Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 28 (1982)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\delta}$ COHOMOLOGY SPACES

Autor: Williams, Floyd L.

Kapitel: 2. Induction and reciprocity

DOI: https://doi.org/10.5169/seals-52231

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 08.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

2. Induction and reciprocity

The notion of induced representations for finite groups was introduced in 1898 by G. Frobenius in the paper [37]. In the same paper Frobenius established what is now called the Frobenius reciprocity relation. We recall his basic construction which is fundamental in the entire theory of group representations. [1])

Let G be a finite group and let P be a subgroup of G. Let π be a representation of G on a finite dimensional vector space V. That is $\pi\colon G \to GL(V)$ is a homomorphism of G into the group of non-singular endomorphisms of V. We shall also refer to V as a (left) G module. By restriction V is also a P module. Conversely there is a functor I which converts P modules to G modules: Given a P module W the G module IW is defined to be the space of functions $f: G \to W$ such that $f(ap) = p^{-1} \cdot f(a)$ for every (a, p) in $G \times P$. The action of G on IW is defined by

$$(a \cdot f)(x) = f(a^{-1}x)$$

for (f, a, x) in $(I\,W) \times G \times G$. $I\,W$ is called the G module *induced* by the P module W. Induction and restriction are related in the following way.

THEOREM 2.1 (Frobenius reciprocity relation, 1898). *If W is a P module and if V is a G module then*

$$\mathrm{Hom}_G(V, I\,W) = \mathrm{Hom}_P(V, W).$$

We wish to consider extensions or analogues of this relation in a wider context. For this it is most convenient first of all to re-describe the G module $I\,W$. The following "geometric" interpretation of $I\,W$ is well-known. Consider the right action of P on $G \times W$ given by

$$(a, w) \cdot p = (ap, p^{-1}w)$$

for (a, p, w) in $G \times P \times W$. Let

$$E_W = \text{orbit space } (G \times W)/P = G \times_P W. \tag{2.2}$$

Let $\gamma: E_W \to G/P$ be the canonical (well-defined) map $[a, w] \to aP$, where $[a, w]$ is the orbit of $(a, w) \in G \times W$. For each $a \in G$ the map $w \to [a, w]$ of W to $\gamma^{-1}\{aP\}$ is a bijection. That is we may identify W as the fibre over each point of

[1]) For the theory of induced representations of locally compact groups see G. Mackey [55], [56].

G/P. G acts naturally on E_W and G/P on the left. γ is an equivariant map. Let $\Gamma(E_W)$ be the space of sections of E_W. That is $s \in \Gamma(E_W)$ is a map from G/P to E_W satisfying $\gamma \circ s = 1$; hence s maps each point to the fibre over it. $\Gamma(E_W)$ is a left G module:

$$(a \cdot s)(x) = a \cdot s(a^{-1} \cdot x) \tag{2.3}$$

for (a, s, x) in $G \times \Gamma(E_W) \times G/P$. Moreover

PROPOSITION 2.4. *There is a natural G module isomorphism $s \to f^s$ of $\Gamma(E_W)$ onto $I\,W$ such that for every a in G, $s(aP) = [a, f^s(a)]$. Hence by Theorem 2.1*

$$\mathrm{Hom}_G(V, \Gamma(E_W)) = \mathrm{Hom}_P(V, W). \tag{2.5}$$

This sets the stage for a possible extension of Frobenius. Namely, following Bott, we consider the following data. G is a complex Lie group, P is a closed complex Lie subgroup (thus the injection $P \to G$ is holomorphic), and W is a finite dimensional holomorphic P module (i.e. for each w in W and f in the complex dual space of W the map $p \to f(p \cdot w)$ of P to the complex numbers is holomorphic). We define E_W exactly as above. Then E_W has the structure of a holomorphic vector bundle over the complex manifold G/P. Let $\Gamma(E_W)$ now denote the space of C^∞ sections with the G module structure given by (2.3) and let $\Gamma_{\mathrm{hol}}(E_W)$ denote the G stable subspace of holomorphic sections. Since all of our data is now holomorphic the most natural question to ask, considering (2.5), is: When is it true that

$$\mathrm{Hom}_G(V, \Gamma_{\mathrm{hol}}(E_W)) = \mathrm{Hom}_P(V, W) \tag{2.6}$$

