Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: IDENTITIES FOR PRODUCTS OF GAUSS SUMS OVER FINITE FIELDS

Autor: Evans, Ronald J.

Kapitel: 6. Proof of (4)

DOI: https://doi.org/10.5169/seals-51748

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 24.01.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

6. Proof of (4)

For characters $\psi_1, \ldots, \psi_m$ on $GF(q)$, define the Jacobi sum

$$J(\psi_1, \ldots, \psi_m) = (-1)^{m+1} \sum_{\substack{x_1, \ldots, x_m \in GF(q) \\ x_1 + \ldots + x_m = 1}} \psi_1(x_1) \ldots \psi_m(x_m).$$

We will use the well-known fact that if $\psi_1 \psi_2 \ldots \psi_m$ is nontrivial, then

$$J(\psi_1, \ldots, \psi_m) = G(\psi_1 \ldots \psi_m)^{-1} \prod_{i=1}^{m} G(\psi_i). \tag{18}$$

Let S denote the left side of (4). If χ_1, χ_2, or χ_3^2 is trivial, then it is easy to verify (4) directly, with use of (18) and (26) below. Thus assume that χ_1, χ_2, and χ_3^2 are nontrivial. With the change of variables

$$u = xy, \qquad v = x + y,$$

we have

$$S = \sum_{u, v \in GF(q)} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) \{1 + \phi(v^2-4u)\}.$$

It therefore remains to show that

$$S_1 = \sum_{u, v} \chi_1(u) \chi_2(1+u-v) \chi_3(v^2-4u) = R(\chi_1, \chi_2, \chi_3\phi). \tag{19}$$

Replace v by $u + 1 - v$ to get

$$S_1 = \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(1+u^2+v^2-2u-2v-2uv). \tag{20}$$

Replace u by u/t, and v by v/t, to get

$$\begin{aligned} S_2 &= -S_1 G(\chi_1\chi_2\chi_3^2) \\ &= \sum_{t \neq 0} \sum_{u, v} \chi_1(u) \chi_2(v) \chi_3(t^2+u^2+v^2-2ut-2vt-2uv) \zeta^{T(t)}. \end{aligned} \tag{21}$$

Since $\chi_1\chi_2\chi_3^2$ is nontrivial, the restriction $t \neq 0$ may be dropped. Then replace t by $t + u + v$ to get

$$S_2 = \sum_{t, u, v} \chi_1(u) \chi_2(v) \chi_3(t^2-4uv) \zeta^{T(t+u+v)}.$$

Replace u by ua and v by vb to get

$$\begin{aligned} S_3 &= S_2 \overline{G(\chi_1)}\, \overline{G(\chi_2)} \\ &= \sum_{t} \sum_{a, b, u, v \neq 0} \chi_1(u) \chi_2(v) \chi_3(t^2-4uvab) \zeta^{T(t+a(u-1)+b(v-1))}. \end{aligned} \tag{22}$$

Replace a by $a/(4uvb)$ to get

$$S_3 = \sum_t \sum_{a,b,u,v\neq 0} \chi_1(u)\chi_2(v)\chi_3(t^2-a)\zeta^{T\left(t+b(v-1)+\frac{a(u-1)}{4uvb}\right)}.$$

Since χ_1 is nontrivial, the restriction $a \neq 0$ may be dropped. Then replace a by $t^2 - a$ to get

$$S_3 = \sum_{a,t} \sum_{b,u,v\neq 0} \chi_1(u)\chi_2(v)\chi_3(a)\zeta^{T\left(t+b(v-1)+\frac{(1-u)(a-t^2)}{4uvb}\right)}$$

$$= -G(\chi_3) \sum_{u\neq 0,1} \sum_{b,v\neq 0} \chi_1(u)\chi_2(v)\chi_3\left(\frac{4uvb}{1-u}\right)\zeta^{T(b(v-1))} \sum_t \zeta^{T\left(t+\frac{t^2(u-1)}{4uvb}\right)}.$$

The inner sum on t equals $-\zeta^{T\left(\frac{uvb}{1-u}\right)}\phi\left(\frac{4uvb}{u-1}\right)G(\phi)$.

Hence

$$S_4 = S_3\left(G(\chi_3)G(\phi)\chi_3(-1)\right)^{-1} \tag{23}$$

$$= \sum_{u\neq 0,1} \sum_{b,v\neq 0} \chi_1(u)\chi_2(v)\chi_3\phi\left(\frac{4uvb}{u-1}\right)\zeta^{T\left(b(v-1)+\frac{uvb}{1-u}\right)}.$$

Therefore,

$$S_4 = \sum_{u\neq 0,1} \sum_{v\neq 0} \chi_1\chi_3\phi(u)\chi_2\chi_3\phi(v)\bar{\chi}_3\phi\left(\frac{u-1}{4}\right)\sum_b \chi_3\phi(b)\zeta^{T\left(b(v-1)+\frac{buv}{1-u}\right)}.$$

Since χ_2 and $\chi_3\phi$ are nontrivial,

$$S_4 = -G(\chi_3\phi)\sum_{u,v} \chi_1\chi_3\phi(u)\chi_2\chi_3\phi(v)\bar{\chi}_3\phi\left(\frac{1-u-v}{4}\right),$$

so

$$S_4 = -\chi_3(4)G(\chi_3\phi)J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \bar{\chi}_3\phi). \tag{24}$$

Combining (21)-(24), we get

$$S_1 = \frac{\chi_3(-4)G(\chi_3)G(\phi)G(\chi_3\phi)J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \bar{\chi}_3\phi)}{G(\chi_1\chi_2\chi_3^2)\overline{G(\chi_1)}\,\overline{G(\chi_2)}}. \tag{25}$$

Applying (7) with $l = 2$, we find that for any character χ_3,

$$\chi_3(-4)G(\chi_3)G(\phi)G(\chi_3\phi) = \chi_3\phi(-1)qG(\chi_3^2). \tag{26}$$

Since χ_1 and χ_2 are nontrivial, it follows from (25) and (26) that

$$S_1 = \frac{\chi_3\phi(-1)G(\chi_3^2)G(\chi_1)G(\chi_2)J(\chi_1\chi_3\phi, \chi_2\chi_3\phi, \bar{\chi}_3\phi)}{q\,G(\chi_1\chi_2\chi_3^2)}. \tag{27}$$

Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1, χ_2. or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1}, then replacing v by vu^{-1}, to obtain

$$S_1 = \sum_{u,\,v} \bar{\chi}_1\bar{\chi}_2\bar{\chi}_3^2(u)\,\chi_3(1+u^2+v^2-2u-2v-2uv)\,\chi_2(v)\,.$$

Finally, suppose that χ_1, χ_2, χ_3^2, and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. Proof of (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,\,y\neq 0\\ x+y\neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right)\chi_2\chi_3\left(\frac{1+y}{x}\right)\chi_1\chi_2(y-x)\,.$$

Set $\quad t = \dfrac{1+x}{y}, \qquad u = \dfrac{1+y}{x}, \quad$ so

$$x = \frac{t+1}{ut-1}, \qquad y = \frac{u+1}{ut-1}.$$

Then

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{u,\,t\neq -1\\ ut\neq 1}} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2\left(\frac{t-u}{1-ut}\right)$$

$$= \sum_{u,\,t\neq -1} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,.$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,.$$

Replace t by t/u to obtain

$$E = \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1^2(u)\,\chi_1\chi_2(t-u^2)\,\bar{\chi}_1\bar{\chi}_2(1-t)$$

$$= \sum_{u,\,t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(t-u)\,\{1+\phi(u)\}\,.$$