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ON THE GENUS OF GENERALIZED FLAG MANIFOLDS

by Henry H. GLOVER and Guido MISLIN

INTRODUCTION

Let X be a nilpotent space of finite type. We denote by $G(X)$ the genus of X , i.e. the set of all homotopy types Y (nilpotent, of finite type) with p -localizations $Y_p \simeq X_p$ for all primes p , (cf. [HMR]). The set $G(X)$ has been studied extensively in case of X an H -space. In particular it is known that for the special unitary group $SU(n)$ one has

$$|G(SU(n))| \geq \prod_{1 \leq m \leq n} (\phi(m!)/2)$$

where ϕ is the Euler function [Z, p. 152]. We are interested in this note in finding non-trivial examples X with $G(X) = \{[X]\}$ and we call such spaces *generically rigid*. A large family of such generically rigid spaces is provided by certain generalized flag manifolds. Let

$$G = U(n_1 + n_2 + \dots + n_k)$$

and

$$H = U(n_1) \times U(n_2) \times \dots \times U(n_k),$$

embedded in G in the obvious way. Then

$$M = M(n_1, n_2, \dots, n_k) = G/H$$

is a generalized flag manifold (generalizing the standard complex flag manifold $U(n)/T^n$ which corresponds to $M(1, 1, \dots, 1)$). We will show essentially that whenever the homotopy rigidity result for linear actions holds for M (cf. [L1], [L2], [EL]), then M is also generically rigid. These two seemingly unrelated rigidity results are tied up by certain results on $E(X)$ and $E(X_0)$, the groups of homotopy classes of self equivalences of X and X_0 , X_0 the rationalization of X .

To make our result more precise, we need some further notation. For

$$M = M(n_1, \dots, n_k) = G/H$$

as above, we write $N(H)$ for the normalizer of H in G . The finite group $N(H)/H$ acts on M in an obvious way and it is well known that through that action, $N(H)/H$ is faithfully represented in $H^*(M; \mathbf{Q})$. We can therefore consider $N(H)/H$ as a subgroup of $E(M)$ or $E(M_0)$. By Theorem 1.1 of [GH2] the canonical map

$$E(M_0) \rightarrow \text{Aut}_{\text{alg}} H^*(M; \mathbf{Q})$$

is a group isomorphism. In particular, the grading automorphisms

$$g(q): H^*(M; \mathbf{Q}) \rightarrow H^*(M; \mathbf{Q})$$

defined by $g(q)x = q^i x$ for $x \in H^{2i}(M; \mathbf{Q})$ and $q \in \mathbf{Q}^*$, lift to unique self equivalences of M_0 (which we denote also by $g(q)$), and thus

$$\text{Gr}(M_0) = \{g(q) \mid q \in \mathbf{Q}^*\} \subset E(M_0)$$

is a central subgroup isomorphic to \mathbf{Q}^* .

In all cases of generalized flag manifolds for which $E(M_0)$ has been computed, the subgroup generated by $\text{Gr}(M_0)$ and $N(H)/H$,

$$\langle \text{Gr}(M_0), N(H)/H \rangle \subset E(M_0)$$

is all of $E(M_0)$. The following conjecture is thus plausible.

Conjecture C. Let $M = M(n_1, n_2, \dots, n_k)$ be a generalized flag manifold. Then

$$E(M_0) = \langle \text{Gr}(M_0), N(H)/H \rangle.$$

A similar conjecture appears in [L1, Conjecture C] but the relationship between the two conjectures is not entirely clear.

The Conjecture C has been verified in the following cases:

- 1) $n_1 = n_2 = \dots = n_k = 1$ (compare the proof of Thm. 1 in [EL])
- 2) $n_1 = n_2 = \dots = n_{k-1} = 1, n_k \geq k - 1$ (compare the proof of Theorem 9 in [L1])
- 3) $n_1 = 2$ and $k = 2$ (follows from [O])
- 4) $n_2 > n_1$ and $k = 2$ ([GH1], [Br])
- 5) $n_1 = 1, n_2 > 1, n_3 \geq 2n_2^2 - 1$ and $k = 3$ ([GH2])

The Conjecture C holds therefore for instance for all complex Grassmann manifolds $G_p(\mathbf{C}^{p+q}) = M(p, q)$ with $p \neq q$ (since $M(p, q) \simeq M(q, p)$), and for the classical flag manifolds $U(n)/T^n$.

Our main theorem may be stated as follows.

