## 2. Induction and reciprocity

The notion of induced representations for finite groups was introduced in 1898 by G. Frobenius in the paper [37]. In the same paper Frobenius established what is now called the Frobenius reciprocity relation. We recall his basic construction which is fundamental in the entire theory of group representations. [1])

Let $G$ be a finite group and let $P$ be a subgroup of $G$. Let $\pi$ be a representation of $G$ on a finite dimensional vector space $V$. That is $\pi: G \to GL(V)$ is a homomorphism of $G$ into the group of non-singular endomorphisms of $V$. We shall also refer to $V$ as a (left) $G$ module. By restriction $V$ is also a $P$ module. Conversely there is a functor $I$ which converts $P$ modules to $G$ modules: Given a $P$ module $W$ the $G$ module $IW$ is defined to be the space of functions $f: G \to W$ such that $f(ap) = p^{-1} \cdot f(a)$ for every $(a, p)$ in $G \times P$. The action of $G$ on $IW$ is defined by

$$(a \cdot f)(x) = f(a^{-1}x)$$

for $(f, a, x)$ in $(I\,W) \times G \times G$. $I\,W$ is called the $G$ module *induced* by the $P$ module $W$. Induction and restriction are related in the following way.

THEOREM 2.1 (Frobenius reciprocity relation, 1898). *If $W$ is a $P$ module and if $V$ is a $G$ module then*

$$\mathrm{Hom}_G(V, I\,W) = \mathrm{Hom}_P(V, W).$$

We wish to consider extensions or analogues of this relation in a wider context. For this it is most convenient first of all to re-describe the $G$ module $I\,W$. The following "geometric" interpretation of $I\,W$ is well-known. Consider the right action of $P$ on $G \times W$ given by

$$(a, w) \cdot p = (ap, p^{-1}w)$$

for $(a, p, w)$ in $G \times P \times W$. Let

$$E_W = \text{orbit space } (G \times W)/P = G \times_P W. \tag{2.2}$$

Let $\gamma: E_W \to G/P$ be the canonical (well-defined) map $[a, w] \to aP$, where $[a, w]$ is the orbit of $(a, w) \in G \times W$. For each $a \in G$ the map $w \to [a, w]$ of $W$ to $\gamma^{-1}\{aP\}$ is a bijection. That is we may identify $W$ as the fibre over each point of

[1]) For the theory of induced representations of locally compact groups see G. Mackey [55], [56].

$G/P$. $G$ acts naturally on $E_W$ and $G/P$ on the left. $\gamma$ is an equivariant map. Let $\Gamma(E_W)$ be the space of sections of $E_W$. That is $s \in \Gamma(E_W)$ is a map from $G/P$ to $E_W$ satisfying $\gamma \circ s = 1$; hence $s$ maps each point to the fibre over it. $\Gamma(E_W)$ is a left $G$ module:

$$(a \cdot s)(x) = a \cdot s(a^{-1} \cdot x) \tag{2.3}$$

for $(a, s, x)$ in $G \times \Gamma(E_W) \times G/P$. Moreover

PROPOSITION 2.4. *There is a natural $G$ module isomorphism $s \to f^s$ of $\Gamma(E_W)$ onto $I\,W$ such that for every $a$ in $G$, $s(aP) = [a, f^s(a)]$. Hence by Theorem 2.1*

$$\mathrm{Hom}_G(V, \Gamma(E_W)) = \mathrm{Hom}_P(V, W). \tag{2.5}$$

This sets the stage for a possible extension of Frobenius. Namely, following Bott, we consider the following data. $G$ is a complex Lie group, $P$ is a closed complex Lie subgroup (thus the injection $P \to G$ is holomorphic), and $W$ is a finite dimensional holomorphic $P$ module (i.e. for each $w$ in $W$ and $f$ in the complex dual space of $W$ the map $p \to f(p \cdot w)$ of $P$ to the complex numbers is holomorphic). We define $E_W$ exactly as above. Then $E_W$ has the structure of a holomorphic vector bundle over the complex manifold $G/P$. Let $\Gamma(E_W)$ now denote the space of $C^\infty$ sections with the $G$ module structure given by (2.3) and let $\Gamma_{\mathrm{hol}}(E_W)$ denote the $G$ stable subspace of holomorphic sections. Since all of our data is now holomorphic the most natural question to ask, considering (2.5), is: When is it true that

