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FROBENIUS RECIPROCITY AND LIE GROUP REPRESENTATIONS ON $\bar{\partial}$ COHOMOLOGY SPACES [1]

by Floyd L. WILLIAMS

To the memory of Dr. Walter R. Talbot

1. INTRODUCTION

The general theme of this lecture is Lie group representations on complex $\bar{\partial}$ cohomology spaces and extensions thereby of the classical Frobenius reciprocity theorem. Particular but not exclusive attention will be focused on the representations of semisimple groups. We shall survey some results ranging, historically, from the Borel-Weil-Bott-Kostant theorem to the recent theorem of W. Schmid which confirms the Kostant-Langlands conjecture. We shall also discuss, along these lines, recent results of Moscovici, Verona, Rosenberg and Penney for nilpotent groups. Before developing the particular ideas of the lecture we begin with some broader remarks which may serve as a more general frame of reference.

The finite dimensional representation theory of compact semisimple Lie groups is now a well established chapter in classical mathematics. The theory is due to E. Cartan and H. Weyl [16], [93]. Using non-algebraic methods, Weyl showed the complete reducibility of all (finite dimensional) representations. That is, every representation is the direct sum of irreducible representations. A modern algebraic proof of this fact can be accomplished using Lie algebra cohomology. Cartan classified the irreducible representations by setting up a 1-1 correspondence with the equivalence classes of such representations and the so-called dominant integral linear forms on a Cartan subalgebra of the Lie algebra of the group. This is the celebrated "highest weight" theory. Cartan's case by case approach depended on the classification of the simple Lie algebras. A more

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[1]) This is an expanded version of an invited address delivered at the 769th meeting of the American Mathematical Society in Washington D.C. on October 20, 1979.

uniform and elegant treatment was given later by Harish-Chandra [18] and C. Chevalley (unpublished), independently. Harish-Chandra's arguments were simplified by N. Jacobson [48].

Another triumph of the theory was the derivation by Weyl of his famous character formula [93]. This gives in particular the dimension of an irreducible module in terms of its highest weight and it permits an explicit determination of the Plancherel theorem, and thus the harmonic analysis, for compact Lie groups. As first noted by Bott and Kostant [12], [50], the character formula may also be derived by cohomological considerations. It is in fact, in the proper context, an Euler-Poincaré formula expressing the equality of an alternating sum of traces on the cochain and cohomology level.

Given a compact semisimple group K and the irreducible representations of it, corresponding to highest weights, there naturally arises the question of realizing these representations in some concrete manner. By the principle of analytic continuation (Weyl's "unitary trick") we may consider equivalently the irreducible holomorphic representations of the complexification $K^{\mathbf{C}}$, when K is simply connected. One particular realization is given by the classical Borel-Weil theorem [11], [12] which asserts, in effect, that all such representations of $K^{\mathbf{C}}$ occur as the modules of holomorphic sections of appropriate homogeneous line bundles. That is, the modules occur "in zero dimensional cohomology". The Borel-Weil theorem may be viewed therefore as a type of imprimitivity theorem in the frame-work of holomorphic induction. We remark that the irreducible representations of compact groups definitely are *not* induced in the sense of Frobenius and Mackey, although the corresponding statement is valid, say for simply connected nilpotent Lie groups. However *in a holomorphic context* analogues of the Frobenius identity remain valid. [1]) The validity of this identity for non-compact groups as well for representations "in higher cohomology" is precisely the focal point of this lecture.

For a non-compact semisimple group G one must consider infinite dimensional representations (in contrast with the compact case there are no finite dimensional unitary representations of such a group other than the trivial one-dimensional representation) for the purposes of harmonic analysis. Much attention has been devoted in this direction in recent years. The foundations here have been layed principally by Harish-Chandra in some profoundly deep work. In a long and deep study of the local structure of invariant eigendistributions, which he showed to be locally integrable functions which are even analytic on a dense open set in G, Harish-Chandra constructed all the characters (these are

[1]) For compact groups this observation is due to Bott.

distributions on G) of the *discrete series* of representations of G, for G which has a compact Cartan subgroup H. By definition the latter representations are the irreducible subrepresentations of $L^2(G)$. Their existence, as shown by Harish-Chandra, coincides precisely with the existence in G of such a subgroup [26]. These discrete series representations are, in a very definite sense, the analogues (in the non-compact case) of the irreducible representations of compact groups. In fact by the Peter-Weyl theorem all of the irreducible representations of a compact group K occur in $L^2(K)$. Also the discrete series satisfy the "orthogonality relations" which for irreducible representations of compact groups are well-known.

The great importance of the discrete series representations is the following (which is in fact Harish-Chandra's guiding principle in his approach to the harmonic analysis on G): Every Cartan subgroup of G makes a specific contribution to the Plancherel formula which is determined by the discrete series of a suitable reductive group. The contributions of the compact Cartan subgroups (when they exist) is to the discrete part of the Plancherel formula.

Just as in the compact case, there naturally arises the question of realizing these discrete series representations (or even certain "continuous series"). It was Kostant and Langlands [51], [54] who first suggested that, in analogy with the Borel-Weil theorem, the discrete series representations should occur on L^2-cohomology spaces of homogeneous holomorphic line bundles over the complex manifold G/H. This important conjecture was recently confirmed by W. Schmid [82]. In our discussion of Schmid's results, we shall see (as in the compact case) a holomorphic version of Frobenius reciprocity.

Besides the work of Schmid, fundamental break-throughs towards the verification of the Kostant-Langlands conjecture appeared in the work of M. S. Narasimhan and K. Okamoto [60]. An alternative to the L^2-cohomology realization of the discrete series is the realization by means of invariant differential operators. In this regard very attractive results have been obtained in particular by R. Parthasarathy by way of the Dirac operator [65]. Recent analogues of the Kostant-Langlands conjecture for nilpotent Lie groups have been proposed and proved by H. Moscovici, A. Verona, J. Rosenberg and R. Penney [59], [74] [69]. For the realizations of non-discrete series representations of reductive groups on "partially holomorphic cohomology spaces" the reader may consult the important AMS Memoir of J. Wolf [99].

This lecture is dedicated to the memory of my undergraduate mathematics teacher and dear friend Dr. Walter R. Talbot.