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We can attempt to generalise the definition of the above vacuous hierarchy by allowing the number of "alternations" to increase with the number of indeterminates.

Let t be any polynomial. Define $t\text{-}D^0$ to be the class of t -computable families. For $i > 0$ let $t\text{-}D^i$ be the class of families that are defined by some family in $t\text{-}D^{i-1}$ in the sense of Definition 3. Finally PD^* is the class of all families P such that for some t

$$P = \{P_i \mid P_i = Q_i \text{ for some } Q \in t\text{-}D^{(i)}\}.$$

THEOREM 6. $PD^* = PD^1$

Proof. Similar to previous theorem. □

The above two results should be contrasted with the Boolean case where they still hold formally, but are no longer natural. The above definition of the successive levels PD^i is only natural if each level is a robust closure class. In Boolean algebra, however, PD^i is not known to be closed under complementation for any $i \geq 1$. Analogues of PD^i and PD^* where complementation is allowed at each level of alternation are not known to collapse, and are merely finite versions of the Meyer-Stockmeyer hierarchy, and PSPACE respectively [10].

A simple application of Theorem 5 is in recognising such polynomials as $\# HG$ as being p -definable. An intriguing open question is whether HG itself is p -definable for each F . If it is not then $P \neq NP$ (see Proposition 4 in [13]). If it is then the Meyer-Stockmeyer hierarchy and PSPACE can be simulated within p -definable families of polynomials.

6. UNIVERSALITY OF LINEAR PROGRAMMING

Here we consider a Boolean function family LP that corresponds to a linear programming problem and show that every p -computable family is the p -projection of it. Thus for computing discrete functions in polynomial time a package for LP for each input size is sufficient and no further programming is required. If we fix certain of the arguments of LP_i according to the particular function and input size being computed, the package becomes a program for the required function. That LP is itself p -computable follows from the recent result of Khachian [8].

The reader should note that several tractable problems in combinatorial optimisation are already known to have linear programming formula-

tions [9]. Our result shows that this is a universal phenomenon. It is related to the result in [3].

We define $LP_{2n(n+1)}$ to be the following Boolean function of arguments $\{a_{ij}, b_{ij}, e_i, d_i \mid 1 \leq i, j \leq n\}$:

$$LP(a_{ij}, b_{ij}, e_i, d_i) = 1$$

if and only if the set of inequalities

$$\sum (\tilde{a}_{ij}x_j - \tilde{b}_{ij}x_j) \geq \tilde{e}_i - \tilde{d}_i$$

has a solution in real numbers, where each number $\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{e}_i, \tilde{d}_i$ is 1 or 0 according to whether the corresponding Boolean variable a_{ij}, b_{ij}, e_i, d_i is 1 or 0.

THEOREM 7. *Any p -computable family P of Boolean functions is the p -projection of LP .*

Proof. Consider some $P_m \in P$ with indeterminates y_1, \dots, y_m , and a minimal program for it. The latter consists of a sequence of instructions of the form $v_i \leftarrow v_j \wedge v_k$ and $v_i \leftarrow v_j \vee v_k$, where $1 \leq i \leq C$ and each v_n with $n \leq 0$ equals some y_r or \bar{y}_r .

For any fixed assignment of truth values to y_1, \dots, y_m we can define a set E_0 of linear inequalities:

$$E_0 = \{x_r \leq 0 \mid r < 0 \text{ and } v_r \text{ has value } 0\} \\ \cup \{x_r \geq 1 \mid r < 0 \text{ and } v_r \text{ has value } 1\}$$

For each sequence v_1, v_2, \dots, v_i we define E_i by induction from E_0 :

$$E_i = \begin{cases} E_{i-1} \cup \{x_j - x_i \geq 0, x_k - x_i \geq 0, x_i + 1 - x_j - x_k \geq 0\} \\ \quad \text{if } v_i \leftarrow v_j \wedge v_k, \\ E_{i-1} \cup \{x_j + x_k - x_i \geq 0, x_i - x_j \geq 0, x_i - x_k \geq 0\} \\ \quad \text{if } v_i \leftarrow v_j \vee v_k \end{cases}$$

Claim 1. For any i, j ($j < i$) every solution of E_i has $x_j \leq 0$, or every solution of E_i has $x_j \geq 1$.

Proof. The claim is true for E_0 by definition. Assume inductively that it is true for E_{i-1} . (a) If $v_i \leftarrow v_j \wedge v_k$ then $x_j \leq 0$ implies that $x_i \leq 0$ since $x_j - x_i \geq 0$. Similarly if $x_k \leq 0$. In the remaining case $x_j, x_k \geq 1$ inequality $x_i + 1 - x_j - x_k \geq 0$ ensures that $x_i \geq 1$. (b) If $v_i \leftarrow v_j \vee v_k$ then $x_j \geq 1$

implies that $x_i \geq 1$ since $x_i - x_j \geq 0$. Similarly if $x_k \geq 1$. If $x_j, x_k \leq 0$ then $x_j + x_k - x_i \geq 0$ ensures that $x_i \leq 0$. \square

Claim 2. If $\text{val}(v_i) = 0$ then $E_i \cup \{x_i \leq 0\}$ has a solution. If $\text{val}(v_i) = 1$ then $E_i \cup \{x_i \geq 1\}$ has a solution.

Proof. By induction on i it is easy to see that the point

$$x_j = \begin{cases} 1 & \text{if } \text{val}(v_j) = 1 \\ 0 & \text{if } \text{val}(v_j) = 0 \end{cases}$$

for $1 \leq j \leq i$ is a solution of E_i . \square

Claim 3. If for some i, j ($j \leq i$) $E_i \cup \{x_j \geq 1\}$ has a solution in reals then $\text{val}(v_j) = 1$.

Proof. By Claim 1, if $E_i \cup \{x_j \geq 1\}$ has a solution then $E_i \cup \{x_j \leq 0\}$ has no solution. Hence by Claim 2 $\text{val}(v_j) = 1$. \square

Finally we observe that the given program of size C for P_m translates to $3C + 2m$ inequalities in E_C , of which the $2m$ of E_o depend on the values of y_1, \dots, y_m , while the remaining $3C$ are fixed. It remains to note that P_m is the projection under σ of $LP_{2n(n+1)}$ for $n = 3C + 2m$, where σ maps $3C$ of the inequalities to those of $E_C - E_o$, and the remaining $2m$ values of i as follows. If v_i equals y_j or \bar{y}_j then: $\sigma(a_{ik}) = \sigma(b_{ik}) = 0$ if $j \neq k$, $\sigma(d_i) = 0$, $\sigma(a_{ij}) = \sigma(e_i) = v_i$, $\sigma(b_{ij}) = \bar{v}_i$. \square

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APPENDIX 1

We show here that in the concept of p -definability it is immaterial whether the defining polynomials allowed are the p -computable ones or merely those of p -bounded formula size. We shall suppose that the family P is p -definable in the sense of Definition 3, i.e.

$$P_n(x_1, \dots, x_n) = \sum_{b \in \{0,1\}^{m-n}} Q_m(x_1, \dots, x_n, b_{n+1}, \dots, b_m)$$

It will suffice to prove that any p -computable family, such as Q , is p -definable in the sense of Definition 4. By Theorem 5 it then follows that P itself is also p -definable in the sense of Definition 4.