THEOREM. Let $M = M(n_1, \dots, n_k)$ be a generalized flag manifold for which the Conjecture C holds. Then

$$G(M) = \{[M]\}.$$

In particular the Grassmann manifolds $G_p(\mathbb{C}^{p+q})$ for $p \neq q$ and the flag manifolds $U(n)/T^n$ are all generically rigid.

§1. GENUS AND SELF MAPS

Let P denote a fixed set of primes. Two P -sequences

$$S_1, S_2: P \rightarrow E(X_0)$$

are called *equivalent*, if there exist maps $h(0) \in E(X_0)$ and

$$h(p) \in \text{im}(E(X_p) \xrightarrow{\text{can}} E(X_0))$$

such that for all $p \in P$ one has

$$h(0) S_1(p) = S_2(p) h(p).$$

Definition 1.1. We denote by $P\text{-Seq}(E(X_0))$ the set of equivalence classes of P -sequences in $E(X_0)$.

If P is a finite set of primes and X a nilpotent space of finite type, then there is a canonical map

$$\theta: G(X) \rightarrow P\text{-Seq}(E(X_0)).$$

It is defined as follows. Let $Y \in G(X)$ and $P = \{p_1, \dots, p_n\}$. Then the localization Y_P is a pull-back of maps $X_{p_i} \xrightarrow{\lambda_i} X_0$, i.e. $Y_P \simeq \text{hoinvlim} \{X_{p_i} \xrightarrow{\lambda_i} X_0\}$. The maps λ_i induce equivalences $\bar{\lambda}_i \in E(X_0)$ and we put

$$\theta(Y) = \{[\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n]\}.$$

If Y_P may also be represented by $\text{hoinvlim} \{X_{p_i} \xrightarrow{\mu_i} X_0\}$, then there exist maps $h(0) \in E(X_0)$ and $\tilde{h}(p_i) \in E(X_{p_i})$, $i \in \{1, \dots, n\}$ rendering the diagrams

$$\begin{array}{ccc}
 X_{p_i} & \xrightarrow{\tilde{h}(p_i)} & X_{p_i} \\
 \lambda_i \downarrow & & \downarrow \mu_i \\
 X_0 & \xrightarrow{h(0)} & X_0
 \end{array}$$

homotopy commutative and thus inducing $\text{hoinvlim } \{\lambda_i\} \simeq \text{hoinvlim } \{\mu_i\}$. Hence

$$\{[\bar{\lambda}_1, \dots, \bar{\lambda}_n]\} = \{[\bar{\mu}_1, \dots, \bar{\mu}_n]\} \in P\text{-Seq}(E(X_0))$$

and therefore θ is well defined.

LEMMA 1.2. Let X be a nilpotent space of finite type and let P denote a finite set of primes. Then

$$\theta: G(X) \rightarrow P\text{-Seq}(E(X_0))$$

is surjective with fibers of the form

$$\theta^{-1}(\theta(Y)) = \{Z \in G(X) \mid Z_P \simeq Y_P\}.$$

Proof. Let $P = \{p_1, \dots, p_n\}$ and

$$\{[\bar{f}_1, \dots, \bar{f}_n]\} \in P\text{-Seq}(E(X_0)).$$

Let $e_i: X_{p_i} \rightarrow X_0$ denote the canonical maps and put

$$f_i = \bar{f}_i \circ e_i: X_{p_i} \rightarrow X_0.$$

Define $W = \text{hoinvlim } \{f_i\}$; W comes equipped with a canonical map $f: W \rightarrow X_0$. Let Z be the homotopy pull back of $W \xrightarrow{f} X_0 \xleftarrow{\text{can}} X_{\bar{P}}$, where \bar{P} denotes the set of primes complementary to P . Then $Z \in G(X)$ and

$$\theta(Z) = \{[\bar{f}_1, \dots, \bar{f}_n]\};$$

thus θ is surjective. It is clear from the definition of θ that for $Y, Z \in G(X)$ one has $\theta(Y) = \theta(Z)$ if and only if $Y_P \simeq Z_P$.

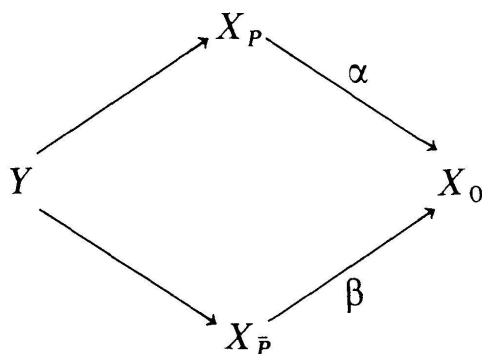
The next lemma provides a sufficient condition for θ to be monic "at the basepoint".

