrals in the latter being in the sense of Bochner, and $\hat{\mathscr{M}}_{\mathscr{X}}(G)$ $(\supset \hat{L}^1_{\mathscr{X}}(G))$, $\mathscr{M}_{\mathscr{X}}(G)$ being the space of vector measures on G into $\mathscr{X}$ with semivariation norm.

The following result contains the desired extension:

Theorem 7.2. *Let G be an LCA group, $\mathscr{X}$ a reflexive separable Banach space, and $X : G \to \mathscr{X}$ be bounded. Then X is a generalized Fourier transform of a vector measure ν on $\hat{G}$ into $\mathscr{X}$ iff for each $p \in \hat{L}^1(\hat{G})$ the mapping Y_p*

$= (Xp) : G \to \mathscr{X}$ *is in* $\hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$, *i.e., iff* Y_p *is the Fourier transform of a vector measure on* $\hat{G}$ *into* $\mathscr{X}$.

Proof. Suppose X is a generalized Fourier transform of ν on $\hat{G}$ to $\mathscr{X}$, so that

$$X(g) = \int_{\hat{G}} \langle g, s\rangle \nu(ds), \qquad g \in G. \tag{77}$$

By hypothesis $p \in \hat{L}^1(\hat{G})$ so that $p = \hat{f}$ for a unique $f \in L^1(\hat{G})$. Hence $X(g)p(g)$ is well defined, and if $l \in \mathscr{X}^*$, then by the scalar theory one has

$$l(X(g)\cdot p(g)) = p(g)l(X(g)) = \int_{\hat{G}} \langle g, s\rangle f(s)ds \int_{\hat{G}} \langle g, t\rangle\, l\circ\nu(dt)$$

$$= \int_{\hat{G}} \langle g, s\rangle (l\circ\nu * f)ds, \text{ since } (l\circ\nu * f)\hat{} = (l\circ\nu)\hat{}\cdot \hat{f} \text{ the “}*\text{” denoting}$$

convolution,

$$= \int_{\hat{G}} \langle g, s\rangle k_l(s)ds, \tag{78}$$

where $k_l = l\circ\nu * f \in L^1(G)$ by the classical theory (cf. [24], p. 122 and p. 142). Also $k_{(\cdot)}(s) : \mathscr{X}^* \to \mathbf{C}$ is additive, and

$$\| k_l(\cdot) \|_1 \leqslant \| f \|_1 \cdot \| l \| \cdot \| \nu \| (\hat{G}) \to 0$$

as $l \to 0$ in $\mathscr{X}^*$. Hence $k_l(s) \to 0$ as $l \to 0$ for $a\cdot a\cdot(s)$, so that $k_l(s) = \tilde{k}(s)(l)$ for a $\tilde{k}(s) \in \mathscr{X}^{**} = \mathscr{X}$ by reflexivity, and for $a\cdot a\cdot(s)$. Thus $\tilde{k}(\cdot)$ is Pettis integrable on $\hat{G}$, and the mapping $Z_p(\cdot) : A \mapsto \int_A \tilde{k}(s)ds$, defines a σ-additive bounded set function into $\mathscr{X}$, a vector measure, by known results in Abstract Analysis. Consequently,

$$l(X(g))\cdot p(g) = \int_{\hat{G}} \langle g, s\rangle\, l\circ Z_p(ds)$$

$$= l\left(\int_{\hat{G}} \langle g, s\rangle Z_p(ds)\right), \qquad l \in \mathscr{X}^*. \tag{79}$$

Since Z_p is a vector measure, $\| Z_p \| (\hat{G}) < \infty$, and $l \in \mathscr{X}^*$ is arbitrary, one has

$$Y_p(g) = (X\cdot p)(g) = \int_{\hat{G}} \langle g, s\rangle Z_p(ds) \in \mathscr{X}, \qquad g \in G, \tag{80}$$

to be well-defined. Also

$$| Y_p(g) |_{\mathscr{X}} = | p(g) |\, | X(g) |_{\mathscr{X}} \leqslant \| f \|_1 \cdot | X(g) |_{\mathscr{X}}$$

so that $\| Y_p \|_\infty \leqslant \| f \|_1 \| X \|_\infty < \infty$ and by (80) Y_p is the Fourier transform of the vector measure Z_p on $\hat{G}$ into $\mathscr{X}$. Hence $Y_p \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$. This proves the direct part. The converse implication is more involved.

Thus, for the converse, let $Xp = Y_p \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$ for each $p \in \hat{L}^1(\hat{G})$. Since $\mathscr{X}$ is reflexive, by Proposition 7.1, it is enough to establish that the (weakly continuous) X is V-bounded (cf. Definition 4.1). This is accomplished in two stages.

Let us first define an operator $\tau : L^1(\hat{G}) \to L^1_{\mathscr{X}}(\hat{G})$ by the equation:

$$(\tau f)\hat{} = p \cdot X = Y_p, \quad p = \hat{f}, \quad f \in L^1(\hat{G}). \tag{81}$$

Then $(\tau f)\hat{} \in \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$ by hypothesis for each $f \in L^1(\hat{G})$. Clearly τ is linear. It is also bounded. To see this, let us show that it is closed so that the desired assertion follows by the closed graph theorem. So let $f_n, f \in L^1(\hat{G})$, $f_n \to f$ in norm, and $h_n = \tau f_n \to h$ in $\hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$. Then (cf. [21], p. 268)

$$\| \hat{f}_n - \hat{f} \|_u \leqslant \| f_n - f \|_1 \to 0 \text{ and } \| \hat{h}_n - \hat{h} \|_u \leqslant \| h_n - h \|_1 \to 0,$$

as $n \to \infty$. But then

$$\hat{h}_n = (\tau f_n)\hat{} = X \cdot \hat{f}_n \to \hat{h} \text{ and } \hat{f}_n \to \hat{f} \text{ uniformly.}$$

$$\begin{aligned} \| X\hat{f} - \hat{h} \| (s) &\leqslant \| X(\hat{f}_n - \hat{f}) \| (s) + \| X\hat{f}_n - \hat{h} \| (s) \\ &\leqslant \| X(s) \| \, | \hat{f}_n - \hat{f} | (s) + \| \hat{h}_n - \hat{h} \| (s) \to 0, \text{ as } n \to \infty, s \in \hat{G}. \end{aligned}$$

Hence $X\hat{f} = \hat{h} = (\tau f)\hat{}$, and $\tau f = h$ (by uniqueness). So τ is closed.

Next let us verify the key property of V-boundedness for X. Since Y_p is continuous for each $p \in \hat{L}^1(\hat{G})$, it follows that X is weakly continuous. Let $h \in L^1(G)$. Consider the operator $T : L^1(G) \to \mathscr{X}$ defined by

$$T(h) = \tilde{T}(\hat{h}) = \int_G X(g)h(g)dg, \qquad \| \hat{h} \|_u \leqslant 1. \tag{82}$$

Since the correspondence $h \leftrightarrow \hat{h}$ is $1-1$, $\tilde{T}$ is well defined on $\hat{L}^1(G)$, and it is to be shown that $\tilde{T} : \hat{L}^1(G) \to \mathscr{X}$ is bounded when the former is endowed with the uniform norm. [Note: h below is different from h above!]

