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Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1 , χ_2 , or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1} , then replacing v by vu^{-1} , to obtain

$$S_1 = \sum_{u,v} \bar{\chi}_1 \bar{\chi}_2 \bar{\chi}_3^2 (u) \chi_3 (1+u^2+v^2-2u-2v-2uv) \chi_2 (v).$$

Finally, suppose that χ_1 , χ_2 , χ_3^2 , and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. PROOF OF (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,y \neq 0 \\ x+y \neq -1}} \chi_1\chi_3 \left(\frac{1+x}{y} \right) \chi_2\chi_3 \left(\frac{1+y}{x} \right) \chi_1\chi_2 (y-x).$$

Set $t = \frac{1+x}{y}$, $u = \frac{1+y}{x}$, so

$$x = \frac{t+1}{ut-1}, \quad y = \frac{u+1}{ut-1}.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,t \neq -1 \\ ut \neq 1}} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2 \left(\frac{t-u}{1-ut} \right) \\ &= \sum_{u,t \neq -1} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2 (1-ut). \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,t \neq 0} \chi_1\chi_3(t) \chi_2\chi_3(u) \chi_1\chi_2(t-u) \bar{\chi}_1 \bar{\chi}_2 (1-ut).$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1^2(u) \chi_1\chi_2(t-u^2) \bar{\chi}_1 \bar{\chi}_2 (1-t) \\ &= \sum_{u,t \neq 0} \chi_1\chi_3(t) \bar{\chi}_1 \bar{\chi}_2 (1-t) \bar{\chi}_1(u) \chi_1\chi_2(t-u) \{1 + \phi(u)\}. \end{aligned}$$

Now replace u by tu to get

$$(28) \quad E = \sum_{u,t} \chi_1 \chi_2 \chi_3 (t) \bar{\chi}_1 \bar{\chi}_2 (1-t) \bar{\chi}_1 (u) \chi_1 \chi_2 (1-u) \{1 + \phi (ut)\} \\ = J (\chi_1 \chi_2 \chi_3, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1, \chi_1 \chi_2) + J (\chi_1 \chi_2 \chi_3 \phi, \bar{\chi}_1 \bar{\chi}_2) J (\bar{\chi}_1 \phi, \chi_1 \chi_2),$$

where the Jacobi sums J are defined above (18). Since $\chi_1^2, \chi_2^2, \chi_3^2$, and $\chi_1 \chi_2$ are nontrivial, (5) now follows from (18).

Remark. If $\chi_1 \chi_2, \chi_1 \chi_3$, or $\chi_2 \chi_3$ is trivial, we can easily evaluate E directly from its definition. Otherwise, E can be evaluated simply from (28).

8. CHARACTER SUM ANALOGUES OF (1), (1a) AND (1b)

Let $\chi_1, \chi_2, \chi_3, \phi$ be characters on $GF(q)$, where ϕ has order 2, $p > 2$. Set $t_0 = 1$. The discriminant of the polynomial

$$F(y) = \sum_{i=0}^n t_i y^{n-i}$$

is a polynomial in t_1, \dots, t_n which shall be denoted by D_n . Write

$$E_n = \sum_{i=0}^n (-1)^i t_i.$$

We conjecture that the following analogues of (1), (1a), (1b) hold for each $n \geq 1$:

$$(29) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_2 (E_n) \chi_3 \phi (D_n) = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j) G(\chi_2 \chi_3^j)}{G(\chi_3) G(\chi_1 \chi_2 \chi_3^{n+j-1})},$$

provided that the n characters $\chi_1 \chi_2 \chi_3^{n+j-1}$ ($0 \leq j \leq n-1$) are all nontrivial;

$$(29a) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_1 (t_n) \chi_3 \phi (D_n) \zeta^{T(t_1)} = \prod_{j=0}^{n-1} \frac{-G(\chi_3^{j+1}) G(\chi_1 \chi_3^j)}{G(\chi_3)}$$

for all χ_1, χ_3 ; and

$$(29b) \quad \sum_{t_1, \dots, t_n \in GF(q)} \chi_3 \phi (D_n) \zeta^{\frac{p+1}{2} T(t_1^2 - 2t_2)} = \prod_{j=0}^{n-1} \frac{-\phi(2) G(\phi) G(\chi_3^{j+1})}{G(\chi_3)}$$

for all χ_3 .

Formulas (29), (29a), and (29b) have been verified by computer for some small primes q with $n = 3, 4$. Of course the formulas are well known for $n = 1$. For