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We can attempt to generalise the definition of the above vacuous hierarchy by allowing the number of "alternations" to increase with the number of indeterminates.

Let t be any polynomial. Define t- $D^0$  to be the class of t-computable families. For i > 0 let t- $D^i$  be the class of families that are defined by some family in t- $D^{i-1}$  in the sense of Definition 3. Finally  $PD^*$  is the class of all families P such that for some t

$$P = \{P_i \mid P_i = Q_i \text{ for some } Q \in t\text{-}D^{t(i)}\}$$
.

THEOREM 6.  $PD^* = PD^1$ 

*Proof.* Similar to previous theorem.

The above two results should be contrasted with the Boolean case where they still hold formally, but are no longer natural. The above definition of the successive levels  $PD^i$  is only natural if each level is a robust closure class. In Boolean algebra, however,  $PD^i$  is not known to be closed under complementation for any  $i \ge 1$ . Analogues of  $PD^i$  and  $PD^*$  where complementation is allowed at each level of alternation are not known to collapse, and are merely finite versions of the Meyer-Stockmeyer hierarchy, and PSPACE respectively [10].

A simple application of Theorem 5 is in recognising such polynomials as # HG as being p-definable. An intriguing open question is whether HG itself is p-definable for each F. If it is not then  $P \neq NP$  (see Proposition 4 in [13]). If it is then the Meyer-Stockmeyer hierarchy and PSPACE can be simulated within p-definable families of polynomials.

# 6. Universality of Linear Programming

Here we consider a Boolean function family LP that corresponds to a linear programming problem and show that every p-computable family is the p-projection of it. Thus for computing discrete functions in polynomial time a package for LP for each input size is sufficient and no further programming is required. If we fix certain of the arguments of  $LP_i$  according to the particular function and input size being computed, the package becomes a program for the required function. That LP is itself p-computable follows from the recent result of Khachian [8].

The reader should note that several tractable problems in combinatorial optimisation are already known to have linear programming formula-

tions [9]. Our result shows that this is a universal phenomenon. It is related to the result in [3].

We define  $LP_{2n(n+1)}$  to be the following Boolean function of arguments  $\{a_{ij}, b_{ij}, e_i, d_i \mid 1 \le i, j \le n\}$ :

$$LP(a_{ij}, b_{ij}, e_i, d_i) = 1$$

if and only if the set of inequalities

$$\sum_{i=1}^{\infty} (a_{ij}x_j - \tilde{b}_{ij}x_j) \geqslant \tilde{e}_i - \tilde{d}_i$$

has a solution in real numbers, where each number  $a_{ij}$ ,  $b_{ij}$ ,  $e_i$ ,  $d_i$  is 1 or 0 according to whether the corresponding Boolean variable  $a_{ij}$ ,  $b_{ij}$ ,  $e_i$ ,  $d_i$  is 1 or 0.

Theorem 7. Any p-computable family P of Boolean functions is the p-projection of LP.

*Proof.* Consider some  $P_m \in P$  with indeterminates  $y_1, ..., y_m$ , and a minimal program for it. The latter consists of a sequence of instructions of the form  $v_i \leftarrow v_j \wedge v_k$  and  $v_i \leftarrow v_j \vee v_k$ , where  $1 \le i \le C$  and each  $v_n$  with  $n \le 0$  equals some  $y_r$  or  $\bar{y}_r$ .

For any fixed assignment of truth values to  $y_1, ..., y_m$  we can define a set  $E_0$  of linear inequalities:

$$E_0 = \{x_r \le 0 \mid r < 0 \text{ and } v_r \text{ has value } 0\}$$
  
 
$$\cup \{x_r \ge 1 \mid r < 0 \text{ and } v_r \text{ has value } 1\}$$

For each sequence  $v_1, v_2, ..., v_i$  we define  $E_i$  by induction from  $E_0$ :

$$E_{i} = \begin{cases} E_{i-1} \cup \{x_{j} - x_{i} \geqslant 0, x_{k} - x_{i} \geqslant 0, x_{i} + 1 - x_{j} - x_{k} \geqslant 0\} \\ \text{if } v_{i} \leftarrow v_{j} \wedge v_{k} \\ E_{i-1} \cup \{x_{j} + x_{k} - x_{i} \geqslant 0, x_{i} - x_{j} \geqslant 0, x_{i} - x_{k} \geqslant 0\} \\ \text{, if } v_{i} \leftarrow v_{j} \vee v_{k} \end{cases}$$