LEMMA 1.3. Let X be a nilpotent space of finite type. Suppose that there exists a finite set of primes P with complement \bar{P} such that

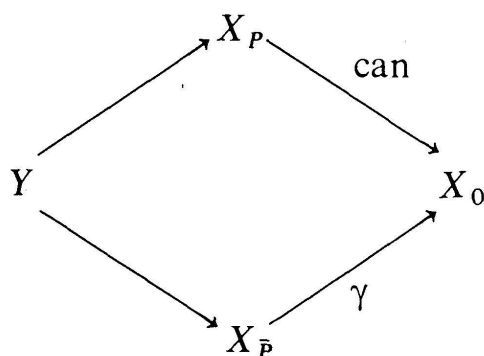
- a) $Y \in G(X)$ implies $Y_{\bar{P}} \simeq X_{\bar{P}}$
- b) every $f \in E(X_0)$ can be written as $f_1 \circ f_2$ with $f_1 \in \text{im}(E(X_P) \xrightarrow{\text{can}} E(X_0))$ and $f_2 \in \text{im}(E(X_{\bar{P}}) \rightarrow E(X_0))$.

Then for $\theta: G(X) \rightarrow P\text{-Seq}(E(X_0))$ as above, one has $\theta^{-1}(\theta(X)) = \{X\}$.

Proof. Let $Y \in G(X)$ with $\theta(Y) = \theta(X)$. Then $Y_P \simeq X_P$ by the definition of θ , and $Y_{\bar{P}} \simeq X_{\bar{P}}$ by assumption. Hence Y may be obtained as a homotopy pull back of the form



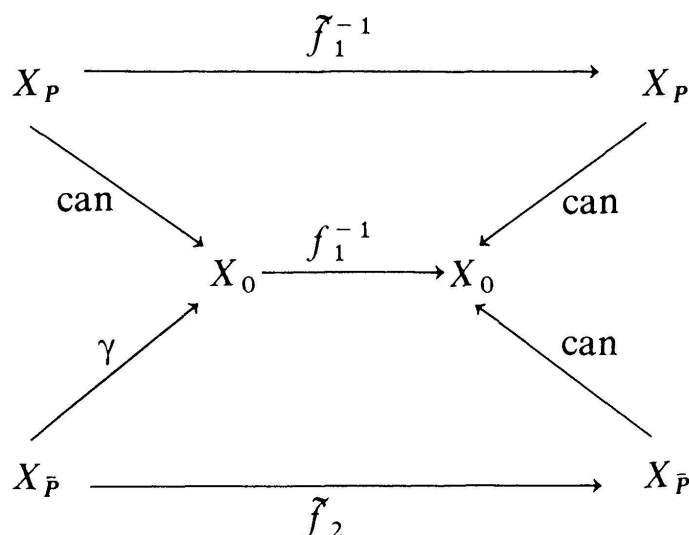
If α induces $\bar{\alpha} \in E(X_0)$ and if $\gamma = \bar{\alpha}^{-1} \circ \beta$, then Y is also a pull back of the form



Let $\bar{\gamma} \in E(X_0)$ be the map induced by γ and write $\bar{\gamma} = f_1 f_2$ with

$$f_1 \in \text{im}(E(X_P) \rightarrow E(X_0)), \quad f_2 \in \text{im}(E(X_{\bar{P}}) \rightarrow E(X_0)).$$

Choose a lift $\tilde{f}_1^{-1} \in E(X_P)$ of f_1^{-1} and a lift $\tilde{f}_2 \in E(X_{\bar{P}})$ of f_2 . Then $f_1^{-1} \gamma = \text{can} \circ \tilde{f}_2$ and one can form a commutative diagram,



which shows that $Y \simeq X$.

§2. THE CASE OF GENERALIZED FLAG MANIFOLDS

The following result is an easy consequence of [F].

LEMMA 2.1. Let M be a generalized flag manifold. Then the following holds.

- If $g(\lambda) \in Gr(M_0)$ is a grading map with $\lambda \in \mathbb{Z}_Q^*$ for some (not necessarily finite) set of primes Q , then $g(\lambda)$ lifts to a homotopy equivalence $\tilde{g}(\lambda): M_Q \rightarrow M_Q$.
- Let P be an arbitrary set of primes with complement \bar{P} . Then every

$$f \in \langle Gr(M_0), N(H)/H \rangle$$

may be written in the form $f = f_1 \circ f_2$ with

$$f_1 \in \text{im}(E(M_P) \rightarrow E(M_0))$$

and

$$f_2 \in \text{im}(E(M_{\bar{P}}) \rightarrow E(M_0)).$$

Proof. Let $\lambda = k/l$ with k and l relatively prime integers. Then $g(k)$ and $g(l)$ lift to equivalences

