ON THE GENUS OF GENERALIZED FLAG MANIFOLDS

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INTRODUCTION

Let X be a nilpotent space of finite type. We denote by $G(X)$ the genus of X, i.e. the set of all homotopy types Y (nilpotent, of finite type) with p-localizations $Y_p \simeq X_p$ for all primes p, (cf. [HMR]). The set $G(X)$ has been studied extensively in case of X an H-space. In particular it is known that for the special unitary group $SU(n)$ one has

$$|G(SU(n))| \geqslant \prod_{1<m<n} (\phi(m!)/2)$$

where ϕ is the Euler function [Z, p. 152]. We are interested in this note in finding non-trivial examples X with $G(X) = \{[X]\}$ and we call such spaces *generically rigid*. A large family of such generically rigid spaces is provided by certain generalized flag manifolds. Let

$$G = U(n_1 + n_2 + \dots + n_k)$$

and

$$H = U(n_1) \times U(n_2) \times \dots \times U(n_k),$$

embedded in G in the obvious way. Then

$$M = M(n_1, n_2, \dots, n_k) = G/H$$

is a generalized flag manifold (generalizing the standard complex flag manifold $U(n)/T^n$ which corresponds to $M(1, 1, \dots, 1)$). We will show essentially that whenever the homotopy rigidity result for linear actions holds for M (cf. [L1], [L2], [EL]), then M is also generically rigid. These two seemingly unrelated rigidity results are tied up by certain results on $E(X)$ and $E(X_0)$, the groups of homotopy classes of self equivalences of X and X_0, X_0 the rationalization of X.

To make our result more precise, we need some further notation. For

$$M = M(n_1, \dots, n_k) = G/H$$

as above, we write $N(H)$ for the normalizer of H in G. The finite group $N(H)/H$ acts on M in an obvious way and it is well known that through that action, $N(H)/H$ is faithfully represented in $H^*(M; \mathbf{Q})$. We can therefore consider $N(H)/H$ as a subgroup of $E(M)$ or $E(M_0)$. By Theorem 1.1 of [GH2] the canonical map

$$E(M_0) \to \mathrm{Aut}_{alg} H^*(M; \mathbf{Q})$$

is a group isomorphism. In particular, the grading automorphisms

$$g(q): H^*(M; \mathbf{Q}) \to H^*(M; \mathbf{Q})$$

defined by $g(q)x = q^i x$ for $x \in H^{2i}(M; \mathbf{Q})$ and $q \in \mathbf{Q}^*$, lift to unique self equivalences of M_0 (which we denote also by $g(q)$), and thus

$$Gr(M_0) = \{g(q) \mid q \in \mathbf{Q}^*\} \subset E(M_0)$$

is a central subgroup isomorphic to $\mathbf{Q}^*$.

In all cases of generalized flag manifolds for which $E(M_0)$ has been computed, the subgroup generated by $Gr(M_0)$ and $N(H)/H$,

$$\langle Gr(M_0), N(H)/H \rangle \subset E(M_0)$$

is all of $E(M_0)$. The following conjecture is thus plausible.

Conjecture C. Let $M = M(n_1, n_2, \ldots, n_k)$ be a generalized flag manifold. Then

$$E(M_0) = \langle Gr(M_0), N(H)/H \rangle .$$

A similar conjecture appears in [L1, Conjecture C] but the relationship between the two conjectures is not entirely clear.

The Conjecture C has been verified in the following cases:

1) $n_1 = n_2 = \ldots = n_k = 1$ (compare the proof of Thm. 1 in [EL])

2) $n_1 = n_2 = \ldots = n_{k-1} = 1, n_k \geqslant k - 1$ (compare the proof of Theorem 9 in [L1])

3) $n_1 = 2$ and $k = 2$ (follows from [O])

4) $n_2 > n_1$ and $k = 2$ ([GH1], [Br])

5) $n_1 = 1, n_2 > 1, n_3 \geqslant 2n_2^2 - 1$ and $k = 3$ ([GH2])

The Conjecture C holds therefore for instance for all complex Grassmann manifolds $G_p(\mathbf{C}^{p+q}) = M(p, q)$ with $p \neq q$ (since $M(p, q) \simeq M(q, p)$), and for the classical flag manifolds $U(n)/T^n$.

Our main theorem may be stated as follows.