Let $h \in L^1(G)$ be arbitrarily fixed and $\{e_\alpha, \alpha \in I\} \subset L^1(\hat{G})$ be an approximate unit (cf. [24], p. 124) so that $\| e_\alpha \|_1 = 1$, $e_\alpha \geqslant 0$ and $\| (e_\alpha - e_\beta) * h \|_1 \to 0$ as $\alpha, \beta \nearrow$ "∞". Now $(\tau e_\alpha)\hat{} = X \cdot \hat{e}_\alpha$ $(= X_\alpha$, say). The hypothesis implies $X_\alpha \in \hat{\mathscr{M}}_{\mathscr{X}}(G)$, $\alpha \in I$, and

$$\begin{aligned} \| (X_\alpha - X_\beta)\hat{h} \| (t) &= \| \tau(e_\alpha - e_\beta)\hat{}\, \hat{h} \| (t) \leqslant \| \tau((e_\alpha - e_\beta) * h) \|_1 \\ &\leqslant \| \tau \| \, \| (e_\alpha - e_\beta) * h \|_1 \to 0, \qquad t \in G, \end{aligned} \tag{83}$$

since τ was shown to be bounded. Thus $X_\alpha \to X$ uniformly. Since $\hat{h} \in \hat{L}^1(G) \subset C_0(\hat{G})$ and is uniformly dense in the latter, it follows that $\| X_\alpha \|_u \leqslant C < \infty$, and the operator T_α defined below is bounded uniformly in α:

$$T_\alpha(h) = \int_G X_\alpha(t)h(t)dt, \qquad h \in L^1(G). \tag{84}$$

But X is the uniform limit of X_α's so it is also bounded, and hence T of (82) is bounded. Moreover, for $f \in C_{00}(G)$ $(\subset C_0(G))$ of compact supports,

$$\| T(hf) - T_\alpha(hf) \|_{\mathscr{X}} = \| \int_G (X - X_\alpha)(t) h(t) f(t) dt \|_{\mathscr{X}}$$
$$\leqslant \| (X - X_\alpha) f \|_u \cdot \int_G | h(t) | dt \to 0,$$

by (83), as $\alpha \nearrow$ "∞". Hence $\| T_\alpha(hf) \|_{\mathscr{X}} \to \| T(hf) \|_{\mathscr{X}}$, and

$$T(hf) = \lim_\alpha \int_G X_\alpha(t) h(t) f(t) dt \qquad (= \int_G X(t) h(t) f(t) dt). \tag{85}$$

If $l \in \mathscr{X}^*$, (85) implies, with $h \in L^1(G) \cap C_{00}(G) = C_{00}(G)$,

$$(l \circ T)(h) = \lim_\alpha \int_G l(X_\alpha(t)) h(t) dt \qquad (= \lim_\alpha (l \circ T_\alpha)(h)).$$

On the other hand,

$$\begin{aligned}(l \circ T_\alpha) &= \int_G l(X_\alpha(t)) h(t) dt = \int_G (l((\tau e_\alpha)\hat{}\,) h)(t) dt \\ &= \int_G h(t) \cdot l(\int_{\hat{G}} \langle g, t \rangle (\tau e_\alpha)(g) dg) dt \\ &= \int_{\hat{G}} \int_G h(t) \langle g, t \rangle\, l(\tau e_\alpha)(g) dt dg, \text{ by Fubini's theorem,} \\ &= \int_{\hat{G}} l(\tau e_\alpha)(g) \hat{h}(g) dg, \text{ by Fubini again.}\end{aligned}$$

Thus for all $h \in C_{00}(g) \subset L^1(G)$,

$$| (l \circ T_\alpha)(h) | \leqslant \| \hat{h} \|_u \| l(\tau e_\alpha) \|_1 \leqslant \| \hat{h} \|_u \cdot \| l \| \| \tau \| \cdot \| e_\alpha \|_1 . \tag{86}$$

Taking suprema on $\| l \| \leqslant 1$, and noting that $\| e_\alpha \|_1 = 1$, (86) implies

$$\| T_\alpha(h) \| \leqslant \| \hat{h} \|_u \| \tau \| . \tag{87}$$

Thus (85) and (87) yield that $\| T(h) \| \leqslant c \| \hat{h} \|_u$ with $c = \| \tau \| < \infty$. Since $C_{00}(G)$ is dense in $L^1(G)$, the same holds for all $h \in L^1(G)$. So X is V-bounded. Since $\mathscr{X}$ is reflexive, Proposition 7.1 now applies and yields (77) for a unique vector measure ν on $\hat{G}$ into $\mathscr{X}$. This completes the proof.

Remark. The necessity proof also holds (and thus the theorem) if $\hat{L}^1(\hat{G})$ is replaced by

$$\hat{\mathscr{M}}(\hat{G}) = \{\hat{\mu} : \hat{\mu}(t) = \int_{\hat{G}} \langle g, t \rangle \mu(dg), \mu \in \mathscr{M}(\hat{G}), t \in G\},$$

where $\mathscr{M}(\hat{G})$ is the space of regular signed Borel measures on $\hat{G}$. In fact let $Y_p = \hat{\mu} X$, where $p = \hat{\mu}$ (is a function), so that for $l \in \mathscr{X}^*$,

$$\begin{aligned}l(Y_p(t)) &= \int_{\hat{G}} \langle g, t \rangle \mu(dg) \cdot \int_{\hat{G}} \langle s, t \rangle\, l \circ Z(ds) = (\hat{\mu} \cdot \widehat{l \circ Z})(t) \\ &= (\mu * l \circ Z)\hat{}\,(t) = l(\int_{\hat{G}} \langle g, t \rangle (\mu * Z)(dg)),\end{aligned}$$

using the convolution products appropriately (cf., e.g. [21]). Hence $\mu * Z$ is a vector measure on $\hat{G}$ and

$$\| \mu * Z \| (\hat{G}) \leqslant | \mu | (\hat{G}) \| Z \| (\hat{G}) < \infty .$$

Thus Y_p is a Fourier transform of $\mu * Z$. Identifying $L^1(\hat{G}) \hookrightarrow \mathscr{M}(\hat{G})$ as $\tilde{\mu} : A \mapsto \int_A f(t)dt$, the sufficiency proof of theorem and the above lines show that $\hat{L}^1(\hat{G})$ can be replaced by $\hat{\mathscr{M}}(\hat{G})$ every where in that result.

Taking $\mathscr{X} = L_0^2(P)$ so that V-boundedness is the same as weak harmonizability, the above theorem together with Theorems 3.3, 6.3, yield the following two summary statements on characterizations of weakly harmonizable random fields.

THEOREM 7.3. *Let* G *be an LCA group,* $\mathscr{X} = L_0^2(P)$ *be separable and* $X : G \to \mathscr{X}$ *be a weakly continuous mapping. Then the following statements are equivalent:*

(i) X *is weakly harmonizable.*

(ii) X *is V-bounded.*

(iii) X *is the Fourier transform of a regular vector measure on* $\hat{G}$ *into* $\mathscr{X}$.

(iv) *for each* $p \in \hat{L}^1(\hat{G})$, *the process* $Y_p = Xp : G \to L_0^2(P)$ *is weakly harmonizable and bounded.*

Furthermore, the following implies (i)-(iv):

(v) *if* $\mathscr{H} = \overline{sp}\{X(g), g \in G\} \subset \mathscr{X}$, *then there exists a weakly continuous contractive positive type family of operators* $\{T(g), g \in G\} \subset B(\mathscr{H})$ *such that* $T(0) =$ *identity, and* $X(g) = T(g)X(0), g \in G$.

In order to present a similar description of the dilation results, these individual statements should be couched in terms of classes. Let us therefore define various classes with $\mathscr{X} = L_0^2(P)$, separable.

$\mathscr{V}$ = the set of weakly continuous V-bounded random fields on G.

$\mathscr{W}$ = the set of weakly harmonizable random fields on G.

$\mathscr{F}$ = the class of all random fields which are Fourier transforms of regular vector measures on $\hat{G} \to \mathscr{X}$.