for a holomorphic G module V? (2.6) would then represent an exact holomorphic analogue of Frobenius reciprocity. It turns out that (2.6) is valid if the space G/P is sufficiently nice. For example suppose that G/P is a compact simply connected Kahler manifold. Group theoretically this means that G is a connected complex semisimple Lie group and P is a parabolic subgroup. Then it is due to Bott [12] that (2.6) is valid. In fact in [12] Bott proves considerably more: Let SE_W be the sheaf of germs of local holomorphic sections of E_W and let $H^*(G/P, SE_W)$ be the cohomology of G/P with coefficients in SE_W. Then we have

THEOREM 2.7 (R. Bott, 1957). *Suppose G is a connected complex semisimple Lie group and P is a parabolic subgroup of G. Let p be the Lie algebra of P and let V, W be finite dimensional holomorphic G and P modules respectively. Then*

$$\text{(2.8)} \qquad \mathrm{Hom}_G\left(V, H^j(G/P, SE_W)\right) = H^j\left(p, p \cap \bar{p}, \mathrm{Hom}(V, W)\right)$$

for each $j \geqslant 0$.

The bar – denotes conjugation of G with respect to a maximal compact subgroup K of G and the right hand side of (2.8) is the *relative* Lie algebra cohomology of p (in the sense of Hochschild, Serre [44]). Here $H^j(G/P, SE_W)$ [1]) has the G module structure induced by the left action of G on E_W and Hom (V, W) has the p module structure defined by

$$\text{(2.9)} \qquad (x \cdot \phi)(v) = -\phi(x \cdot v) + x \cdot \phi(v)$$

for (x, ϕ, v) in $p \times \mathrm{Hom}(V, W) \times V$.

Remarks. (i) For $j = 0$, $H^0\left(p, p \cap \bar{p}, \mathrm{Hom}(V, W)\right)$ is independent of the subalgebra $p \cap \bar{p}$ of p and has the value $\mathrm{Hom}(V, W)^P$ (the space of invariants) which is precisely $\mathrm{Hom}_p(V, W) = \mathrm{Hom}_P(V, W)$ by (2.9) (P is connected). Also $H^0(G/P, SE_W)$ is precisely $\Gamma_{\mathrm{hol}}(E_W)$. Thus taking $j = 0$ in (2.8) we get

$$\mathrm{Hom}_G\left(V, \Gamma_{\mathrm{hol}}(E_W)\right) = \mathrm{Hom}_P(V, W)$$

which is (2.6). This shows that (2.8) represents a rather remarkable extension of Frobenius reciprocity to higher cohomology. Here the induction functor is $I: W \to H^*(G/P, SE_W)$.

(ii) As shown by Bott (2.8) is valid, more generally, for C-spaces G/P in the sense of Wang [90]. The latter need not be Kahler, as we have assumed for our purposes.

The functor I in remark (i) can be explicated by the use of differential forms: Let $\Lambda^{0,j}(G/P, E_W)$ denote the space of E_W valued C^∞ differential forms on G/P of pure type $(0, j)$. That is

$$\omega \in \Lambda^{0,j}(G/P, E_W)$$

assigns to each $x \in G/P$ a skew-symmetric j linear map

$$\omega_x : T_x(G/P)^{\mathbf{C}} \times \ldots \times T_x(G/P)^{\mathbf{C}} \to (E_W)_x = \gamma^{-1}\{x\}$$

on the complexified tangent space $T_x(G/P)^{\mathbf{C}}$ of G/P at x to the fiber $(E_W)_x$ over x such that (*a*) given smooth vector fields $X_1, \ldots, X_j$ on G/P the map

$$\omega(X_1, \ldots, X_j) : x \to \omega_x(X_{1x}, \ldots, X_{jx})$$

is C^∞—i.e. it belongs to $\Gamma(E_W)$ and (*b*) for each real number θ,

$$\omega(U_\theta X_1, \ldots, U_\theta X_j) = e^{-\sqrt{-1}\, j\theta}\, \omega(X_1, \ldots, X_j)$$

---

[1]) Since G/P is compact $H^j(G/P, SE_W)$ is known to be finite-dimensional.

