

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: IDENTITIES FOR PRODUCTS OF GAUSS SUMS OVER FINITE FIELDS
Autor: Evans, Ronald J.
Kapitel: 7. Proof of (5)
DOI: <https://doi.org/10.5169/seals-51748>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 13.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1 , χ_2 , or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1} , then replacing v by vu^{-1} , to obtain

$$S_1 = \sum_{u,v} \bar{\chi}_1 \bar{\chi}_2 \bar{\chi}_3^2(u) \chi_3(1+u^2+v^2-2u-2v-2uv) \chi_2(v).$$

Finally, suppose that χ_1 , χ_2 , χ_3^2 , and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. PROOF OF (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,y \neq 0 \\ x+y \neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right) \chi_2\chi_3\left(\frac{1+y}{x}\right) \chi_1\chi_2(y-x).$$

Set $t = \frac{1+x}{y}$, $u = \frac{1+y}{x}$, so

$$x = \frac{t+1}{ut-1}, \quad y = \frac{u+1}{ut-1}.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,t \neq -1 \\ ut \neq 1}} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2\left(\frac{t-u}{1-ut}\right) \\ &= \sum_{u,t \neq -1} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2(1-ut). \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,t \neq 0} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2(1-ut).$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,t \neq 0} \chi_1\chi_3\left(\frac{t}{u}\right) \bar{\chi}_1^2(u) \chi_1\chi_2(t-u^2) \bar{\chi}_1 \bar{\chi}_2(1-t) \\ &= \sum_{u,t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1 \bar{\chi}_2(1-t) \bar{\chi}_1(u) \chi_1\chi_2(t-u) \{1 + \phi(u)\}. \end{aligned}$$

Now replace u by tu to get

$$(28) \quad E = \sum_{u,t} \chi_1 \chi_2 \chi_3 (t) \bar{\chi}_1 \bar{\chi}_2 (1-t) \bar{\chi}_1 (u) \chi_1 \chi_2 (1-u) \{1 + \phi (ut)\} \\ = J (\chi_1 \chi_2 \chi_3, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1, \chi_1 \chi_2) + J (\chi_1 \chi_2 \chi_3 \phi, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1 \phi, \chi_1 \chi_2),$$

where the Jacobi sums J are defined above (18). Since $\chi_1^2, \chi_2^2, \chi_3^2$, and $\chi_1 \chi_2$ are nontrivial, (5) now follows from (18).

Remark. If $\chi_1 \chi_2, \chi_1 \chi_3$, or $\chi_2 \chi_3$ is trivial, we can easily evaluate E directly from its definition. Otherwise, E can be evaluated simply from (28).

8. CHARACTER SUM ANALOGUES OF (1), (1a) AND (1b)

Let $\chi_1, \chi_2, \chi_3, \phi$ be characters on $GF(q)$, where ϕ has order 2, $p > 2$. Set $t_0 = 1$. The discriminant of the polynomial

$$F(y) = \sum_{i=0}^n t_i y^{n-i}$$

is a polynomial in t_1, \dots, t_n which shall be denoted by D_n . Write

$$E_n = \sum_{i=0}^n (-1)^i t_i.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each $n \geq 1$:

$$(29) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_2 (E_n) \chi_3 \phi (D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j) G(\chi_2 \chi_3^j)}{G(\chi_3) G(\chi_1 \chi_2 \chi_3^{n+j-1})},$$

provided that the n characters $\chi_1 \chi_2 \chi_3^{n+j-1}$ ($0 \leq j \leq n-1$) are all nontrivial;

$$(29a) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_3 \phi (D_n) \zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j)}{G(\chi_3)}$$

for all χ_1, χ_3 ; and

$$(29b) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_3 \phi (D_n) \zeta^{\frac{p+1}{2} T(t_1^2 - 2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2) G(\phi) G(\chi_3^{j+1})}{G(\chi_3)}$$

for all χ_3 .

Formulas (29), (29a), and (29b) have been verified by computer for some small primes q with $n = 3, 4$. Of course the formulas are well known for $n = 1$. For