$\mathscr{M}$ = the module over $\hat{L}^1(\hat{G})$ of all functions on $G \to \mathscr{X}$ which belong to $\hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})$, i.e., $\mathscr{M} = \{X : G \to \mathscr{X} \mid X \cdot \hat{L}^1(\hat{G}) \subset \hat{\mathscr{M}}_{\mathscr{X}}(\hat{G})\}$.

$\mathscr{P}$ = the class of all random fields on $G \to \mathscr{X}$ which are projections of stationary fields on $G \to \mathscr{K}$, where $\mathscr{K} \supset \mathscr{X}$ is some extension (or super) Hilbert space of $\mathscr{X}$.

Then the following result obtains:

THEOREM 7.4. *With the above notation, one has* $\mathscr{F} = \mathscr{M} = \mathscr{P} = \mathscr{V} = \mathscr{W}$

These two theorems embody essentially all the known as well as new results on the structure of weakly harmonizable processes or fields. Some applications and extensions will be indicated in the rest of the paper.

8. Associated spectra and consequences

For a large class of nonstationary processes, including the (strongly) harmonizable ones, it is possible to associate a (nonnegative) spectral measure and study some of the key properties of the process through it. One such reasonably large class, isolated by Kampé de Fériet and Frankiel ([15]-[17]), called *class* (KF) in [35], is the desired family. This was also considered under the name "asymptotic stationarity" by E. Parzen [32] (cf. also [14] with the same name for a subclass), and by Rozanov ([40], p. 283) without a name. All these authors, motivated by applications, arrived at the concept independently. But it is Kampé de Fériet and Frankiel who emphasized the importance of this class and made a deep study. This was further analyzed in [35].

If $X : \mathbf{R} \to L_0^2(P)$ is a process with covariance $k(s, t) = E(X(s)\overline{X}(t))$, then it is said to be of *class* (KF), after its authors [15]-[17], provided the following limit exists for all $h \in \mathbf{R}$:

$$r(h) = \lim_{T\to\infty} \frac{1}{T}\int_0^{T-|h|} k(s, s+|h|)ds = \lim_{T\to\infty} r_T(h)\,. \tag{88}$$

It is easy to see that $r_T(\cdot)$, hence $r(\cdot)$, is a positive definite function when $X(\cdot)$ is a measurable process. If $X(\cdot)$ is continuous in mean square, the latter is implied. It is clear that stationary processes are in class (KF). By the classical theorem of Bochner (or its modified form by F. Riesz) there is a unique bounded increasing function $F : \mathbf{R} \to \mathbf{R}^+$ such that

$$r(h) = \int_{\mathbf{R}} e^{ith}\, F(dt), \qquad a\cdot a\cdot (h)\cdot(\text{Leb})\,. \tag{89}$$

This F is termed the *associated spectral function* of the process X. Every strongly harmonizable process is of class (KF). This is not obvious, but was shown in ([40], p. 283), and in [35] as a consequence of the membership of a more general class called *almost (strongly) harmonizable.* The latter is not necessarily V-bounded and so the weakly harmonizable class is not included. (Almost harmonizable need not imply weakly harmonizable.) Since the bimeasure of (30) is not necessarily of bounded variation, the elementary proof of [40] given for the strongly harmonizable process does not extend. Perhaps for this reason, Rozanov (cf. [40], footnote on p. 283) felt that the weakly harmonizable processes may not be in class (KF). However, a positive solution can be obtained as follows:

THEOREM 8.1. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be weakly harmonizable. Then* $X \in$ class (KF), *so that it has a well defined associated spectral function.*

Proof: Since X is weakly harmonizable,

$$X(t) = \int_{\mathbf{R}} e^{it\lambda} Z(d\lambda), \qquad t \in \mathbf{R},$$

for a stochastic measure Z on $\mathbf{R}$ into $L_0^2(P)$, and if

$$F(A, B) = (Z(A), Z(B)),$$

then $F : \mathscr{B} \times \mathscr{B} \to \mathbf{C}$ is a bounded bimeasure. Considering (88) for $h \geqslant 0$ (the case $h < 0$ being similar), one has with $k(s, t) = E(X(s)\bar{X}(t))$

$$r_T(h) = \frac{T-h}{T} \cdot \frac{1}{T-h} \int_0^{T-h} k(s, s+h) ds .$$

To show that $\lim_{T \to \infty} r_T(h)$ exists it suffices to consider

$$\frac{1}{T}\int_0^T k(s, s+h)ds = \frac{1}{T}\int_0^T E(X(s) \cdot \bar{X}(s+h))ds$$

$$= E\left(\frac{1}{T}\int_0^T ds \int_{\mathbf{R}} e^{is\lambda} Z(d\lambda) \int_{\mathbf{R}} e^{-i(s+h)\lambda'} Z(d\lambda')\right) \tag{90}$$

and show that the right side has a limit as $T \to \infty$. Let $\mathscr{X} = \mathscr{Y} = L_0^2(P)$, and $\mathscr{Z} = L^1(P)$. Since $Z : \mathscr{B} \to \mathscr{X}$, $\tilde{Z} = Z : \mathscr{B} \to \mathscr{Y}$ are stochastic measures, one can define a product measure on $\mathbf{R} \times \mathbf{R}$ into $\mathscr{Z}$, using the bilinear mapping $(x, y) \to xy$, of $\mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$, as the pointwise product which is continuous in their respective norm topologies. Under these conditions and identifications, the product measure $Z \otimes \tilde{Z} : \mathscr{B} \times \mathscr{B} \to \mathscr{Z}$ is defined and satisfies (D-S integrals):

$$\int_{\mathbf{R}\times\mathbf{R}} f(s, t)\,(Z \otimes \tilde{Z})\,(ds, dt) = \int_{\mathbf{R}} Z(ds) \int_{\mathbf{R}} f(s, t)\tilde{Z}(dt)$$

$$= \int_{\mathbf{R}} \tilde{Z}(dt) \int_{\mathbf{R}} f(s, t) Z(ds) , \tag{91}$$

for all $f \in C_b(\mathbf{R} \times \mathbf{R})$, by ([5], p. 388). In most of the work on product vector measures, Dinculeanu assumes that they are "dominated". However, as shown in a separate Remark (cf. [5], p. 388; cf. also [7], Cor. 3), such a product measure as in (91) is well defined even though it need not be "dominated". It has finite semivariation: indeed,

$$\| Z \otimes \tilde{Z} \| (\mathbf{R} \times \mathbf{R}) \leqslant \| Z \| (\mathbf{R}) \cdot \| \tilde{Z} \| (\mathbf{R}) = (\| Z \| (\mathbf{R}))^2 < \infty ,$$

so that $Z \otimes Z$ is again a stochastic measure. Letting

$$f_{s,h}(\lambda, \lambda') = e^{is\lambda} \cdot e^{-i(s+h)\lambda'} ,$$

so $f_{s,h} \in C_b(\mathbf{R} \times \mathbf{R})$, (91) becomes:

$$\int_{\mathbf{R}} e^{is\lambda} Z(d\lambda) \int_{\mathbf{R}} e^{-i(s+h)\lambda'} Z(d\lambda') = \int_{\mathbf{R} \times \mathbf{R}} e^{is(\lambda-\lambda') - ih\lambda'} Z \otimes Z(d\lambda, d\lambda') , \tag{92}$$

the right side being an element of $L^1(P)$. Applying the same calculation to the measures $Z \otimes Z : \mathscr{B}(\mathbf{R} \times \mathbf{R}) \to \mathscr{X}$ and $\mu : \mathscr{B}([0, T]) \to \mathbf{R}^+$ (μ is Lebesgue measure), with $(x, a) \to ax$ being the mapping of $\mathscr{X} \times \mathbf{R} \to \mathscr{X}$, one can define

$$\mu \otimes (Z \otimes Z) : \mathscr{B}(0, T) \times \mathscr{B}(\mathbf{R} \times \mathbf{R}) \to \mathscr{X}$$

and, with $\underline{\lambda}$ for the pair (λ, λ'),

$$\int_0^T \mu(dt) \int_{\mathbf{R} \times \mathbf{R}} f(t, \underline{\lambda}) Z \otimes Z(d\underline{\lambda}) = \int_{\mathbf{R} \times \mathbf{R}} Z \otimes Z(d\underline{\lambda}) \int_0^T f(t, \underline{\lambda}) \mu(dt) . \tag{93}$$