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W) \tag{2.6}$$

for a holomorphic $G$ module $V$? (2.6) would then represent an exact holomorphic analogue of Frobenius reciprocity. It turns out that (2.6) is valid if the space $G/P$ is sufficiently nice. For example suppose that $G/P$ is a compact simply connected Kahler manifold. Group theoretically this means that $G$ is a connected complex semisimple Lie group and $P$ is a parabolic subgroup. Then it is due to Bott [12] that (2.6) is valid. In fact in [12] Bott proves considerably more: Let $SE_W$ be the sheaf of germs of local holomorphic sections of $E_W$ and let $H^*(G/P, SE_W)$ be the cohomology of $G/P$ with coefficients in $SE_W$. Then we have

THEOREM 2.7 (R. Bott, 1957). *Suppose $G$ is a connected complex semisimple Lie group and $P$ is a parabolic subgroup of $G$. Let $\mathfrak{p}$ be the Lie algebra of $P$ and let $V, W$ be finite dimensional holomorphic $G$ and $P$ modules respectively. Then*

(2.8) $$\mathrm{Hom}_G(V, H^j(G/P, SE_W)) = H^j(p, p \cap \bar{p}, \mathrm{Hom}(V, W))$$

*for each* $j \geqslant 0$.

The bar – denotes conjugation of $G$ with respect to a maximal compact subgroup $K$ of $G$ and the right hand side of (2.8) is the *relative* Lie algebra cohomology of $p$ (in the sense of Hochschild, Serre [44]). Here $H^j(G/P, SE_W)$ [1]) has the $G$ module structure induced by the left action of $G$ on $E_W$ and $\mathrm{Hom}(V, W)$ has the $p$ module structure defined by

(2.9) $$(x \cdot \phi)(v) = -\phi(x \cdot v) + x \cdot \phi(v)$$

for $(x, \phi, v)$ in $p \times \mathrm{Hom}(V, W) \times V$.

*Remarks.* (i) For $j = 0$, $H^0(p, p \cap \bar{p}, \mathrm{Hom}(V, W))$ is independent of the subalgebra $p \cap \bar{p}$ of $p$ and has the value $\mathrm{Hom}(V, W)^P$ (the space of invariants) which is precisely $\mathrm{Hom}_p(V, W) = \mathrm{Hom}_P(V, W)$ by (2.9) ($P$ is connected). Also $H^0(G/P, SE_W)$ is precisely $\Gamma_{\mathrm{hol}}(E_W)$. Thus taking $j = 0$ in (2.8) we get

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W)$$

which is (2.6). This shows that (2.8) represents a rather remarkable extension of Frobenius reciprocity to higher cohomology. Here the induction functor is $I: W \to H^*(G/P, SE_W)$.

(ii) As shown by Bott (2.8) is valid, more generally, for $C$-spaces $G/P$ in the sense of Wang [90]. The latter need not be Kahler, as we have assumed for our purposes.

The functor $I$ in remark (i) can be explicated by the use of differential forms: Let $\Lambda^{0,j}(G/P, E_W)$ denote the space of $E_W$ valued $C^\infty$ differential forms on $G/P$ of pure type $(0, j)$. That is

$$\omega \in \Lambda^{0,j}(G/P, E_W)$$

assigns to each $x \in G/P$ a skew-symmetric $j$ linear map

$$\omega_x : T_x(G/P)^{\mathbf{C}} \times \dots \times T_x(G/P)^{\mathbf{C}} \to (E_W)_x = \gamma^{-1}\{x\}$$

on the complexified tangent space $T_x(G/P)^{\mathbf{C}}$ of $G/P$ at $x$ to the fiber $(E_W)_x$ over $x$ such that (*a*) given smooth vector fields $X_1, \dots, X_j$ on $G/P$ the map

$$\omega(X_1, \dots, X_j): x \to \omega_x(X_{1_x}, \dots, X_{j_x})$$

is $C^\infty$—i.e. it belongs to $\Gamma(E_W)$ and (*b*) for each real number $\theta$,

$$\omega(U_\theta X_1, \dots, U_\theta X_j) = e^{-\sqrt{-1}\, j\theta}\, \omega(X_1, \dots, X_j)$$

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[1]) Since $G/P$ is compact $H^j(G/P, SE_W)$ is known to be finite-dimensional.