where

$$U_\theta X_l = \cos\theta\, X_l + \sin\theta\, JX_l$$

and J is the complex structure tensor on G/P. Let $\bar\partial\colon \Lambda^{0,j} \to \Lambda^{0,j+1}$ denote, as usual, the Cauchy-Riemann operator so that $\bar\partial^2 = 0$. If f is a C^∞ function on G/P and X is a C^∞ vector field on G/P then

$$(\bar\partial f)(X) = \frac{1}{2}[Xf + \sqrt{-1}\,(JX)\,f]\,. \tag{2.10}$$

Since $\bar\partial^2 = 0$ let $H^{0,j}_{\bar\partial}(G/P, E_W)$ denote the corresponding $\bar\partial$ cohomology:

$$\begin{aligned} &H^{0,j}_{\bar\partial}(G/P, E_W) \\ &= \frac{\ker \bar\partial\colon \Lambda^{0,j}(G/P, E_W) \to \Lambda^{0,j+1}(G/P, E_W)}{\bar\partial\Lambda^{0,j-1}(G/P, E_W)}\,. \end{aligned} \tag{2.11}$$

By Dolbeault's theorem [35]

$$H^j(G/P, SE_W) = H^{0,j}_{\bar\partial}(G/P, E_W)\,. \tag{2.12}$$

The induced action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$ is given explicitly as follows. First G acts on $\Lambda^{0,j}(G/P, E_W)$ by

$$\begin{aligned} &(a\cdot\omega)_x(L_1, \ldots, L_j) \\ &= a\cdot\omega_{a^{-1}x}\left(dl_{a^{-1}x}(L_1), \ldots, dl_{a^{-1}x}(L_j)\right) \end{aligned} \tag{2.13}$$

where

$$(a, \omega, x) \in G \times \Lambda^{0,j}(G/P, E_W) \times G/P\,,$$

each $L_l \in T_x(G/P)^{\mathbf{C}}$ and dl_{a_x} is the derivative of left translation $l_a\colon G/P \to G/P$ on G/P at x. Note that (2.13) generalizes the action of G on

$$\Gamma(E_W) = \Lambda^{0,0}(G/P, E_W)$$

given in (2.3). Because left translation is holomorphic the diagram

$$\begin{array}{ccc} \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ \Lambda^{0,j}(G/P, E_W) & \xrightarrow{\bar\partial} & \Lambda^{0,j+1}(G/P, E_W) \end{array}$$

is commutative for each a in G. Thus (2.13) induces a well-defined action of G on $H^{0,j}_{\bar\partial}(G/P, E_W)$. We may now write (2.8) as

$$\mathrm{Hom}_G(V, H^{0,j}_{\bar\partial}(G/P, E_W)) = H^j(p, p\cap\bar p, \mathrm{Hom}(V, W))\,. \tag{2.14}$$

Now assume that W is in fact irreducible. The parabolic subalgebra p has a decomposition $p = (p \cap \bar{p}) \oplus n$ into a reductive part $p \cap \bar{p}$ and a nilpotent part n = an ideal in p. By general principles

$$H^j(p, p \cap \bar{p}, \mathrm{Hom}(V, W)) = H^j(n, \mathrm{Hom}(V, W))^{p \cap \bar{p}}$$
$$= H^j(n, V^* \otimes W)^{p \cap \bar{p}} = (H^j(n\, V^*) \otimes W)^{p \cap \bar{p}}.$$

The last statement of equality follows by the irreducibility of W since by Lie's theorem, W is a trivial n module. Now

$$(H^j(n, V^*) \otimes W)^{p \cap \bar{p}} = \mathrm{Hom}_{p \cap \bar{p}}(W^*, H^j(n, V^*)).$$

From (2.14) we obtain (see [50]).

THEOREM 2.15 (Bott-Kostant reciprocity, 1960). *Let G, P be as in Theorem 2.7, let n be the nilradical of the parabolic subalgebra p, and let W be a finite dimensional irreducible holomorphic P module. Then for any finite dimensional holomorphic G module V we have*

$$\mathrm{Hom}_G(V, H^{0,j}_{\bar{\partial}}(G/P, E_W)) = \mathrm{Hom}_{p \cap \bar{p}}(W^*, H^j(n, V^*)). \tag{2.16}$$

Again $p \cap \bar{p}$ is the reductive part of p where the bar denotes conjugation of $G = K^{\mathbf{C}}$ with respect to a maximal compact subgroup K. We refer to (2.16) as "the debut of n cohomology"! Since 1960 it has played some rather important roles in both finite dimensional and infinite dimensional representation theory. There is an equivalent version of (2.16): The G module structure on $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ induced by (2.13) may be restricted to K. Let $\hat{K}$ denote, as usual, the equivalence classes of the irreducible unitary representations of K and let V_π be the representation space of $\pi \in \hat{K}$. Then we have (again for W irreducible).