$$\tilde{g}(k), \tilde{g}(l): M_Q \rightarrow M_Q.$$

since necessarily $k, l \in \mathbf{Z}_Q^*$ (compare [F]). Thus $\tilde{g}(k) \tilde{g}(l)^{-1}$ is a lift of $g(\lambda)$. For b) we note that $f = g(\rho) \circ \sigma$ for some $\rho \in \mathbf{Q}^*$ and

$$\sigma \in N(H)/H.$$

If we write $\rho = \rho_1 \cdot \rho_2$ with $\rho_1 \in \mathbf{Z}_P^*$ and $\rho_2 \in \mathbf{Z}_P^*$, then

$$f = g(\rho_1) \cdot (g(\rho_2) \sigma)$$

and we may choose

$$f_1 = g(\rho_1), f_2 = g(\rho_2) \sigma.$$

Since σ lifts even to $E(M)$, we infer by using a) that f_1 and f_2 lift as desired.

A final step towards proving the Theorem formulated in the introduction consists in the following.

LEMMA 2.2. Let M be a generalized flag manifold for which Conjecture C holds. Then for every finite set of primes P ,

$$P\text{-Seq}(E(M_0)) = \{[1, 1, \dots, 1]\}.$$

Proof. Let $\{[\mu_1, \dots, \mu_n]\} \in P\text{-Seq}(E(M_0))$, where $P = \{p_1, \dots, p_n\}$ and

$$\mu_i \in \text{im}(E(M_{p_i}) \rightarrow E(M_0))$$

for all i . Then $\mu_i = g(\lambda_i) \circ \sigma_i$ with $\lambda_i \in \mathbf{Q}^*$ and

$$\sigma_i \in N(H)/H \subset E(M_0).$$

Define $\lambda \in \mathbf{Q}^*$ by $\lambda = \prod p_i^{m_i}$, where $m_i \in \mathbf{Z}$ is such that $p_i^{m_i} \lambda_i \in \mathbf{Z}_{p_i}^*$. Then $g(\lambda) \mu_i = g(\lambda \lambda_i) \sigma_i$ with $\lambda \lambda_i \in \mathbf{Z}_{p_i}^*$. By Lemma 2.1 a) we know that $g(\lambda \lambda_i)$ lifts to M_{p_i} , and since σ_i lifts even to M we conclude that

$$h(p_i) = g(\lambda \lambda_i) \sigma_i \in \text{im}(E(M_{p_i}) \rightarrow E(M_0))$$

for all i . The equation

$$g(\lambda) \mu_i = h(p_i), i \in \{1, \dots, n\}$$

show that $\{[\mu_1, \dots, \mu_n]\} = \{[1, \dots, 1]\} \in P\text{-Seq}(E(M_0))$.

The proof of the main Theorem:

Let M be a generalized flag manifold for which the Conjecture C holds. Since M is a formal space we can find for every $N \in G(M)$ a rational equivalence

$f(N): N \rightarrow M$. Let $P(M)$ denote the set of primes which appear in any of the orders of

$$\ker(f(N)_*: H_*(N; \mathbb{Z}) \rightarrow H_*(M; \mathbb{Z}))$$

or $\text{coker } f(N)_*$, N ranging over $G(M)$. The set $P(M)$ is finite, since each $\ker f(N)_*$ and $\text{coker } f(N)_*$ is finite and since $G(M)$ is a finite set by [W]. Consider now the map

$$\theta: G(M) \rightarrow P\text{-Seq } E(M_0)$$

with respect to this finite set of primes $P(M) = P$. Since P is finite,

$$P\text{-Seq } (E(M_0))$$

consists of only one element (Lemma 2.2). It remains to show that

$$\theta^{-1}(\theta(M)) = \{M\}.$$

For this we apply Lemma 1.3. Note that $N \in G(M)$ implies $N_{\bar{P}} \simeq M_{\bar{P}}$ since $f(N): N \rightarrow M$ is a \bar{P} -equivalence. Moreover, the condition b) of 1.3 is satisfied in view of Lemma 2.1 b). Therefore we conclude that $G(M) = \{[M]\}$ and the proof is completed.

Note added in proof. Since this paper went to press, we have been informed that Conjecture C has been proved for the case $k = 2, n_1 = n_2$, by M. Hoffman: "Cohomology endomorphisms of complex flag manifolds", Ph.D. dissertation, MIT 1981. As a consequence, it follows that all complex Grassmann manifolds are generically rigid.

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