where

$$U_\theta X_l = \cos\theta\, X_l + \sin\theta\, JX_l$$

and $J$ is the complex structure tensor on $G/P$. Let $\bar{\partial}\colon \Lambda^{0,j} \to \Lambda^{0,j+1}$ denote, as usual, the Cauchy-Riemann operator so that $\bar{\partial}^2 = 0$. If $f$ is a $C^\infty$ function on $G/P$ and $X$ is a $C^\infty$ vector field on $G/P$ then

$$(2.10) \qquad (\bar{\partial} f)(X) = \frac{1}{2}[Xf + \sqrt{-1}\,(JX)\, f]\,.$$

Since $\bar{\partial}^2 = 0$ let $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ denote the corresponding $\bar{\partial}$ cohomology:

$$(2.11) \qquad H^{0,j}_{\bar{\partial}}(G/P, E_W) = \frac{\ker \bar{\partial}\colon \Lambda^{0,j}(G/P, E_W) \to \Lambda^{0,j+1}(G/P, E_W)}{\bar{\partial}\Lambda^{0,j-1}(G/P, E_W)}\,.$$

By Dolbeault's theorem [35]

$$(2.12) \qquad H^j(G/P, SE_W) = H^{0,j}_{\bar{\partial}}(G/P, E_W)\,.$$

The induced action of $G$ on $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ is given explicitly as follows. First $G$ acts on $\Lambda^{0,j}(G/P, E_W)$ by

$$(2.13) \qquad (a \cdot \omega)_x (L_1, \ldots, L_j) = a \cdot \omega_{a^{-1}x}\left(dl_{a^{-1}x}(L_1), \ldots, dl_{a^{-1}x}(L_j)\right)$$

where

$$(a, \omega, x) \in G \times \Lambda^{0,j}(G/P, E_W) \times G/P\,,$$

each $L_l \in T_x(G/P)^{\mathbf{C}}$ and $dl_{a_x}$ is the derivative of left translation $l_a\colon G/P \to G/P$ on $G/P$ at $x$. Note that (2.13) generalizes the action of $G$ on

$$\Gamma(E_W) = \Lambda^{0,0}(G/P, E_W)$$

given in (2.3). Because left translation is holomorphic the diagram

$$\begin{array}{ccc} \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar{\partial}} & \Lambda^{0,j+1}(G/P, E_W) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar{\partial}} & \Lambda^{0,j+1}(G/P, E_W) \end{array}$$

is commutative for each $a$ in $G$. Thus (2.13) induces a well-defined action of $G$ on $H^{0,j}_{\bar{\partial}}(G/P, E_W)$. We may now write (2.8) as

$$(2.14) \qquad \operatorname{Hom}_G(V, H^{0,j}_{\bar{\partial}}(G/P, E_W)) = H^j\left(\mathfrak{p}, \mathfrak{p}\cap\bar{\mathfrak{p}}, \operatorname{Hom}(V, W)\right).$$

Now assume that $W$ is in fact irreducible. The parabolic subalgebra $p$ has a decomposition $p = (p \cap \bar{p}) \oplus n$ into a reductive part $p \cap \bar{p}$ and a nilpotent part $n =$ an ideal in $p$. By general principles

$$\begin{aligned} H^j(p, p \cap \bar{p}, \operatorname{Hom}(V, W)) &= H^j(n, \operatorname{Hom}(V, W))^{p \cap \bar{p}} \\ &= H^j(n, V^* \otimes W)^{p \cap \bar{p}} = \left(H^j(n\, V^*) \otimes W\right)^{p \cap \bar{p}}. \end{aligned}$$

The last statement of equality follows by the irreducibility of $W$ since by Lie's theorem, $W$ is a trivial $n$ module. Now

$$\left(H^j(n, V^*) \otimes W\right)^{p \cap \bar{p}} = \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V^*)\right).$$

From (2.14) we obtain (see [50]).

THEOREM 2.15 (Bott-Kostant reciprocity, 1960). *Let $G$, $P$ be as in Theorem 2.7, let $n$ be the nilradical of the parabolic subalgebra $p$, and let $W$ be a finite dimensional irreducible holomorphic $P$ module. Then for any finite dimensional holomorphic $G$ module $V$ we have*

$$\operatorname{Hom}_G\left(V, H_{\bar{\partial}}^{0,j}(G/P, E_W)\right) = \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V^*)\right). \tag{2.16}$$

Again $p \cap \bar{p}$ is the reductive part of $p$ where the bar denotes conjugation of $G = K^{\mathbf{C}}$ with respect to a maximal compact subgroup $K$. We refer to (2.16) as "the debut of $n$ cohomology"! Since 1960 it has played some rather important roles in both finite dimensional and infinite dimensional representation theory. There is an equivalent version of (2.16): The $G$ module structure on $H_{\bar{\partial}}^{0,j}(G/P, E_W)$ induced by (2.13) may be restricted to $K$. Let $\hat{K}$ denote, as usual, the equivalence classes of the irreducible unitary representations of $K$ and let $V_\pi$ be the representation space of $\pi \in \hat{K}$. Then we have (again for $W$ irreducible).