Writing $\mu(dt)$ as dt, (90)-(93) yield:

$$E\left(\frac{1}{T}\int_0^T ds \int_{\mathbf{R} \times \mathbf{R}} e^{is(\lambda-\lambda') - ih\lambda'} Z \otimes Z(d\lambda, d\lambda')\right)$$

$$= E\left(\int_{\mathbf{R} \times \mathbf{R}} e^{-ih\lambda'} Z \otimes Z(d\lambda, d\lambda') \cdot \frac{1}{T}\int_0^T e^{is(\lambda-\lambda')} ds\right)$$

$$= E\left(\int_{\mathbf{R} \times \mathbf{R}} e^{-ih\lambda'} \left[\frac{e^{iT(\lambda-\lambda')} - 1}{iT(\lambda - \lambda')} \chi_{[\lambda \neq \lambda']} + \delta_{\lambda\lambda'}\right] Z \otimes Z(d\lambda, d\lambda')\right) \tag{94}$$

But the quantity inside the expectation symbol E is bounded for all T, and since the dominated convergence is valid for the D-S integral ([8], IV.10.10), constants being $Z \otimes Z$-integrable, one can pass the limit as $T \to \infty$ under the expectation as well as the D-S integral in (94). Hence

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T k(s, s+h)ds = E\left(\int_{\mathbf{R}\times\mathbf{R}} e^{-ih\lambda'}\,\delta_{\lambda\lambda'}\, Z\otimes Z(d\lambda, d\lambda')\right)$$
$$= \int_{\mathbf{R}\times\mathbf{R}} e^{-ih\lambda'}\,\delta_{\lambda\lambda'}\, E(Z\otimes Z(d\lambda, d\lambda'))$$
$$= \int_{[\lambda=\lambda']} e^{-ih\lambda}\, F(d\lambda, d\lambda'),$$

where F is the bimeasure of Z. Hence $\lim_{T\to\infty} r_T(h) = r(h)$ exists and $r(h) = \int_{\mathbf{R}} e^{-ih\lambda}\, G(d\lambda)$, where $G : A \mapsto \int_{\pi^{-1}(A)} \delta_{\lambda\lambda'}\, F(d\lambda, d\lambda')$, $A \in \mathscr{B}$, is a positive finite measure which therefore is the associated spectral measure of $X \in$ class (KF). (Here $\pi : \mathbf{R}^2 \to \mathbf{R}$ is the coordinate projection.) This completes the proof.

The above result implies that several other considerations of [40] hold for weakly harmonizable processes.

As another application of the present work, especially as a consequence of Theorem 6.1, the following precise version of a result stated in ([40], Thm. 3.2) will be deduced from the corresponding classical stationary case.

THEOREM 8.2. *Let* $X : \mathbf{R} \to L_0^2(P)$ *be a weakly harmonizable process with* $Z : \mathscr{B} \to L_0^2(P)$ *as its representing stochastic measure. Then for any* $-\infty < \lambda_1 < \lambda_2 < \infty$, *writing* $Z(\lambda)$ *for* $Z((-\infty, \lambda))$, *one has*

$$\underset{T\to\infty}{\mathrm{l\cdot i\cdot m}} \int_{-T}^{T} \frac{e^{-it\lambda_2} - e^{-it\lambda_1}}{-it} X(t)dt$$
$$= \frac{Z(\lambda_2+) + Z(\lambda_2-)}{2} - \frac{Z(\lambda_1+) + Z(\lambda_1-)}{2} \tag{95}$$

where $\mathrm{l\cdot i\cdot m}$ *is the* $L^2(P)$*-limit. Further the covariance bimeasure* F *of* Z *can be obtained for intervals* $A = (\lambda_1, \lambda_2), B = (\lambda_1', \lambda_2')$ *as:*

$$\lim_{0\leqslant T_1, T_2\to\infty} \int_{-T_1}^{T_1}\int_{-T_2}^{T_2} \frac{e^{-i\lambda_2 s} - e^{-i\lambda_1 s}}{-is}$$
$$\cdot \frac{e^{i\lambda_2' t} - e^{-i\lambda_1' t}}{it} r(s, t)dsdt = F(A, B), \tag{96}$$

provided A, B *are continuity intervals of* F *in the sense that*

$$F((-\infty, \lambda_j\pm), (-\infty, \lambda_j'\pm)) = F((-\infty, \lambda_j), (-\infty, \lambda_j')), j = 1, 2,$$

and where $r(\cdot,\cdot)$ *is the covariance function of the X-process. In particular, if the mapping* $S:\mathbf{R}\to\mathbf{C}$ *is continuous,* $\frac{1}{T}\int_0^T S(t)dt \to a_0$ *exists as* $T\to\infty$, *and* $\lim_{|s|+|t|\to\infty} r(s,t) = 0$, *then for the observed process* $\tilde{Y}(t) = S(t) + X(t)$, *so that* $S(\cdot)$ *is the nonstochastic "signal" and* $X(\cdot)$ *is the weakly harmonizable "noise", the estimator*

$$\hat{S}_T = \frac{1}{T}\int_0^T \tilde{Y}(t)dt \to a_0$$

in $L_0^2(P)$ *(i.e.,* $E(|\hat{S}_T - a_0|^2)\to 0$*) as* $T\to\infty$. *Thus* $\hat{S}_T$ *is a consistent estimator of* a_0, *and in other terms, both X- and* $\tilde{Y}$*-processes obey the law of large numbers.*

Proof: The key idea of the proof is to reduce the result to the classical stationary case through an application of the dilation theorem. Thus by Theorem 6.1, there exists a probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, with $L_0^2(\tilde{P}) \supset L_0^2(P)$, and a stationary process $Y:\mathbf{R}\to L_0^2(\tilde{P})$ such that $X(t) = QY(t)$, $t\in\mathbf{R}$ where Q is the orthogonal projection on $L_0^2(\tilde{P})$ with range $L_0^2(P)$. There is an orthogonally scattered stochastic measure $\tilde{Z}:\mathcal{B}\to L_0^2(\tilde{P})$ such that

$$Y(t) = \int_{\mathbf{R}} e^{it\lambda}\tilde{Z}(d\lambda), \qquad t\in\mathbf{R}, \tag{97}$$

and $Z(A) = Q\tilde{Z}(A)$, $A\in\mathcal{B}$, with $Z:\mathcal{B}\to L_0^2(P)$ representing the given X-process. Since Q is bounded, as is well-known, it commutes with the integral as well as the l·i·m. Thus (95) is true for the Y-process with $\tilde{Z}$ in place of Z there (cf., e.g., [6], p. 527). Then the result follows on applying Q to both sides and interchanging the l·i·m as well as the integral with Q, which is legitimate. Hence (95) is true as stated.