THEOREM. *Let* $M = M(n_1, ..., n_k)$ *be a generalized flag manifold for which the Conjecture C holds. Then*

$$G(M) = \{[M]\} .$$

In particular the Grassmann manifolds $G_p(\mathbf{C}^{p+q})$ *for* $p \neq q$ *and the flag manifolds* $U(n)/T^n$ *are all generically rigid.*

§1. GENUS AND SELF MAPS

Let P denote a fixed set of primes. Two P-sequences

$$S_1, S_2 : P \to E(X_0)$$

are called *equivalent*, if there exist maps $h(0) \in E(X_0)$ and

$$h(p) \in \operatorname{im}\left(E(X_p) \xrightarrow{\text{can}} E(X_0)\right)$$

such that for all $p \in P$ one has

$$h(0)\, S_1(p) = S_2(p)\, h(p) .$$

Definition 1.1. We denote by P-Seq $\left(E(X_0)\right)$ the set of equivalence classes of P-sequences in $E(X_0)$.

If P is a finite set of primes and X a nilpotent space of finite type, then there is a canonical map

$$\theta : G(X) \to P\text{-Seq}\left(E(X_0)\right) .$$

It is defined as follows. Let $Y \in G(X)$ and $P = \{p_1, ..., p_n\}$. Then the localization Y_P is a pull-back of maps $X_{p_i} \xrightarrow{\lambda_i} X_0$, i.e. $Y_P \simeq \operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\lambda_i} X_0\}$. The maps λ_i induce equivalences $\bar{\lambda}_i \in E(X_0)$ and we put

$$\theta(Y) = \{[\bar{\lambda}_1, \bar{\lambda}_2, ..., \bar{\lambda}_n]\} .$$

If Y_P may also be represented by $\operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\mu_i} X_0\}$, then there exist maps $h(0) \in E(X_0)$ and $\tilde{h}(p_i) \in E(X_{p_i})$, $i \in \{1, ..., n\}$ rendering the diagrams

$$\begin{array}{ccc} X_{p_i} & \xrightarrow{\tilde{h}(p_i)} & X_{p_i} \\ \lambda_i \downarrow & & \downarrow \mu_i \\ X_0 & \xrightarrow[h(0)]{} & X_0 \end{array}$$

homotopy commutative and thus inducing hoinvlim $\{\lambda_i\} \simeq$ hoinvlim $\{\mu_i\}$. Hence

$$\{[\bar{\lambda}_1, \ldots, \bar{\lambda}_n]\} = \{[\bar{\mu}_1, \ldots, \bar{\mu}_n]\} \in P\text{-Seq}\,(E\,(X_0))$$

and therefore θ is well defined.

LEMMA 1.2. Let X be a nilpotent space of finite type and let P denote a finite set of primes. Then

$$\theta : G\,(X) \to P\text{-Seq}\,(E\,(X_0))$$

is surjective with fibers of the form

$$\theta^{-1}\,(\theta\,(Y)) = \{Z \in G\,(X) \mid Z_P \simeq Y_P\}\,.$$

Proof. Let $P = \{p_1, \ldots, p_n\}$ and

$$\{[\bar{f}_1, \ldots, \bar{f}_n]\} \in P\text{-Seq}\,(E\,(X_0))\,.$$

Let $e_i : X_{p_i} \to X_0$ denote the canonical maps and put

$$f_i = \bar{f}_i \circ e_i : X_{p_i} \to X_0\,.$$

Define $W =$ hoinvlim $\{f_i\}$; W comes equipped with a canonical map $f : W \to X_0$. Let Z be the homotopy pull back of $W \xrightarrow{f} X_0 \xleftarrow{\text{can}} X_{\bar{P}}$, where $\bar{P}$ denotes the set of primes complementary to P. Then $Z \in G\,(X)$ and

$$\theta\,(Z) = \{[\bar{f}_1, \ldots, \bar{f}_n]\}\,;$$

thus θ is surjective. It is clear from the definition of θ that for $Y, Z \in G\,(X)$ one has $\theta\,(Y) = \theta\,(Z)$ if and only if $Y_P \simeq Z_P$.

The next lemma provides a sufficient condition for θ to be monic "at the basepoint".

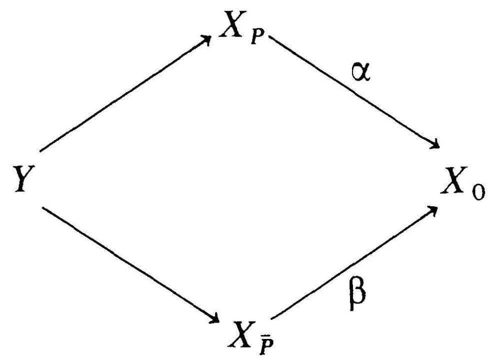
LEMMA 1.3. Let X be a nilpotent space of finite type. Suppose that there exists a finite set of primes P with complement $\bar{P}$ such that