THEOREM 2.17 (B. Kostant). *The decomposition of $H_{\bar{\partial}}^{0,j}(G/P, E_W)$ as a $K$ module is*

$$\begin{aligned} H_{\bar{\partial}}^{0,j}(G/P, E_W) &= \sum_{\pi \in \hat{K}} V_\pi \otimes \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V_\pi^*)\right) \\ &= \sum_{\pi \in \hat{K}} V_\pi^* \otimes \operatorname{Hom}_{p \cap \bar{p}}\left(W^*, H^j(n, V_\pi)\right). \end{aligned} \tag{2.18}$$

In the direct sum on the right hand side the action of $K$ on a summand is $\pi \otimes 1$ or $\pi^* \otimes 1$ in the second equation.

From (2.18) (or from (2.16)) we see that the multiplicity of an irreducible $K$ module $V_\pi$ in $H_{\bar{\partial}}^{0,j}(G/P, E_W)$ is governed precisely by the $n$ cohomology

$H^j(n, V_\pi^*)$. Here, by analytic continuation, we consider $V_\pi$ also as a representation of the complex Lie algebra of $G$. Its $n$ module structure is the restriction thereof to $n$.

*Remarks.* (i) In contrast to remark (ii) made earlier, following Theorem 2.7, Theorems (2.15) and (2.17) do require that $G/P$ should be Kahler.

(ii) One knows that $K$ acts transitively on $G/P$ so that $G/P$ is diffeomorphic to $K/K \cap P$.

Now Kostant in [50] has computed the Lie algebra cohomology groups $H^j(n, V_\pi^*)$. Two outstanding consequences of his results, among others, which we shall briefly discuss are (a) Weyl's character formula and (b) Bott's generalized Borel-Weil theorem. Suppose more generally that $g$ is any complex semisimple Lie algebra (for example $g$ could be the Lie algebra of $G$ above). Let $h \subset g$ be a Cartan subalgebra of $g$, let $\Delta$ be the set of non-zero roots of $(g, h)$, and let $\Delta^+$ be a choice of positive roots. The equivalence classes of finite dimensional irreducible representations of $g$ (over the complex numbers) correspond univalently to linear functionals $\Lambda$ on $h$ which satisfy the condition that $2\dfrac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer for each $\alpha$ in $\Delta^+$. That is $\Lambda$ is $\Delta^+$ *dominant integral*; $(\ ,\ )$ denotes the Killing form on $g$. This is Cartan's highest weight theory alluded to in the introduction. Let $\pi_\Delta$ be a finite dimensional irreducible representation of $g$ with corresponding highest weight $\Lambda \in h^*$. Its *character* $X_\Lambda \colon h \to \mathbf{C}$ is defined to be the function $H \to \operatorname{trace} \exp \pi_\Lambda(H)$, $H \in h$. This definition is independent of the choice of Cartan subalgebra $h$ since any two are conjugate. We consider the special "minimal" parabolic subalgebra $p \subset g$ whose nilradical is

$$n = \sum_{\alpha \in \Delta^+} g_\alpha \tag{2.19}$$

and whose reductive part is $h$ where $g_\alpha$ is the root space of $\alpha \in \Delta$. That is $p$ is just the Borel subalgebra $h + n$. Let $V_\Lambda$ denote the representation space of $\pi_\Lambda$. Then by restriction to $n$ we again form the Lie algebra cohomology groups $H^j(n, V_\Lambda)$. Let $\theta$ denote the adjoint representation of $h$ on $\Lambda n^*$. Then $\theta \otimes \pi_\Lambda$ defines a representation of $h$ on the cochain complex $\Lambda n^* \otimes V_\Lambda$. This $h$ action commutes with the coboundary operator and therefore passes to cohomology. Applying the Euler-Poincaré principle one gets