THEOREM 2.17 (B. Kostant). *The decomposition of $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ as a K module is*

$$H^{0,j}_{\bar{\partial}}(G/P, E_W) = \sum_{\pi \in \hat{K}} V_\pi \otimes \mathrm{Hom}_{p \cap \bar{p}}(W^*, H^j(n, V_\pi^*)) \tag{2.18}$$
$$= \sum_{\pi \in \hat{K}} V_\pi^* \otimes \mathrm{Hom}_{p \cap \bar{p}}(W^*, H^j(n, V_\pi)).$$

In the direct sum on the right hand side the action of K on a summand is $\pi \otimes 1$ or $\pi^* \otimes 1$ in the second equation.

From (2.18) (or from (2.16)) we see that the multiplicity of an irreducible K module V_π in $H^{0,j}_{\bar{\partial}}(G/P, E_W)$ is governed precisely by the n cohomology

$H^j(n, V_\pi^*)$. Here, by analytic continuation, we consider V_π also as a representation of the complex Lie algebra of G. Its n module structure is the restriction thereof to n.

Remarks. (i) In contrast to remark (ii) made earlier, following Theorem 2.7, Theorems (2.15) and (2.17) do require that G/P should be Kahler.

(ii) One knows that K acts transitively on G/P so that G/P is diffeomorphic to $K/K \cap P$.

Now Kostant in [50] has computed the Lie algebra cohomology groups $H^j(n, V_\pi^*)$. Two outstanding consequences of his results, among others, which we shall briefly discuss are (a) Weyl's character formula and (b) Bott's generalized Borel-Weil theorem. Suppose more generally that g is any complex semisimple Lie algebra (for example g could be the Lie algebra of G above). Let $h \subset g$ be a Cartan subalgebra of g, let Δ be the set of non-zero roots of (g, h), and let Δ^+ be a choice of positive roots. The equivalence classes of finite dimensional irreducible representations of g (over the complex numbers) correspond univalently to linear functionals Λ on h which satisfy the condition that $2\dfrac{(\Lambda, \alpha)}{(\alpha, \alpha)}$ is a non-negative integer for each α in Δ^+. That is Λ is Δ^+ *dominant integral*; $(\ ,\ )$ denotes the Killing form on g. This is Cartan's highest weight theory alluded to in the introduction. Let π_Δ be a finite dimensional irreducible representation of g with corresponding highest weight $\Lambda \in h^*$. Its *character* $X_\Lambda : h \to \mathbf{C}$ is defined to be the function $H \to \operatorname{trace} \exp \pi_\Lambda(H)$, $H \in h$. This definition is independent of the choice of Cartan subalgebra h since any two are conjugate. We consider the special "minimal" parabolic subalgebra $p \subset g$ whose nilradical is

$$n = \sum_{\alpha \in \Delta^+} g_\alpha \tag{2.19}$$

and whose reductive part is h where g_α is the root space of $\alpha \in \Delta$. That is p is just the Borel subalgebra $h + n$. Let V_Λ denote the representation space of π_Λ. Then by restriction to n we again form the Lie algebra cohomology groups $H^j(n, V_\Lambda)$. Let θ denote the adjoint representation of h on Λn^*. Then $\theta \otimes \pi_\Lambda$ defines a representation of h on the cochain complex $\Lambda n^* \otimes V_\Lambda$. This h action commutes with the coboundary operator and therefore passes to cohomology. Applying the Euler-Poincaré principle one gets

$$\sum_{j=0}^{\dim n} (-1)^j \operatorname{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{\Lambda^j n^* \otimes V_\Lambda} = \sum_{j=0}^{\dim n} (-1)^j \operatorname{trace} \exp \theta \otimes \pi_\Lambda(H) \Big|_{H^j(n, V_\Lambda)} \tag{2.20}$$

for each H in h. One evaluates the left hand side of (2.20) by general principles and the right hand side using Kostant's main theorem, Theorem 5.14 of [50]. Actually Theorem 5.14 of [50] gives the h_1 module structure of $H^j(n_1, V_\Lambda)$ for an *arbitrary* parabolic $p_1 = h_1 + n_1$ of g with reductive and nilpotent parts h_1, n_1 respectively. For the derivation of Weyl's formula only the simplest case $p_1 = p = h + n$ is needed, where n is given in (2.19). Thus we shall state only a special case of Kostant's result.