Next consider the left hand side (LHS) of (96). With (95) it can be expressed as:

$$\text{LHS} = \lim_{T_1,T_2\to\infty} E\left(\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\left[\frac{e^{-is\lambda_2}-e^{-is\lambda_1}}{-is}X(s)\right]\cdot\left[\frac{e^{-it\lambda_2'}-e^{-it\lambda_1'}}{-it}X(t)\right]^{-} dsdt\right)$$

$$= \lim_{T_1,T_2\to\infty} E\left[\left(\int_{-T_1}^{T_1}\frac{e^{-is\lambda_2}-e^{-is\lambda_1}}{-is}X(s)ds\right)\left(\int_{-T_2}^{T_2}\frac{e^{-it\lambda_2'}-e^{-it\lambda_1'}}{-it}X(t)\right)^{-} dt\right]$$

$$= E\left[\left(\frac{Z(\lambda_2+)+Z(\lambda_2-)}{2}-\frac{Z(\lambda_1+)+Z(\lambda_1-)}{2}\right)\right.$$
$$\left.\cdot\left(\frac{Z(\lambda_2'+)+Z(\lambda_2'-)}{2}-\frac{Z(\lambda_1'+)+Z(\lambda_1'-)}{2}\right)^-\right]$$
$$= F(A,B),$$

by the continuity hypothesis on F, after expanding and taking expectations. This proves (96).

Finally, if $\tilde{Y}(t) = S(t) + X(t)$, $t \in \mathbf{R}$, let

$$a_T = E(\hat{S}_T) = \frac{1}{T}\int_0^T S(t)dt .$$

Noting that $\tilde{Y} \in$ class (KF) since X does (cf. Thm. 8.1), and $a_T \to a_0$, by hypothesis, as $T \to \infty$,

$$E(|\hat{S}_T - a_0|^2) = \frac{2}{T^2}\int_0^T\int_0^T r(s,t)dsdt + 2\,|\,a_T - a_0\,|^2$$
$$= \frac{1}{2T}\int_{-T}^T r_T(h)dh + 2\,|\,a_T - a_0\,|^2 , \qquad (98)$$

where, as usual, $r_T(\cdot)$ is given by (88). Since $r_T(h) \to r(h)$ due to the fact that $\tilde{Y} \in$ class (KF), and since $r(s, s+h) \to 0$ as $|\,s\,| \to \infty$ by hypothesis together with the fact that

$$|\,r(s,t\,|\leqslant (r(s,s)r(t,t))^{1/2} \leqslant M^2 < \infty$$

where $\|\,X(t)\,\| \leqslant M < \infty$ (X being V-bounded), one can invoke a classical result on Cesarò summability (cf., [8], IV.13.83(a)). By this result $r(h) = 0$ for each $h \in \mathbf{R}$. Actually $r_T(h) \to r(h)$ $(=0)$, uniformly in h on compact sets of $\mathbf{R}$. It follows that $E(|\hat{S}_T - a_0|^2) \to 0$, and this completes the proof of the theorem.

Remark. The key reduction for (95), which is used in (96), is possible in the above proof since the linear operation of Q on the process mattered. However, for Theorem 8.1, the dilation result itself is not immediately applicable since the problem there is nonlinear, and one had to use alternate arguments as was done there. Also since Fubini's theorem is not available for the MT-integral (cf. [27], §8), a special computation has to be used for this special case. Thus the point of the general theory here is to clarify the structure of these processes, and a reduction to the stationary case is not always possible.

9. Multivariate extension and related problems

Here a multidimensional extension of weakly harmonizable processes and the filtering problem on them will be briefly discussed. Even though some results have direct k-dimensional analogs ($k \geqslant 2$), there are some new and non-trivial problems in this case for a successful application of the theory. The infinite dimensional case will not be considered here since the key finite dimensional problems are not well-understood and resolved.

Let $L_0^2(P, \mathbf{C}^k)$ $(= L_0^2(\Omega, \Sigma, P; \mathbf{C}^k))$ be the space of equivalence classes of measurable functions $f : \Omega \to \mathbf{C}^k$, the complex k-space, such that (i) $|f|^2 = \sum_{i=1}^{k} |f_i|^2$ is P-integrable, and (ii) $E(f) = \int_\Omega f(\omega)P(d\omega) = 0$, or equivalently,

$$E(f_i) = \int_\Omega f_i(\omega)P(d\omega) = 0, \qquad i = 1, \ldots, k\,,$$

where $f = (f_1, \ldots, f_k)$, $|f|$ is the Euclidean norm of f in $\mathbf{C}^k$, and (Ω, Σ, P) is a probability space. If $f, g \in L_0^2(P, \mathbf{C}^k)$, define $\| f \|_2^2 = (f, f)$ where the inner product is given by

$$(f, g) = \int_\Omega (f(\omega), g(\omega))P(d\omega) = \sum_{i=1}^{k} \int_\Omega f_i(\omega)\bar{g}_i(\omega)P(d\omega)\,.$$

Then $\mathscr{X} = L_0^2(P, \mathbf{C}^k)$ becomes a Hilbert space of k-vectors with zero means. If $k = 1$, one has the space considered in the preceding sections $(\mathscr{H} = L_0^2(P, \mathbf{C}))$.

Definition 9.1. Let G be an LCA group. Then a mapping $X : G \to \mathscr{X}$ is a *weakly* or *strongly harmonizable vector* (or k-dimensional) *random field* (or process) if for each $a = (a_1, \ldots, a_k) \in \mathbf{C}^k$, the mapping

$$Y_a = a \cdot X\Big(= \sum_{i=1}^{k} a_i X_i\Big) : G \to \mathscr{H}$$

is a (scalar) weakly or strongly harmonizable random field (or process).

Similarly a vector stationary, Karhunen, or class (C), processes are defined by reducing to the scalar cases.

It is immediate from this definition that the component processes are also harmonizably or stationarily etc. correlated according to the class they belong. Thus if r_a is the covariance function of the Y_a-process and R is the covariance matrix of the X-process, so that $r_a(g, h) = E(Y_a(g)\overline{Y_a(h)})$ and $R(g, h) = E(X^t(g)\bar{X}(h))$ where $X(g)$ is a k-th order (row) vector and "t" denotes the usual transpose of a vector or matrix, then $r_a(g, h) = aR(g, h)a^t$. With this notation, the integral representations of multivariate weakly and strongly harmonizable random fields can be obtained, using Theorem 3.3, in a straightforward manner.

THEOREM 9.2. *Let G be an LCA group and $X : G \to \mathscr{X} = L_0^2(P, \mathbf{C}^k)$, a weakly continuous bounded mapping. Then X is weakly harmonizable iff there is a stochastic measure $\tilde{Z}$ on $\hat{G} \to \mathscr{X}$ (or if $\tilde{Z}(A) = (Z_1(A), \ldots, Z_k(A)), A \subset \hat{G}$ is a Borel set, then each Z_j is a stochastic measure on $\hat{G} \to \mathscr{H}, j = 1, \ldots, k$), such that*

$$X(g) = \int_{\hat{G}} \langle g, s\rangle \tilde{Z}(ds), \qquad g \in G, \tag{99}$$

where $\hat{G}$ is the dual group of G. The mapping X is strongly harmonizable if further the matrix $F = (F_{jl}, j, l = 1, \ldots, k)$ with

$$F(A, B) = \big((Z_j(A), Z_l(B)), j, l = 1, \ldots, k\big)$$

is of bounded variation on $\hat{G}$, or equivalently each F_{jl} is of bounded variation on $\hat{G}$. The covariance matrix R is representable as:

$$R(g, h) = \int_{\hat{G}} \int_{\hat{G}} \langle g, s\rangle \overline{\langle h, t\rangle} F(ds, dt), \qquad g, h \in G, \tag{100}$$

where the right side is the MT-integral, or the Lebesgue-Stieltjes integral, defined componentwise, accordingly as X is weakly or strongly harmonizable, and where F is a positive definite matrix of bounded bimeasures or of Lebesgue-Stieltjes measures. Conversely, if $R(\cdot, \cdot)$ is a positive definite matrix representable as (100), *then it is the covariance matrix of a multivariate harmonizable random field.*