$$\sum_{j=0}^{\dim n} (-1)^j \operatorname{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{\Lambda^j n^* \otimes V_\Lambda} = \sum_{j=0}^{\dim n} (-1)^j \operatorname{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{H^j(n, V_\Lambda)} \tag{2.20}$$

for each $H$ in $h$. One evaluates the left hand side of (2.20) by general principles and the right hand side using Kostant's main theorem, Theorem 5.14 of [50]. Actually Theorem 5.14 of [50] gives the $h_1$ module structure of $H^j(n_1, V_\Lambda)$ for an *arbitrary* parabolic $p_1 = h_1 + n_1$ of $g$ with reductive and nilpotent parts $h_1, n_1$ respectively. For the derivation of Weyl's formula only the simplest case $p_1 = p = h + n$ is needed, where $n$ is given in (2.19). Thus we shall state only a special case of Kostant's result.

THEOREM 2.21 (B. Kostant, 1960). *The decomposition of* $H^j(n, V_\Lambda)$ *as a* $h$ *module is*

$$H^j(n, V_\Lambda) = \sum V_{\Lambda, \sigma},$$

$$\sigma \in \text{Weyl group } \mathscr{W} \text{ of } (g, h) \text{ such that } l(\sigma) = j,$$

*where each summand* $V_{\Lambda, \sigma}$ *in the direct sum is one-dimensional and* $H \in h$ *acts on* $V_{\Lambda, \sigma}$ *by the scalar* $[\sigma(\Lambda+\delta)-\delta](H)$.

Here by definition $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ and $l(\sigma)$ (the *length* of $\sigma$) is the cardinality of the set $\Delta^+ \cap \sigma(-\Delta^+)$. From the remarks following (2.20) and the knowledge of $n$ cohomology given by Theorem 2.21 one derives Weyl's famous character formula [93]:

THEOREM 2.22 (H. Weyl, 1926). *For* $H \in h$

$$X_\Lambda(H) = \frac{\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{[\sigma(\Lambda+\delta)](H)}}{\prod_{\alpha \in \Delta^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}.$$

The denominator is also given by the sum $\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{(\sigma\delta)(H)}$ (this fact can be proved too using $n$ cohomology) and $\det \sigma = (-1)^{l(\sigma)}$. As a corollary of Theorem 2.22 one obtains Weyl's formula for the dimension of the irreducible module $V_\Lambda$ in terms of its highest weight $\Lambda$. The result is

$$\dim V_\Lambda = \frac{\prod_{\alpha \in \Delta^+} (\Lambda+\delta, \alpha)}{\prod_{\alpha \in \Delta^+} (\delta, \alpha)}. \tag{2.23}$$

Kostant's result on $n$ cohomology can also be used to derive the generalized Borel-Weil theorem. Here one may apply formula (2.18) decisively. Let $g$ now denote the Lie algebra of $G$. Extend a maximal abelian subalgebra of the Lie algebra of $K$ to a Cartan subalgebra $h$ of $g$. Again let $\Delta^+ \subset \Delta$ be a choice of positive roots where $\Delta$ is the set of non-zero roots of $(g, h)$ and let $2\delta = \sum_{\alpha \in \Delta^+} \alpha$.

We choose the parabolic $P$ such that its Lie algebra $p$ contains the Borel subalgebra $h + \sum_{\alpha \in \Delta^+} g_{-\alpha} \cdot h$ is also a Cartan subalgebra of the reductive Lie algebra $p \cap \bar{p}$ so that we have the decompositions

$$(2.24) \qquad \begin{aligned} p &= (p \cap \bar{p}) \oplus n \,, \qquad p \cap \bar{p} = h + \sum_{\alpha \in \Delta(p \cap \bar{p})} g_\alpha \\ n &= \sum_{\alpha \in \Delta^+ - \Delta(p \cap \bar{p})} g_{-\alpha} \end{aligned}$$

where $\Delta(p \cap \bar{p})$ is the set of roots of $(p \cap \bar{p}, h)$.

Let $W$ be an irreducible holomorphic $P$ module. Then $W$ is an irreducible $p \cap \bar{p}$ module thereby such that $n \cdot W = 0$. We let $\Lambda$ denote its highest weight relative to the positive system $\Delta^+ \cap \Delta(p \cap \bar{p})$ for $p \cap \bar{p}$. Applying Kostant's $n$ cohomology theorem to (2.18) one obtains (see [12], [50]).