Claim 1. For any i, j (j < i) every solution of  $E_i$  has  $x_j \le 0$ , or every solution of  $E_i$  has  $x_j \ge 1$ .

*Proof.* The claim is true for  $E_o$  by definition. Assume inductively that it is true for  $E_{i-1}$ . (a) If  $v_i \leftarrow v_j \wedge v_k$  then  $x_j \leqslant 0$  implies that  $x_i \leqslant 0$  since  $x_j - x_i \geqslant 0$ . Similarly if  $x_k \leqslant 0$ . In the remaining case  $x_j, x_k \geqslant 1$  inequality  $x_i + 1 - x_j - x_k \geqslant 0$  ensures that  $x_i \geqslant 1$ . (b) If  $v_i \leftarrow v_j \vee v_k$  then  $x_j \geqslant 1$ 

implies that  $x_i \ge 1$  since  $x_i - x_j \ge 0$ . Similarly if  $x_k \ge 1$ . If  $x_j, x_k \le 0$  then  $x_j + x_k - x_i \ge 0$  ensures that  $x_i \le 0$ .

Claim 2. If val  $(v_i) = 0$  then  $E_i \cup \{x_i \le 0\}$  has a solution. If val  $(v_i) = 1$  then  $E_i \cup \{x_i \ge 1\}$  has a solution.

*Proof.* By induction on i it is easy to see that the point

$$x_j = \begin{cases} 1 & \text{if val } (v_j) = 1 \\ 0 & \text{if val } (v_j) = 0 \end{cases}$$

for  $1 \le j \le i$  is a solution of  $E_i$ .

Claim 3. If for some  $i, j (j \le i)$   $E_i \cup \{x_j \ge 1\}$  has a solution in reals then val  $(v_i) = 1$ .

*Proof.* By Claim 1, if  $E_i \cup \{x_j \ge 1\}$  has a solution then  $E_i \cup \{x_j \le 0\}$  has no solution. Hence by Claim 2 val  $(v_j) = 1$ .

Finally we observe that the given program of size C for  $P_m$  translates to 3C + 2m inequalities in  $E_C$ , of which the 2m of  $E_o$  depend on the values of  $y_1, ..., y_m$ , while the remaining 3C are fixed. It remains to note that  $P_m$  is the projection under  $\sigma$  of  $LP_{2n(n+1)}$  for n = 3C + 2m, where  $\sigma$  maps 3C of the inequalities to those of  $E_C - E_o$ , and the remaining 2m values of i as follows. If  $v_i$  equals  $y_j$  or  $\bar{y}_j$  then:  $\sigma(a_{ik}) = \sigma(b_{ik}) = 0$  if  $j \neq k$ ,  $\sigma(d_i) = 0$ ,  $\sigma(a_{ij}) = \sigma(e_i) = v_i$ ,  $\sigma(b_{ij}) = \bar{v}_i$ .

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## APPENDIX 1

We show here that in the concept of p-definability it is immaterial whether the defining polynomials allowed are the p-computable ones or merely those of p-bounded formula size. We shall suppose that the family P is p-definable in the sense of Definition 3, i.e.

$$P_n(x_1, ..., x_n) = \sum_{b \in \{0,1\}^{m-n}} Q_m(x_1, ..., x_n, b_{n+1}, ..., b_m)$$

It will suffice to prove that any p-computable family, such as Q, is p-definable in the sense of Definition 4. By Theorem 5 it then follows that P itself is also p-definable in the sense of Definition 4.