THEOREM 2.21 (B. Kostant, 1960). *The decomposition of* $H^j(n, V_\Lambda)$ *as a* h *module is*

$$H^j(n, V_\Lambda) = \sum V_{\Lambda,\sigma},$$
$$\sigma \in \text{Weyl group } \mathscr{W} \text{ of } (g, h) \text{ such that } l(\sigma) = j,$$

where each summand $V_{\Lambda,\sigma}$ *in the direct sum is one-dimensional and* $H \in h$ *acts on* $V_{\Lambda,\sigma}$ *by the scalar* $[\sigma(\Lambda+\delta)-\delta](H)$.

Here by definition $2\delta = \sum_{\alpha \in \Delta^+} \alpha$ and $l(\sigma)$ (the *length* of σ) is the cardinality of the set $\Delta^+ \cap \sigma(-\Delta^+)$. From the remarks following (2.20) and the knowledge of n cohomology given by Theorem 2.21 one derives Weyl's famous character formula [93]:

THEOREM 2.22 (H. Weyl, 1926). *For* $H \in h$

$$X_\Lambda(H) = \frac{\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{[\sigma(\Lambda+\delta)](H)}}{\prod_{\alpha \in \Delta^+} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})}.$$

The denominator is also given by the sum $\sum_{\sigma \in \mathscr{W}} (\det \sigma)\, e^{(\sigma\delta)(H)}$ (this fact can be proved too using n cohomology) and $\det \sigma = (-1)^{l(\sigma)}$. As a corollary of Theorem 2.22 one obtains Weyl's formula for the dimension of the irreducible module V_Λ in terms of its highest weight Λ. The result is

$$\dim V_\Lambda = \frac{\prod_{\alpha \in \Delta^+} (\Lambda+\delta, \alpha)}{\prod_{\alpha \in \Delta^+} (\delta, \alpha)}. \tag{2.23}$$

Kostant's result on n cohomology can also be used to derive the generalized Borel-Weil theorem. Here one may apply formula (2.18) decisively. Let g now denote the Lie algebra of G. Extend a maximal abelian subalgebra of the Lie algebra of K to a Cartan subalgebra h of g. Again let $\Delta^+ \subset \Delta$ be a choice of positive roots where Δ is the set of non-zero roots of (g, h) and let $2\delta = \sum_{\alpha \in \Delta^+} \alpha$.

We choose the parabolic P such that its Lie algebra p contains the Borel subalgebra $h + \sum_{\alpha \in \Delta^+} g_{-\alpha} \cdot h$ is also a Cartan subalgebra of the reductive Lie algebra $p \cap \bar{p}$ so that we have the decompositions

$$p = (p \cap \bar{p}) \oplus n, \qquad p \cap \bar{p} = h + \sum_{\alpha \in \Delta(p \cap \bar{p})} g_\alpha$$

(2.24)

$$n = \sum_{\alpha \in \Delta^+ - \Delta(p \cap \bar{p})} g_{-\alpha}$$

where $\Delta(p \cap \bar{p})$ is the set of roots of $(p \cap \bar{p}, h)$.

Let W be an irreducible holomorphic P module. Then W is an irreducible $p \cap \bar{p}$ module thereby such that $n \cdot W = 0$. We let Λ denote its highest weight relative to the positive system $\Delta^+ \cap \Delta(p \cap \bar{p})$ for $p \cap \bar{p}$. Applying Kostant's n cohomology theorem to (2.18) one obtains (see [12], [50]).