Sketch of proof: Let $a \in \mathbf{C}^k$ be arbitrarily fixed and consider

$$Y_a = a \cdot X (= aX^t).$$

If X is weakly harmonizable, so that Y_a is also, then by Theorem 3.3 (trivially extended when $\mathbf{R}$ is replaced by G), there is a stochastic measure Z_a on $\hat{G} \to \mathscr{H}$ such that

$$Y_a(g) = \int_{\hat{G}} \langle g, s\rangle Z_a(ds), \qquad g \in G.$$

From this and the definition of Y_a, it follows that $Z_{(\cdot)}(A) : \mathbf{C}^k \to \mathscr{H}$ is linear and continuous. Hence there is a $\tilde{Z}$ on $\hat{G} \to \mathscr{X}^{**}$ ($= \mathscr{X}$, by reflexivity) such that $Z_a(A) = a \cdot \tilde{Z}(A)$, and it is evident that $\tilde{Z}$ is σ-additive on $\mathscr{B}(\hat{G}) \to \mathscr{X}$ so that it is a stochastic measure. It follows from the properties of the D-S integral that:

$$Y_a(g) = a \cdot X(g) = \int_{\hat{G}} \langle g, s\rangle a \cdot \tilde{Z}(ds) = a \cdot \int_{\hat{G}} \langle g, s\rangle Z(ds), \tag{101}$$

where the last integral defines an element of $\mathscr{X}$. This implies (99) since "a" is arbitrary and $X(\cdot)$ as well as the integral operator are continuous. The converse is similarly deduced from the corresponding part of Theorem 3.3.

If X is strongly harmonizable, then so is Y_a and if F_a is its covariance bimeasure, then $F_a = aF\bar{a}^t$ where

$$F(A, B) = \big((Z_j(A), Z_l(B)), j, l = 1, 2, ..., k\big).$$

Now taking special values for a in $\mathbf{C}^k$, it follows immediately that each component F_{jl} of F is of bounded variation. Interpreting (100) componentwise, the result follows from the scalar case. The same representation holds with the MT-integration for the weakly harmonizable case. All other statements, including the converses, are similarly deduced. This terminates the sketch.

By an analogous reasoning, it is evidently possible to assert that there is a 2-majorant of $\tilde{Z}$, and the X-process has a (vector) stationary dilation. These results are of real interest in the context of the important filtering problem which can be abstractly stated following Bochner [2].

If $X : G \to \mathscr{X}$ is a random field, a (not necessarily bounded) linear operator $\Lambda : \mathscr{X} \to \mathscr{X}$ is called a *filter* of X, if Λ commutes with the translation operator on X, i.e., if $(\tau_h X)(g) = X(hg)$, then $\tau_h(\Lambda X) = \Lambda(\tau_h X)$, where the domain

$$\text{dom}\,(\Lambda) \supset \{\tau_h X(g), g \in G, h \in G\}.$$

The problem is to find solutions X of the equation:

$$\Lambda X = Y(\in \mathscr{X}), \tag{102}$$

such that if Y is a given weakly or strongly harmonizable random field so must X be.

For the stationary case, a general concept of filter was discussed by Hannan [11]. If $k = 1$, $\Lambda = \sum_{i=1}^{m} a_i \Delta_i$ is a reverse shift operator with $G = \mathbf{R}$ (so $\Delta_i X(t) = X(t-i)$) and Y is stationary, then this problem was completely solved by Nagabhushanam [28], and by Kelsh [19] in the strongly harmonizable case. In both these studies, the conditions are on the measure function F of (33). If $k \geqslant 2$, under the usual assumptions on the random fields, the following new questions arise with (99) and (100). Frequently employed general forms of Λ include the constant coefficient difference, differential, or integral operators, or a mixture of these. For instance, if $\Lambda = \sum_{j=0}^{m} A_j D^j$, where the A_j are k-by-k constant matrices, and $D^j = \frac{d^j}{dt^j}$, $(G = \mathbf{R})$ then (102) takes the following form in order that it admit a (weakly) harmonizable solution for a harmonizable Y where $X^{(j)}$ denotes the mean-square j-th derivative (assumed to exist):

$$\int_{\mathbf{R}} e^{it\lambda} Z_y(d\lambda) = Y(t) = (\Lambda X)(t) = \sum_{j=0}^{m} A_j X^{(j)}(k-j)$$
$$= \sum_{j=0}^{m} A_j \int_{\mathbf{R}} e^{i(t-j)\lambda}(i\lambda)^j Z_x(d\lambda)$$
$$= \int_{\mathbf{R}} T(\lambda) \cdot e^{it\lambda} Z_x(d\lambda), \tag{103}$$

where $T(\lambda) = \sum_{j=0}^{m} A_j e^{-ij\lambda}(i\lambda)^j$, called the *generator* of Λ in [2], and Z_x, Z_y are the representing stochastic measures of X- and Y-processes. Clearly the existence of solutions of (102) depends on the coefficients A_j's determining the analytical properties of the generator $T(\cdot)$. If the process is strongly harmonizable then (103) implies (*-denoting conjugate transpose)

$$R_y(s, t) = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda - it\lambda'} F_y(d\lambda, d\lambda')$$
$$= \int_{\mathbf{R}} \int_{\mathbf{R}} e^{is\lambda} T(\lambda) F_x(d\lambda, d\lambda') \left(T(\lambda') e^{it\lambda'}\right)^*, \tag{104}$$

where F_x and F_y are the k-by-k matrix covariance bimeasures of X- and Y-processes. For a special class of strongly harmonizable k-vector processes, recently Kelsh [19] found sufficient conditions on the generator $T(\cdot)$ for a solution of (102) when differential operators are replaced by difference operators so that $\{\lambda : T(\lambda) = 0\}$ is finite. The solution here hinges on the properties of the structure of the space:

$$L^2(F_x) = \{T : \mathbf{R} \to B(\mathbf{C}^k), \| \int_{\mathbf{R}} \int_{\mathbf{R}} T(\lambda) F_x(d\lambda, d\lambda') T^*(\lambda') \| < \infty\}. \tag{105}$$

Since the integral in (105) defines a positive (semi-) definite matrix, its trace gives a semi-norm. The measure function F being a matrix bimeasure, several new problems arise for an analysis of the $L^2(F_x)$-space. For the weakly harmonizable case, an extension of the MT-integration, to include such integrals, should be established. The resulting theory can then be utilized for the multivariate filtering study. Even if $k = 2$, the problem is non-trivial, involving the rank questions of F_x. Application of the dilation results to the filtering problem has some novel features, but it does not materially simplify the problem.

