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Kapitel: 8. Character sum analogues of (1), (1a) and (1b)

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Now replace u by tu to get

$$(28) \qquad E = \sum_{u,t} \chi_1\chi_2\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(1-u)\,\{1 + \phi(ut)\}$$

$$= J(\chi_1\chi_2\chi_3, \bar{\chi}_1\bar{\chi}_2)\,J(\bar{\chi}_1, \chi_1\chi_2) + J(\chi_1\chi_2\chi_3\phi, \bar{\chi}_1\bar{\chi}_2)\,J(\bar{\chi}_1\phi, \chi_1\chi_2),$$

where the Jacobi sums J are defined above (18). Since $\chi_1^2, \chi_2^2, \chi_3^2$, and $\chi_1\chi_2$ are nontrivial, (5) now follows from (18).

Remark. If $\chi_1\chi_2$, $\chi_1\chi_3$, or $\chi_2\chi_3$ is trivial, we can easily evaluate E directly from its definition. Otherwise, E can be evaluated simply from (28).

8. Character sum analogues of (1), (1a) and (1b)

Let $\chi_1, \chi_2, \chi_3, \phi$ be characters on $GF(q)$, where ϕ has order 2, $p > 2$. Set $t_0 = 1$. The discriminant of the polynomial

$$F(y) = \sum_{i=0}^{n} t_i\, y^{n-i}$$

is a polynomial in $t_1, \ldots, t_n$ which shall be denoted by D_n. Write

$$E_n = \sum_{i=0}^{n} (-1)^i\, t_i\,.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each $n \geqslant 1$:

$$(29) \qquad \sum_{t_1,\ldots,t_n \in GF(q)} \chi_1(t_n)\,\chi_2(E_n)\,\chi_3\phi(D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1})\,G(\chi_1\chi_3^j)\,G(\chi_2\chi_3^j)}{G(\chi_3)\,G(\chi_1\chi_2\chi_3^{n+j-1})},$$

provided that the n characters $\chi_1\chi_2\chi_3^{n+j-1}$ $(0 \leqslant j \leqslant n-1)$ are all nontrivial;

$$(29a) \qquad \sum_{t_1,\ldots,t_n \in GF(q)} \chi_1(t_n)\,\chi_3\phi(D_n)\,\zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1})\,G(\chi_1\chi_3^j)}{G(\chi_3)}$$

for all χ_1, χ_3; and

$$(29b) \qquad \sum_{t_1,\ldots,t_n \in GF(q)} \chi_3\phi(D_n)\,\zeta^{\frac{p+1}{2}T(t_1^2 - 2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2)\,G(\phi)\,G(\chi_3^{j+1})}{G(\chi_3)}$$

for all χ_3.

Formulas (29), (29a), and (29b) have been verified by computer for some small primes q with $n = 3, 4$. Of course the formulas are well known for $n = 1$. For

$n = 2$, (29a) and (29b) are not hard to prove, and (29) follows from the proof of (4). (For example, if χ_3 in (29) is trivial, one makes use of (19) and (21)-(23).)

For $n = 3$, we can prove (29b), but not (29) or (29a). The *ad hoc* proof given below appears to shed little light on the general case.

THEOREM. For $n = 3$, (29b) holds.

Proof. All rational fractions below are to be interpreted as integers (mod p); for example, $\frac{1}{2}$ equals $(p+1)/2$. We must show that

$$(30) \qquad A = \sum_{t,u,v} \chi\phi(D_3)\,\zeta^{T\left(\frac{v^2}{2}-u\right)} = -\frac{\phi(2)\,G^3(\phi)\,G(\chi^3)\,G(\chi^2)}{G^2(\chi)}$$

for any character χ on $GF(q)$, where

$$D_3 = u^2v^2 - 4u^3 - 4tv^3 - 27t^2 + 18tuv\,.$$

First suppose that $p = 3$. Then

$$\begin{aligned} A &= \sum_{t,u,v} \chi\phi(u^2v^2-u^3-tv^3)\,\zeta^{T\left(\frac{v^2}{2}-u\right)} \\ &= \sum_{v\neq 0}\sum_{u} \zeta^{T\left(\frac{v^2}{2}-u\right)} \sum_{t} \chi\phi(u^2v^2-u^3-t) + \sum_{t,u} \phi\chi^3(-u)\,\zeta^{T(-u)} \\ &= 0 - q\,G(\phi\chi^3) = -q\,G(\chi\phi)\,, \end{aligned}$$

since $G(\psi^p) = G(\psi)$ for any character ψ. Now (30) follows with the aid of (26).

Next, suppose that $p > 3$. Completing the square in t, one has

$$D_3/27 = c - \left(t + \frac{2v^3}{27} - \frac{uv}{3}\right)^2,$$

where $c = \frac{4}{27}\left(\frac{v^2}{3} - u\right)^3$. Thus

$$\begin{aligned} \bar{\chi}\phi(27)\,A &= \sum_{t,u,v} \chi\phi(c-t^2)\,\zeta^{T\left(\frac{v^2}{2}-u\right)} \\ &= \sum_{u,v} \zeta^{T\left(\frac{v^2}{2}-u\right)} \sum_{t} \chi\phi(c-t)\cdot\{1 + \phi(t)\} \\ &= \sum_{u,v} \zeta^{T\left(\frac{v^2}{2}-u\right)} \sum_{t} \chi\phi(c-t)\,\phi(t) \\ &= K \sum_{\substack{u,v \\ c=0}} \zeta^{T\left(\frac{v^2}{2}-u\right)} + J \sum_{u,v} \chi(c)\,\zeta^{T\left(\frac{v^2}{2}-u\right)}, \end{aligned}$$

where $K = \chi\phi(-1) \sum_t \chi(t)$ and $J = \sum_t \chi\phi(1-t)\,\phi(t)$.

Thus

$$\bar{\chi}\phi(27)\,A = K \sum_v \zeta^{T\left(\frac{v^2}{6}\right)} + \chi\left(\frac{4}{27}\right) J \sum_{u,v} \chi^3\left(\frac{v^2}{3} - u\right) \zeta^{T\left(\frac{v^2}{2} - u\right)}.$$

Replace u by $u + \frac{v^2}{3}$ to obtain

$$\begin{aligned}\bar{\chi}\phi(27)\,A &= -K\phi(6)\,G(\phi) + \chi\left(\frac{4}{27}\right) J \sum_{u,v} \chi^3(-u)\,\zeta^{T\left(\frac{v^2}{6} - u\right)} \\ &= \phi(6)\,G(\phi)\left\{-K + \chi\left(\frac{4}{27}\right) JG(\chi^3)\right\}.\end{aligned}$$

If χ is trivial, then $K = \phi(-1)(q-1)$, $J = -\phi(-1)$, and $G(\chi^3) = 1$, and the desired result (30) follows. If χ is nontrivial, then $K = 0$ and

$$J = -G(\chi\phi)\,G(\phi)/G(\chi)$$

by (18), and (30) follows with the aid of (26).

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