THEOREM 2.25 (R. Bott, 1957). *The spaces* $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ *vanish for all but at most one* $j$. *If*

$$H^{0,j_0}_{\bar{\partial}}(G/P, E_W) \neq 0$$

*then* $H^{0,j_0}_{\bar{\partial}}(G/P, E_W)$ *is an irreducible* $K$ *module.*

More precisely we have the following. Let $\Lambda$ be the highest weight of $W$ (as above) relative to the positive roots in the reductive part of $P$. If $(\Lambda + \delta, \alpha) = 0$ for some $\alpha$ in $\Delta$ then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for every $j$. If $(\Lambda + \delta, \alpha) \neq 0$ for each $\alpha$ in $\Delta$ (i.e. $\Lambda + \delta$ is *regular*) there is a unique element $\sigma$ in the Weyl group of $(g, h)$ such that $(\sigma(\Lambda + \delta), \alpha) > 0$ for every $\alpha \in \Delta^+$. Then $H^{0,j}_{\bar{\partial}}(G/P, E_W) = 0$ for $j \neq l(\sigma)$ where again $l(\sigma)$ is the length of $\sigma$ (see remarks following Theorem 2.21). Moreover $H^{0,l(\sigma)}_{\bar{\partial}}(G/P, E_W)$ is an irreducible $K$ module ($=$ an irreducible $g$ module since $g$ is the complexification of the Lie algebra of $K$) with highest weight $\sigma(\Lambda + \delta) - \delta$ relative to $\Delta^+$.

*Remarks.* (i) By definition of $\sigma$ it follows that

$$\sigma^{-1}\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\} \,.$$

Also since $\Lambda$ is a highest weight $(\Lambda, \alpha) \geqslant 0$ for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}) \Rightarrow (\Lambda + \delta, \alpha) > 0$$

for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}) \,.$$

Hence

$$\begin{aligned} &\{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\} \\ &= \{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda + \delta, \alpha) < 0\} \end{aligned}$$

so that $l(\sigma)$ in Theorem 2.25 has the value

$$|\{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda+\delta, \alpha) < 0\}|\ ^{1)}.$$

$$\Delta^+ - \Delta^+ \cap \Delta(p \cap \bar{p})$$

is the set of roots in the nilradical of the "opposite" parabolic $\bar{p}$. Since

$$(\sigma(\Lambda+\delta), \sigma\alpha) = (\Lambda+\delta, \alpha) > 0$$

for $\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p})$ (as we have just seen) we also conclude that the Weyl group element $\sigma$ in Theorem 2.25 satisfies

$$\Delta^- \cap \Delta(p \cap \bar{p}) \subset \sigma^{-1} \Delta^- .$$

(ii) The irreducible holomorphic $P$ modules $W$ in the statement of Theorem 2.25 can be obtained as follows. Start with an arbitrary irreducible representation $\pi$ of $P \cap K$ on a complex vector space $W$. Since $p \cap \bar{p}$ is the complexification of the Lie algebra of $P \cap K$, $\pi$ defines a unique irreducible representation $\pi$ on $p$ such that $\pi(n) = 0$. This infinitesimal representation can be "integrated" to a representation of $P$ since $P$ and $P \cap K$ have the same fundamental groups. Thus every irreducible representation $\pi$ of $P \cap K$ extends uniquely to an irreducible holomorphic representation of $P$. The highest weight $\Lambda$ of $\pi$ is integral and $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant. Conversely if $G$ is simply connected, any integral $\Lambda \in h^*$ which is $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant is the highest weight of irreducible representation of $P \cap K$ and hence is the highest weight of an irreducible holomorphic representation of $P$.

(iii) Suppose in particular $G$ is simply connected, $p$ is chosen to be

$$h + \sum_{\alpha \in \Delta^+} g_{-\alpha},$$

and that $\Lambda$ is $\Delta^+$ dominant integral. Then in Theorem 2.25 $\sigma = 1$ so that the irreducible $K$, $G$ or $g$ module with highest weight $\Lambda$ is given by $H^{0,0}_{\bar\partial}(G/P, E_W)$ = space of holomorphic sections of the line bundle $E_W$. Indeed $\dim_{\mathbf{C}} W = 1$ since in this case $P \cap K$ is abelian. This gives the geometric realization of $V_\Lambda$ [11].

[1]) $|S|$ denotes the cardinality of a set $S$.