Another interesting point is to seek "weak solutions" of the filtering equation (102) in the sense of distribution theory. This idea is introduced in [2]. If $\mathcal{G}$ is a class of functions on $\mathbf{R}$ (e.g. the Schwartz space $C_{00}^{\infty}(\mathbf{R})$) with a locally convex topology, then one says that (102) has a (weak) solution iff for each $f \in \mathcal{G}$

$$\int_{\mathbf{R}} f(t) Y(t) dt = \int_{\mathbf{R}} f(t) \Lambda X(t) dt = \int_{\mathbf{R}} (\bar{\Lambda} f)(t) X(t) dt, \tag{106}$$

where $\tilde{\Lambda} : \mathscr{G} \to \mathscr{G}$ is an operator, associated with Λ, defined by the last two integrals above. It is an "adjoint" to Λ. For instance, if Λ is a differential operator with $T(\cdot)$ as its generator, if $k = 1$ and X, Y are stationary, then $\tilde{\Lambda}$ is given by

$$(\tilde{\Lambda} f)(t) = \int_{\mathbf{R}} T(t-\lambda) f(\lambda) F_x(d\lambda), \qquad f \in \mathscr{G} \tag{107}$$

where F_x is the spectral measure function of the X-process. Clearly many other possibilities are available. Thus there are a number of directions to pursue the research on these problems, and the paper [2] has a wealth of ideas of great interest here.

This essentially includes what is known about weakly harmonizable random fields and processes, as far as their structure is concerned. Since the class (C) of Cramér and its weak counterpart (cf. Definition 3.1) and the Karhunen class of processes, defined by (31), are natural generalizations of harmonizable and stationary classes, it is reasonable to ask whether the latter is a dilation of the former, i.e., is the analog of Theorem 6.1 true for weakly class (C)? A restricted version can be establshed by the same methods, but the parallel generalization does not hold. (See [38] on this point.) This question will be briefly discussed here in order to include it in the set of problems raised by the present study.

Recall that a mapping $X : \mathbf{R} \to L_0^2(P)$ is a *Karhunen process* if its covariance function $r(\cdot, \cdot)$ admits a representation

$$r(s, t) = \int_{\mathbf{R}} g_s(\lambda) \overline{g_t(\lambda)} F(d\lambda), \qquad s, t \in \mathbf{R},$$

relative to a family $\{g_s(\cdot), s \in \mathbf{R}\}$ of measurable functions and F which defines a locally finite positive regular (or Radon) measure on $\mathbf{R}$ and $g_s \in L^2(F)$ (cf. also [10], p. 241). As an immediate consequence of Theorem 3.2 (cf. Remark 2 following its proof), an integral representation for Karhunen processes can be given.

PROPOSITION 9.3. *Let S be a locally compact space and $X : S \to L_0^2(P)$ be a process of Karhunen class relative to a locally finite positive regular (or Radon) measure F on S and a family $\{g_t, t \in S\} \subset L^2(F)$, the space of all scalar square integrable functions on $(S, \mathscr{B}, F)$. Then there is a locally bounded regular (or Radon) stochastic measure $Z : \mathscr{B}_0 \to L_0^2(P)$ where $\mathscr{B}_0 \subset \mathscr{B}$ is the δ-ring of bounded sets, such that* (i)

$$E(Z(A) \cdot \bar{Z}(B)) = F(A \cap B), \qquad A, B \in \mathscr{B}_0,$$

i.e., Z is orthogonally scattered, and (ii) *one has*

$$X(t) = \int_S g_t(\lambda) Z(d\lambda), \qquad t \in S, \tag{108}$$

where the right side symbol is a D-S *integral (cf. also* [42], §1*). Conversely, if* $X : S \to L_0^2(P)$ *is a process defined by* (108) *relative to an orthogonally scattered measure* Z *on* S *and* $\{g_t, t \in S\}$ *satisfies the above conditions, then it is a Karhunen process with respect to the family* $\{g_t, t \in S\}$ *and* F *defined by* $F(A \cap B) = (Z(A), Z(B))$. *Moreover*

$$\mathscr{H}_X = \overline{sp}\{X(t), t \in S\} \subseteq \mathscr{H}_Z = \overline{sp}\{Z(A), A \in \mathscr{B}_0\} \subset L_0^2(P)$$

and $\mathscr{H}_X = \mathscr{H}_Z$ *iff* $\{g_t, t \in S\}$ *is dense in* $L^2(F)$.

A proof of this result is essentially given in ([10], p. 242) and is a simplification of that of Theorem 3.2. Even a multidimensional version is not difficult, which in fact is analogous to that of Theorem 9.2 above. Actually, the version in [10] is sketched for the k-dimensional case.

It follows from the arguments of the D-S theory of integration that a bounded linear operator T and the vector integral such as that of (108) commute even if Z is of locally finite semivariation on the locally compact space S. This extension of ([8], IV.10) was proved in ([42], p. 79), and shown to be easy. Thus if $X : S \to L_0^2(P)$ is a Karhunen process, so that it is representable as in (108) and if $T \in B(L_0^2(P))$, then it follows that

$$TX(t) = \int_S g_t(\lambda) T \circ Z(d\lambda), \tag{109}$$

and it is simple to see that $\tilde{Z} = T \circ Z$ is a stochastic measure of locally finite semivariation, but not necessarily orthogonally scattered. Hence by Theorem 3.2, TX is weakly of class (C).

In the opposite direction, for a process $\{X(s), s \in S\} \in$ weakly class (C), one cannot apply the theory of Section 5 above if only $\{g_t, t \in S\} \subset L^2(F_x)$, and no further restrictions are imposed, where $L^2(F_x)$ is the space of functions g such that $|g|$ is MT-integrable relative to the covariance bimeasure F_x representing X (cf. (105), with $k = 1$). Suppose now that F_x is such that if each g_t is a *bounded* Borel function and $M(S)$ is the uniformly closed algebra generated by $\{g_t, t \in S\}$ then $M(S) \subset L^2(F_x)$. Let

$$Tg_t = X(t) = \int_S g_t(\lambda) Z(d\lambda)$$

and extend T linearly to $M(S)$. Then $T \in B(M(S), \mathscr{H})$ when $M(S)$ is given the uniform norm. This forces F_x to be of finite semivariation if at least one g_s has noncompact support. Under this assumption T is a 2-absolutely summing, and Proposition 5.6 is applicable. Hence

$$\| Tf \| \leqslant \| f \|_{2,\mu}, \qquad f \in M(S) \tag{110}$$

for a finite measure μ on S. (A similar result seems possible *if* Z is restricted so that $T \in B(L^2(F_x), \mathscr{H})$, defined above is Hilbert-Schmidt by [22], p. 302. But it is not a good assumption here.) Thus one can repeat the proof of Theorem 6.1 essentially verbatim and establish a dilation result. Omitting the details of this computation one obtains the following result. (For related remarks, details and other results, see [38].)

THEOREM 9.4. *Let* S *be a locally compact space and*

$$X : S \to L_0^2(P) = \mathscr{H}$$

be a Karhunen process relative to a Radon measure F *and a family*

$$\{g_t, t \in S\} \subset L^2(F).$$

If $Q : \mathscr{H} \to \mathscr{H}$ *is any (bounded) projection, then* $\tilde{X}(t) = QX(t), t \in S,$ *is a process in weakly class (C). On the other hand if* $\{X(t), t \in S\}$ *is an element of weakly class (C), and so is representable in the form (108) for some family* $\{g_t, t \in S\} \subset L^2(F_x)$ *where* F_x *is a bounded covariance bimeasure of the process* $(L^2(F_x)$ *is defined above), and if each* g_t *is also bounded, then there exists an extension Hilbert space* $\mathscr{K} \supset \mathscr{H}$, *a probability space* $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ *with* $\mathscr{K} = L_0^2(\tilde{P})$, *and a Karhunen process* $Y : S \to \mathscr{K}$ *such that*

$$X(t) = QY(t), \qquad t \in S,$$

where Q *is the orthogonal projection on* $\mathscr{K}$ *with range* $\mathscr{H}$.