a) $Y \in G(X)$ implies $Y_{\bar{P}} \simeq X_{\bar{P}}$

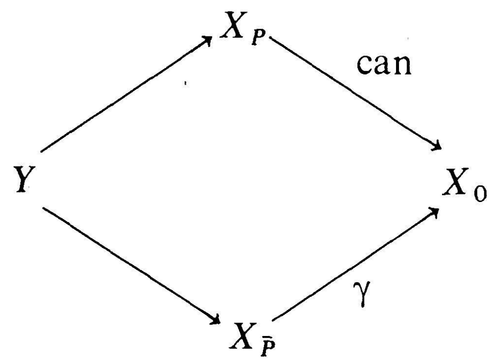
b) every $f \in E(X_0)$ can be written as $f_1 \circ f_2$ with $f_1 \in \operatorname{im}\left(E(X_P) \overset{\text{can}}{\rightarrow} E(X_0)\right)$ and $f_2 \in \operatorname{im}\left(E(X_{\bar{P}}) \rightarrow E(X_0)\right)$.

Then for $\theta: G(X) \rightarrow P\text{-Seq}\left(E(X_0)\right)$ as above, one has $\theta^{-1}(\theta(X)) = \{X\}$.

Proof. Let $Y \in G(X)$ with $\theta(Y) = \theta(X)$. Then $Y_P \simeq X_P$ by the definition of θ, and $Y_{\bar{P}} \simeq X_{\bar{P}}$ by assumption. Hence Y may be obtained as a homotopy pull back of the form



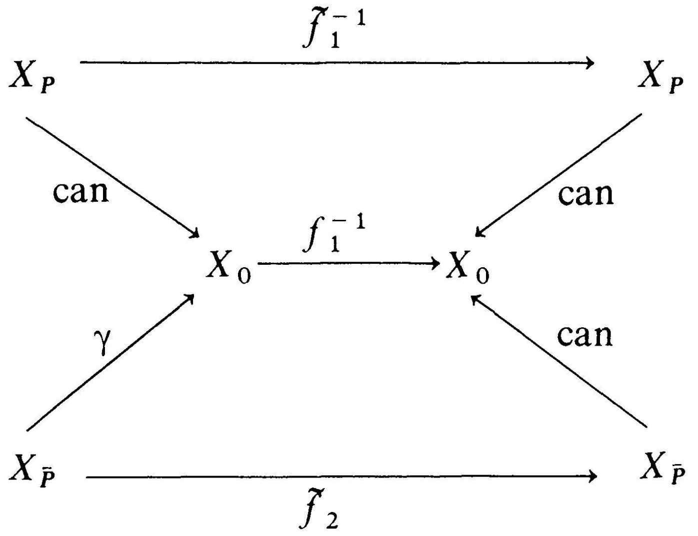
If α induces $\bar{\alpha} \in E(X_0)$ and if $\gamma = \bar{\alpha}^{-1} \circ \beta$, then Y is also a pull back of the form



Let $\bar{\gamma} \in E(X_0)$ be the map induced by γ and write $\bar{\gamma} = f_1 f_2$ with

$$f_1 \in \operatorname{im}\left(E(X_P) \rightarrow E(X_0)\right), \quad f_2 \in \operatorname{im}\left(E(X_{\bar{P}}) \rightarrow E(X_0)\right).$$

Choose a lift $\tilde{f}_1^{-1} \in E(X_P)$ of f_1^{-1} and a lift $\tilde{f}_2 \in E(X_{\bar{P}})$ of f_2. Then $f_1^{-1}\gamma = \text{can} \circ \tilde{f}_2$ and one can form a commutative diagram,



which shows that $Y \simeq X$.

§2. The case of generalized flag manifolds

The following result is an easy consequence of [F].

Lemma 2.1. Let M be a generalized flag manifold. Then the following holds.

a) If $g(\lambda) \in Gr(M_0)$ is a grading map with $\lambda \in \mathbf{Z}_Q^*$ for some (not necessarily finite) set of primes Q, then $g(\lambda)$ lifts to a homotopy equivalence $\tilde{g}(\lambda): M_Q \to M_Q$.

