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Since  $\chi_1\chi_2\chi_3\phi$  and  $\chi_3\phi$  are nontrivial, (19) now follows from (18) and (27).

*Remark.* We evaluated  $S$  (the left side of (4)) only under the assumption that  $\chi_1\chi_2\chi_3^2$  and  $(\chi_1\chi_2\chi_3)^2$  were nontrivial. We now indicate how  $S$  can be simply evaluated in terms of Gauss sums when this assumption is dropped. If  $\chi_1, \chi_2$ , or  $\chi_3^2$  is trivial, one can easily evaluate  $S$  directly from its definition. If  $\chi_1\chi_2\chi_3^2$  is trivial, then one can evaluate  $S_1$  (and hence  $S$ ) from (20), by first replacing  $u$  by  $u^{-1}$ , then replacing  $v$  by  $vu^{-1}$ , to obtain

$$S_1 = \sum_{u, v} \bar{\chi}_1 \bar{\chi}_2 \bar{\chi}_3^2(u) \chi_3(1+u^2+v^2-2u-2v-2uv) \chi_2(v).$$

Finally, suppose that  $\chi_1, \chi_2, \chi_3^2$ , and  $\chi_1\chi_2\chi_3^2$  are nontrivial. Then  $S_1$  can be evaluated from (27).

## 7. PROOF OF (5)

Let  $E$  denote the left side of (5). Since  $\chi_1\chi_2$  is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x, y \neq 0 \\ x+y \neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right) \chi_2\chi_3\left(\frac{1+y}{x}\right) \chi_1\chi_2(y-x).$$

Set  $t = \frac{1+x}{y}$ ,  $u = \frac{1+y}{x}$ , so

$$x = \frac{t+1}{ut-1}, \quad y = \frac{u+1}{ut-1}.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u, t \neq -1 \\ ut \neq 1}} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2\left(\frac{t-u}{1-ut}\right) \\ &= \sum_{u, t \neq -1} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1\bar{\chi}_2(1-ut). \end{aligned}$$

Since  $\chi_1\chi_3$  and  $\chi_2\chi_3$  are nontrivial,

$$E = \sum_{u, t \neq 0} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1\bar{\chi}_2(1-ut).$$

Replace  $t$  by  $t/u$  to obtain

$$\begin{aligned} E &= \sum_{u, t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1^2(u) \chi_1\chi_2(t-u^2) \bar{\chi}_1\bar{\chi}_2(1-t) \\ &= \sum_{u, t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1\bar{\chi}_2(1-t) \bar{\chi}_1(u) \chi_1\chi_2(t-u) \{1 + \phi(u)\}. \end{aligned}$$

Now replace  $u$  by  $tu$  to get

$$(28) \quad E = \sum_{u, t} \chi_1 \chi_2 \chi_3(t) \bar{\chi}_1 \bar{\chi}_2(1-t) \bar{\chi}_1(u) \chi_1 \chi_2(1-u) \{1 + \phi(ut)\} \\ = J(\chi_1 \chi_2 \chi_3, \bar{\chi}_1 \bar{\chi}_2) J(\bar{\chi}_1, \chi_1 \chi_2) + J(\chi_1 \chi_2 \chi_3 \phi, \bar{\chi}_1 \bar{\chi}_2) J(\bar{\chi}_1 \phi, \chi_1 \chi_2),$$

where the Jacobi sums  $J$  are defined above (18). Since  $\chi_1^2, \chi_2^2, \chi_3^2$ , and  $\chi_1 \chi_2$  are nontrivial, (5) now follows from (18).

*Remark.* If  $\chi_1 \chi_2, \chi_1 \chi_3$ , or  $\chi_2 \chi_3$  is trivial, we can easily evaluate  $E$  directly from its definition. Otherwise,  $E$  can be evaluated simply from (28).

### 8. CHARACTER SUM ANALOGUES OF (1), (1a) AND (1b)

Let  $\chi_1, \chi_2, \chi_3, \phi$  be characters on  $GF(q)$ , where  $\phi$  has order 2,  $p > 2$ . Set  $t_0 = 1$ . The discriminant of the polynomial

$$F(y) = \sum_{i=0}^n t_i y^{n-i}$$

is a polynomial in  $t_1, \dots, t_n$  which shall be denoted by  $D_n$ . Write

$$E_n = \sum_{i=0}^n (-1)^i t_i.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each  $n \geq 1$ :

$$(29) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1(t_n) \chi_2(E_n) \chi_3 \phi(D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j) G(\chi_2 \chi_3^j)}{G(\chi_3) G(\chi_1 \chi_2 \chi_3^{n+j-1})},$$

provided that the  $n$  characters  $\chi_1 \chi_2 \chi_3^{n+j-1}$  ( $0 \leq j \leq n-1$ ) are all nontrivial;

$$(29a) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1(t_n) \chi_3 \phi(D_n) \zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j)}{G(\chi_3)}$$

for all  $\chi_1, \chi_3$ ; and

$$(29b) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_3 \phi(D_n) \zeta^{\frac{p+1}{2} T(t_1^2 - 2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2) G(\phi) G(\chi_3^{j+1})}{G(\chi_3)}$$

for all  $\chi_3$ .

Formulas (29), (29a), and (29b) have been verified by computer for some small primes  $q$  with  $n = 3, 4$ . Of course the formulas are well known for  $n = 1$ . For