This result points out clearly the need to consider the domination problem for other Banach spaces than those covered by the results of Section 5. Indeed the associated abstract problem of classifying Banach spaces admitting a positive p-majorizable measure for each vector measure from a probability space into that space is essentially open. Also the preceding theorem and related analysis presumably extend to class$_N$ (C)-processes of Definition 3.4. This will be of independent interest in addition to its use in a treatment of the general filtering theory on these processes. Other problems noted in the main text of the paper are of both methodological and applicational importance for a future study.

REFERENCES

[1] ABREU, J. L. A note on harmonizable and stationary sequences. *Bol. Soc. Mat. Mexicana 15* (1970), 48-51.

[2] BOCHNER, S. Stationarity, boundedness, almost periodicity of random valued functions. *Proc. Third Berkeley Symp. Math. Statist. and Prob. 2* (1956), 7-27.

[3] CRAMÉR, H. A contribution to the theory of stochastic processes. *Proc. Second Berkeley Symp. Math. Statist. and Prob.*, 1951, 329-339.

[4] DAY, M. M. *Normed Linear Spaces.* Springer-Verlag, New York, 1962.

[5] DINCULEANU, N. *Integration on Locally Compact Spaces.* Noordhoff International Publ., Leyden, 1974.

[6] DOOB, J. L. *Stochastic Processes.* J. Wiley & Sons, New York, 1953.

[7] DUCHOŇ, M. and I. KLUVÁNEK. Inductive tensor product of vector valued measures. *Mat. Časopis 17* (1967), 108-112.

[8] DUNFORD, N. and J. T. SCHWARTZ. *Linear Operators, Part I: General Theory.* Interscience, New York, 1958.

[9] GETOOR, R. K. The shift operator for nonstationary stochastic processes. *Duke Math. J. 23* (1956), 175-187.

[10] GIKHMAN, I. I. and A. V. SKOROKHOD. *The Theory of Stochastic Processes–I.* Springer-Verlag, New York, 1974.

[11] HANNAN, E. J. The concept of a filter. *Proc. Cambridge Phil. Soc. 63* (1967), 221-227.

[12] HELSON, H. Isomorphisms of abelian group algebras. *Ark. Mat. 2* (1953), 475-487.

[13] HELSON, H. and D. LOWDENSLAGER. Prediction theory and Fourier series in several variables. *Acta Math. 99* (1958), 165-202.

[14] KAMPÉ DE FÉRIET, J. Correlation and spectrum of asymptotically stationary random functions. *The Math. Student 30* (1962), 55-67.

[15] KAMPÉ DE FÉRIET, J. and F. N. FRENKIEL. Correlation for truncated samples of a random function. *Proc. International Congress Math.*, Amsterdam, Noordhoff, *2* (1954), 291-292.

[16] —— Estimation de la corrélation d'une fonction aléatoire non stationnaire. *C. R. Acad. Sci. (Paris) 249* (1959), 348-351.

[17] —— Correlations and spectra of nonstationary random functions. *Math. Comput. 10* (1962), 1-21.

[18] KARHUNEN, K. Über lineare Methoden in der Wahrscheinlichkeitsrechnung. *Ann. Sci. Fenn. Ser. AI Math. 37* (1947), 3-79.

[19] KELSH, J. P. *Linear Analysis of Harmonizable Time Series.* Ph.D. thesis, Univ. of Calif. at Riverside, 1978.

[20] KHINTCHINE, A. Ya. Korrelationstheorie der stationäre stochastischen Prozesse. *Math. Ann. 109* (1934), 605-615.

[21] KLUVÁNEK, I. Characterization of Fourier-Stieltjes transformations of vector and operator valued measures. *Czech. Math. J. 17* (92) (1967), 261-277.

[22] LINDENSTRAUSS, J. and A. PELCZÝNSKI. Absolutely summing operators in L_p-spaces and their applications. *Studia Math. 29* (1968), 275-326.

[23] LOÈVE, M. *Probability Theory* (3rd ed.). D. Van Nostrand, Princeton, NJ, 1963.

[24] LOOMIS, L. H. *Introduction to Abstract Harmonic Analysis.* D. Van Nostrand, Princeton, NJ, 1953.

[25] MIAMEE, A. G. and H. SALEHI. Harmonizability, *V*-boundedness and stationary dilation of stochastic processes. *Indiana Univ. Math. J. 27* (1978), 37-50.

[26] MORSE, M. and W. TRANSUE. C-bimeasures and their superior integrals. *Rend. Circolo Mat. Palermo (2) 4* (1955), 270-300.

[27] —— C-bimeasures and their integral extensions. *Ann. Math. 64* (1956), 480-504.
[28] NAGABHUSHANAM, K. The primary process of a smoothing relation. *Ark. Mat. 1* (1951), 421-488.
[29] NIEMI, H. Stochastic processes as Fourier transforms of stochastic measures. *Ann. Acad. Sci. Fenn. AI Math. 591* (1975), 1-47.
[30] —— On stationary dilations and the linear prediction of certain stochastic processes. *Comment. Phy.-Math. 45* (1975), 111-130.
[31] —— On orthogonally scattered dilations of bounded vector measures. *Ann. Acad. Sci. Fenn. AI Math. 3* (1977), 43-52.
[32] PARZEN, E. Spectral analysis of asymptotically stationary time series. *Bull. Inst. Internat. Statist. 39* (1962) (livraison 2), 87-103.
[33] PHILLIPS, R. S. On Fourier-Stieltjes integrals. *Trans. Amer. Math. Soc. 69* (1950), 312-323.
[34] RAO, M. M. Characterization and extension of generalized harmonizable random fields. *Proc. Nat. Acad. Sci. 58* (1967), 1213-1219.
[35] —— Covariance analysis of nonstationary time series. In: *Developments in Statistics*, Vol. 1, Academic Press, New York, 1978, 171-225.
[36] —— *Stochastic Processes and Integration.* Sijthoff & Noordhoff, Alphen aan den Rijn, Netherlands, 1979.
[37] —— Representation of weakly harmonizable processes. *Proc. Nat. Acad. Sci. 79* (1981), 5288-89.
[38] —— Domination problem for vector measures and applications to nonstationary processes. *Proc. 1981 Oberwolfach Conf. on Measure Theory*, Springer Lecture Notes in Math. (to appear).
[39] ROSENBERG, M. Minimum-trace quasi-isometric dilations of operator-valued measures. *Abstracts Amer. Math. Soc. 1* (1980), 455.
[40] ROZANOV, Yu. A. Spectral analysis of abstract functions. *Teor. Prob. Appl. 4* (1959), 271-287.
[41] SZ.-NAGY, B. Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe. *Acta. Sci. Math., Szeged 15* (1954), 104-114.
[42] THOMAS, E. L'intégration par rapport à une mesure de Radon vectorielle. *Ann. Inst. Fourier Grenoble 22* (1970), 55-191.
[43] WIENER, N. *Extrapolation, Interpolation, and Smoothing of Stationary Time Series* (with engineering applications). Wiley, New York, 1949.
[44] YAGLOM, A. M. *Introduction to the Theory of Stationary Random Functions.* Prentice-Hall, Englewood Cliffs, NJ, 1962. (English translation).
[45] YLINEN, K. Fourier transforms of noncommutative analogs of vector measures and bimeasures with applications to stochastic processes. *Ann. Acad. Sci. Fenn. Ser. AI Math. 1* (1975), 355-385.
[46] —— On vector bimeasures. *Ann. Mat. Pura Appl.* (4) *117* (1978), 115-138.
[47] ZYGMUND, A. *Trigonometric Series*, Vol. I. Cambridge Univ. Press, London, 1959.

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