b) Let P be an arbitrary set of primes with complement $\bar{P}$. Then every

$$f \in \langle Gr(M_0), N(H)/H \rangle$$

may be written in the form $f = f_1 \circ f_2$ with

$$f_1 \in \mathrm{im}\,\big(E(M_P) \to E(M_0)\big)$$

and

$$f_2 \in \mathrm{im}\,\big(E(M_{\bar{P}}) \to E(M_0)\big)\,.$$

Proof. Let $\lambda = k/l$ with k and l relatively prime integers. Then $g(k)$ and $g(l)$ lift to equivalences

$$\tilde{g}(k), \tilde{g}(l): M_Q \to M_Q$$

since necessarily $k, l \in \mathbf{Z}_Q^*$ (compare [F]). Thus $\tilde{g}(k)\, \tilde{g}(l)^{-1}$ is a lift of $g(\lambda)$. For b) we note that $f = g(\rho) \circ \sigma$ for some $\rho \in \mathbf{Q}^*$ and

$$\sigma \in N(H)/H .$$

If we write $\rho = \rho_1 \cdot \rho_2$ with $\rho_1 \in \mathbf{Z}_P^*$ and $\rho_2 \in \mathbf{Z}_{\bar{P}}^*$, then

$$f = g(\rho_1) \cdot (g(\rho_2)\, \sigma)$$

and we may choose

$$f_1 = g(\rho_1),\ f_2 = g(\rho_2)\, \sigma .$$

Since σ lifts even to $E(M)$, we infer by using a) that f_1 and f_2 lift as desired.

A final step towards proving the Theorem formulated in the introduction consists in the following.

Lemma 2.2. Let M be a generalized flag manifold for which Conjecture C holds. Then for every finite set of primes P,

$$P\text{-Seq}\,(E(M_0)) = \{[1, 1, ..., 1]\} .$$

Proof. Let $\{[\mu_1, ..., \mu_n]\} \in P\text{-Seq}\,(E(M_0))$, where $P = \{p_1, ..., p_n\}$ and

$$\mu_i \in \operatorname{im}\,(E(M_{p_i}) \to E(M_0))$$

for all i. Then $\mu_i = g(\lambda_i) \circ \sigma_i$ with $\lambda_i \in \mathbf{Q}^*$ and

$$\sigma_i \in N(H)/H \subset E(M_0) .$$

Define $\lambda \in \mathbf{Q}^*$ by $\lambda = \Pi p_i^{m_i}$, where $m_i \in \mathbf{Z}$ is such that $p_i^{m_i} \lambda_i \in \mathbf{Z}_{p_i}^*$. Then $g(\lambda)\, \mu_i = g(\lambda\lambda_i)\, \sigma_i$ with $\lambda\lambda_i \in \mathbf{Z}_{p_i}^*$. By Lemma 2.1 a) we know that $g(\lambda\lambda_i)$ lifts to M_{p_i}, and since σ_i lifts even to M we conclude that

$$h(p_i) = g(\lambda\lambda_i)\, \sigma_i \in \operatorname{im}\,(E(M_{p_i}) \to E(M_0))$$

for all i. The equation

$$g(\lambda)\, \mu_i = h(p_i),\ i \in \{1, ..., n\}$$

show that $\{[\mu_1, ..., \mu_n]\} = \{[1, ..., 1]\} \in P\text{-Seq}\,(E(M_0))$.

The proof of the main Theorem:

Let M be a generalized flag manifold for which the Conjecture C holds. Since M is a formal space we can find for every $N \in G(M)$ a rational equivalence

$f(N): N \to M$. Let $P(M)$ denote the set of primes which appear in any of the orders of

$$\ker\big(f(N)_*: H_*(N; \mathbf{Z}) \to H_*(M; \mathbf{Z})\big)$$

or coker $f(N)_*$, N ranging over $G(M)$. The set $P(M)$ is finite, since each ker $f(N)_*$ and coker $f(N)_*$ is finite and since $G(M)$ is a finite set by [W]. Consider now the map

$$\theta: G(M) \to P\text{-Seq } E(M_0)$$

with respect to this finite set of primes $P(M) = P$. Since P is finite,

$$P\text{-Seq}\,\big(E(M_0)\big)$$

consists of only one element (Lemma 2.2). It remains to show that

$$\theta^{-1}\big(\theta(M)\big) = \{M\}.$$

For this we apply Lemma 1.3. Note that $N \in G(M)$ implies $N_{\bar{P}} \simeq M_{\bar{P}}$ since $f(N): N \to M$ is a $\bar{P}$-equivalence. Moreover, the condition b) of 1.3 is satisfied in view of Lemma 2.1 b). Therefore we conclude that $G(M) = \{[M]\}$ and the proof is completed.

Note added in proof. Since this paper went to press, we have been informed that Conjecture C has been proved for the case $k = 2, n_1 = n_2$, by M. Hoffman: "Cohomology endomorphisms of complex flag manifolds", Ph.D. dissertation, MIT 1981. As a consequence, it follows that all complex Grassmann manifolds are generically rigid.

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