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3. THEOREMS OF STICKELBERGER AND DAVENPORT-HASSE

We will make use of the following three classical formulas. First [3, (0.8)],

$$(6) \quad G_{f_m} \left(\chi \frac{q^m - 1}{q - 1} \right) = G_f(\chi)^m,$$

where χ is a character on $GF(q^m)$. Next [3, (0.9)],

$$(7) \quad 1 = \frac{\chi^l(l)}{G_f(\chi^l)} \prod_{j=0}^{l-1} \frac{G_f(\chi\psi^j)}{G_f(\psi^j)},$$

where χ, ψ are characters on $GF(q)$ and ψ has order l . Finally [8], [5, p. 25]

$$(8) \quad \frac{G_f(\chi^\alpha)}{(\zeta - 1)^{s(\alpha)}} \equiv \frac{1}{\gamma(\alpha)} \equiv \frac{(\zeta - 1)^{\alpha - s(\alpha)}}{\alpha!} \pmod{P},$$

where α is an integer, $0 \leq \alpha < q - 1$; $s(\alpha)$ denotes the sum of the p -adic digits of α ; $\gamma(\alpha)$ denotes the product of the factorials of the p -adic digits of α ; P is a prime ideal above p in the ring $\mathcal{O} = \mathbb{Z}[\omega]$, where $\omega = \exp(2\pi i/p(q-1))$; and χ is the character on $\mathcal{O}/P \approx GF(q)$ of order $q - 1$ which maps the coset $\omega + P$ to $\bar{\omega}$.

4. PROOF OF (2)

Let η denote the right side of (2). We must show that $\eta = 1$. Let $\delta = \frac{q^n - 1}{q - 1}$, $\theta = w^{k-1}(cw + i_j)$. Using (6), we have

$$\eta^n = \frac{\chi^{ln}(l) G_{f_n}(\chi^\delta)}{G_{f_n}(\chi^{\delta l})} \prod_{j=1}^e \prod_{k=1}^r \prod_{c=1}^{w^{r-k}} \frac{G_{f_n}^n(\chi^\delta \psi^\theta)}{G_{f_n}^n(\psi^\theta)}.$$

Consider a fixed pair j, k . For each $a \in \{1, 2, \dots, n\}$, $G_{f_n}(\psi^\theta) = G_{f_n}(\psi^{\theta q^a})$, so

$$\prod_{c=1}^{w^{r-k}} G_{f_n}(\psi^\theta) = \prod_{c=1}^{w^{r-k}} G_{f_n}(\psi^{w^{k-1}(cw + i_j q^a)}).$$

Similarly,

$$\prod_{c=1}^{w^{r-k}} G_{f_n}(\chi^\delta \psi^\theta) = \prod_{c=1}^{w^{r-k}} G_{f_n}(\chi^\delta \psi^{w^{k-1}(cw + i_j q^a)}).$$

Thus

$$(9) \quad \eta^n = \frac{\chi^{ln}(l) G_{fn}(\chi^\delta)}{G_{fn}(\chi^{\delta l})} \prod_{j=1}^{l-1} \frac{G_{fn}(\chi^\delta \psi^j)}{G_{fn}(\psi^j)}.$$

Since $n \equiv \delta \pmod{q-1}$, $\chi^{ln}(l) = \chi^{\delta l}(l)$. Therefore, by (7), the right side of (9) equals 1, so

$$(10) \quad \eta^n = 1.$$

By the definition of η and of Gauss sums,

$$\eta^l \equiv \frac{\chi^{l^2}(l) \overline{\chi^l}(l) G_f(\chi^l)}{G_f^l(\chi^l)} \prod_{j=1}^e \prod_{k=1}^r \prod_{c=1}^{w^{r-k}} \frac{\overline{\chi^{\delta l}}(l) G_{fn}(\chi^{\delta l})}{1} \pmod{w},$$

so

$$\eta^l \equiv \frac{\chi^{l^2-l-l\delta(l-1)/n}(l) G_{fn}^{(l-1)/n}(\chi^{\delta l})}{G_f^{l-1}(\chi^l)} \pmod{w}.$$

By (6), $G_{fn}(\chi^{\delta l}) = G_f^n(\chi^l)$; hence

$$(11) \quad \eta^l \equiv 1 \pmod{w}.$$

Thus w divides the norm $N(\eta^l - 1)$. By (10), η^l is an n -th root of unity. Thus if $\eta^l - 1 \neq 0$, then $N(\eta^l - 1)$ divides n , which contradicts the fact that $w + n$. Therefore $\eta^l = 1 = \eta^n$, so since $(l, n) = 1$, $\eta = 1$.

5. PROOF OF (3)

Let η denote the right side of (3). We assume that $0 < \alpha < q - 1$. To see that this presents no loss of generality, we now show that η is unchanged when α is replaced by $\alpha + (q-1)j$, where j is an integer. Clearly $G_f(\chi^\alpha)$ and $\chi^\alpha(l)$ are unchanged, since the restriction $\chi|_{GF(q)}$ has order $q - 1$. Finally, $G_{fl}(\chi^{\alpha\beta})$ is also unchanged, as

$$(12) \quad G_{fl}(\chi^{\alpha\beta}) = G_{fl}(\chi^{\alpha\beta q^j}) = G_{fl}(\chi^{\beta(\alpha+j(q-1))}),$$

where α_j is defined by $\alpha_j \alpha \equiv j \pmod{l}$, $\alpha_j \geq 0$.

Let $\psi = \chi^{\beta(q-1)}$. Using (6), we have

$$\eta^l = \frac{G_{fl}(\chi^{\alpha\beta l})}{\chi^{\alpha l}(l) G_{fl}^l(\chi^{\alpha\beta})} \prod_{j=1}^{l-1} G_{fl}(\psi^j).$$