Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1, χ_2, or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1}, then replacing v by vu^{-1}, to obtain

$$S_1 = \sum_{u,\, v} \bar{\chi}_1\bar{\chi}_2\bar{\chi}_3^2(u)\,\chi_3(1+u^2+v^2-2u-2v-2uv)\,\chi_2(v)\,.$$

Finally, suppose that χ_1, χ_2, χ_3^2, and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. Proof of (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,\, y \neq 0 \\ x+y \neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right)\chi_2\chi_3\left(\frac{1+y}{x}\right)\,\chi_1\chi_2(y-x)\,.$$

Set $t = \dfrac{1+x}{y}$, $\quad u = \dfrac{1+y}{x}$, so

$$x = \frac{t+1}{ut-1}, \qquad y = \frac{u+1}{ut-1}\,.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,\, t \neq -1 \\ ut \neq 1}} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2\left(\frac{t-u}{1-ut}\right) \\ &= \sum_{u,\, t \neq -1} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,. \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,\, t \neq 0} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,.$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,\, t \neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1^2(u)\,\chi_1\chi_2(t-u^2)\,\bar{\chi}_1\bar{\chi}_2(1-t) \\ &= \sum_{u,\, t \neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(t-u)\,\{1 + \phi(u)\}\,. \end{aligned}$$

Now replace u by tu to get

$$(28)\qquad E = \sum_{u,t} \chi_1\chi_2\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(1-u)\,\{1+\phi(ut)\}$$

$$= J(\chi_1\chi_2\chi_3, \bar{\chi}_1\bar{\chi}_2)\,J(\bar{\chi}_1, \chi_1\chi_2) + J(\chi_1\chi_2\chi_3\phi, \bar{\chi}_1\bar{\chi}_2)\,J(\bar{\chi}_1\phi, \chi_1\chi_2),$$

where the Jacobi sums J are defined above (18). Since $\chi_1^2, \chi_2^2, \chi_3^2$, and $\chi_1\chi_2$ are nontrivial, (5) now follows from (18).

Remark. If $\chi_1\chi_2$, $\chi_1\chi_3$, or $\chi_2\chi_3$ is trivial, we can easily evaluate E directly from its definition. Otherwise, E can be evaluated simply from (28).

8. Character sum analogues of (1), (1a) and (1b)

Let $\chi_1, \chi_2, \chi_3, \phi$ be characters on $GF(q)$, where ϕ has order 2, $p > 2$. Set $t_0 = 1$. The discriminant of the polynomial

$$F(y) = \sum_{i=0}^{n} t_i\, y^{n-i}$$

is a polynomial in $t_1, \ldots, t_n$ which shall be denoted by D_n. Write

$$E_n = \sum_{i=0}^{n} (-1)^i\, t_i\,.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each $n \geqslant 1$:

$$(29)\qquad \sum_{t_1,\ldots,t_n\in GF(q)} \chi_1(t_n)\,\chi_2(E_n)\,\chi_3\phi(D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1})\,G(\chi_1\chi_3^j)\,G(\chi_2\chi_3^j)}{G(\chi_3)\,G(\chi_1\chi_2\chi_3^{n+j-1})},$$

provided that the n characters $\chi_1\chi_2\chi_3^{n+j-1}$ $(0 \leqslant j \leqslant n-1)$ are all nontrivial;

$$(29\text{a})\qquad \sum_{t_1,\ldots,t_n\in GF(q)} \chi_1(t_n)\,\chi_3\phi(D_n)\,\zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1})\,G(\chi_1\chi_3^j)}{G(\chi_3)}$$

for all χ_1, χ_3; and

$$(29\text{b})\qquad \sum_{t_1,\ldots,t_n\in GF(q)} \chi_3\phi(D_n)\,\zeta^{\frac{p+1}{2}T(t_1^2-2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2)\,G(\phi)\,G(\chi_3^{j+1})}{G(\chi_3)}$$

for all χ_3.

Formulas (29), (29a), and (29b) have been verified by computer for some small primes q with $n = 3, 4$. Of course the formulas are well known for $n = 1$. For