THEOREM 2.25 (R. Bott, 1957). *The spaces* $H_{\bar{\partial}}^{0,j}(G/P, E_W)$ *vanish for all but at most one* j. *If*

$$H_{\bar{\partial}}^{0,j_0}(G/P, E_W) \neq 0$$

then $H_{\bar{\partial}}^{0,j_0}(G/P, E_W)$ *is an irreducible* K *module.*

More precisely we have the following. Let Λ be the highest weight of W (as above) relative to the positive roots in the reductive part of P. If $(\Lambda + \delta, \alpha) = 0$ for some α in Δ then $H_{\bar{\partial}}^{0,j}(G/P, E_W) = 0$ for every j. If $(\Lambda + \delta, \alpha) \neq 0$ for each α in Δ (i.e. $\Lambda + \delta$ is *regular*) there is a unique element σ in the Weyl group of (g, h) such that $(\sigma(\Lambda + \delta), \alpha) > 0$ for every $\alpha \in \Delta^+$. Then $H_{\bar{\partial}}^{0,j}(G/P, E_W) = 0$ for $j \neq l(\sigma)$ where again $l(\sigma)$ is the length of σ (see remarks following Theorem 2.21). Moreover $H_{\bar{\partial}}^{0,l(\sigma)}(G/P, E_W)$ is an irreducible K module (= an irreducible g module since g is the complexification of the Lie algebra of K) with highest weight $\sigma(\Lambda + \delta) - \delta$ relative to Δ^+.

Remarks. (i) By definition of σ it follows that

$$\sigma^{-1}\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\}.$$

Also since Λ is a highest weight $(\Lambda, \alpha) \geqslant 0$ for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}) \Rightarrow (\Lambda + \delta, \alpha) > 0$$

for

$$\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p}).$$

Hence

$$\{\alpha \in \Delta^+ \mid (\Lambda + \delta, \alpha) < 0\}$$
$$= \{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p})) \mid (\Lambda + \delta, \alpha) < 0\}$$

so that $l(\sigma)$ in Theorem 2.25 has the value

$$|\{\alpha \in \Delta^+ - (\Delta^+ \cap \Delta(p \cap \bar{p}))\,|\,(\Lambda+\delta, \alpha) < 0\}|\ ^{1)}.$$

$$\Delta^+ - \Delta^+ \cap \Delta(p \cap \bar{p})$$

is the set of roots in the nilradical of the "opposite" parabolic $\bar{p}$. Since

$$(\sigma(\Lambda+\delta), \sigma\alpha) = (\Lambda+\delta, \alpha) > 0$$

for $\alpha \in \Delta^+ \cap \Delta(p \cap \bar{p})$ (as we have just seen) we also conclude that the Weyl group element σ in Theorem 2.25 satisfies

$$\Delta^- \cap \Delta(p \cap \bar{p}) \subset \sigma^{-1}\Delta^-.$$

(ii) The irreducible holomorphic P modules W in the statement of Theorem 2.25 can be obtained as follows. Start with an arbitrary irreducible representation π of $P \cap K$ on a complex vector space W. Since $p \cap \bar{p}$ is the complexification of the Lie algebra of $P \cap K$, π defines a unique irreducible representation π on p such that $\pi(n) = 0$. This infinitesimal representation can be "integrated" to a representation of P since P and $P \cap K$ have the same fundamental groups. Thus every irreducible representation π of $P \cap K$ extends uniquely to an irreducible holomorphic representation of P. The highest weight Λ of π is integral and $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant. Conversely if G is simply connected, any integral $\Lambda \in h^*$ which is $\Delta^+ \cap \Delta(p \cap \bar{p})$ dominant is the highest weight of irreducible representation of $P \cap K$ and hence is the highest weight of an irreducible holomorphic representation of P.

(iii) Suppose in particular G is simply connected, p is chosen to be

$$h + \sum_{\alpha \in \Delta^+} g_{-\alpha},$$

and that Λ is Δ^+ dominant integral. Then in Theorem 2.25 $\sigma = 1$ so that the irreducible K, G or g module with highest weight Λ is given by $H_{\partial}^{0,0}(G/P, E_W)$ = space of holomorphic sections of the line bundle E_W. Indeed $\dim_{\mathbf{C}} W = 1$ since in this case $P \cap K$ is abelian. This gives the geometric realization of V_Λ [11].

---

[1]) $|S|$ denotes the cardinality of a set S.