

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 27 (1981)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES
Autor: Simha, R. R.
Kapitel: §5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY
DOI: <https://doi.org/10.5169/seals-51747>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

systems at P and Q , f is the map $Z \rightarrow Z^{e_p+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \dots, Z^{e_p}$ provide an \mathcal{O}_W -basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det(\tau(Z^{i+j})), \quad 0 \leq i, j \leq e = e_p.$$

But

$$\tau(Z^{i+j}) = Z^{i+j} (1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \dots + (\zeta^{i+j})^e),$$

(ζ denoting a primitive $(e+1)$ -st root of unity), hence

$$\begin{aligned} \tau(Z^{i+j}) &= (e+1)Z^{i+j} \quad \text{if } i+j = 0 \quad \text{or } e+1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $\det(\tau(Z^{i+j}))$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X . Then, using the formula $E(X) = \text{number of vertices} - \text{number of edges} + \text{number of faces}$ in any triangulation of X , it is easy to see that $E(X) = rE(Y) - \deg R(Y=\mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X . Then, while r edges or faces lie over each edge or face of Y , the ramification points reduce the number of vertices over certain vertices of Y , and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) **COROLLARY.** $\deg K_X = -E(X) = 2g - 2$, i.e. g is also the topological genus $(1/2)b_1(X)$ of the compact oriented surface X .

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) **(RIEMANN-ROCH THEOREM).** For any line bundle \mathcal{L} on X ,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1.$$

Proof: It is enough to prove

(5.2) for all \mathcal{L} , $h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) \geq \deg \mathcal{L} - g + 1$. For then, replacing \mathcal{L} by $K \otimes \mathcal{L}^{-1}$ changes only the sign of the left side, and the same is true of the right side by (4.1) (cf. [4], p. 147).

Now (5.2) is true if $\deg \mathcal{L} > \deg K$, for then $h^0(K \otimes \mathcal{L}^{-1}) = 0$, and we can use (3.5). Thus, to prove (5.2), we may assume that $\mathcal{L} = \mathcal{O}(D)$ for some $D \in \text{Div } X$, and that (5.2) holds for $\mathcal{L}' = \mathcal{O}(D + P_0)$, $P_0 \in X$. Now it is clear that $h^0(\mathcal{L}') \leq h^0(\mathcal{L}) + 1$, and similarly $h^0(K \otimes \mathcal{L}'^{-1}) \leq h^0(K \otimes \mathcal{L}^{-1}) + 1$ (cf. the proof of (3.4)). So (5.2) fails for \mathcal{L} if and only if (*) $h^0(\mathcal{L}') = h^0(\mathcal{L}) + 1$, and $h^0(K \otimes \mathcal{L}^{-1}) = h^0(K \otimes \mathcal{L}'^{-1}) + 1$. But if (*) holds, there exist

$$\sigma \in H^0(X, \mathcal{L}') - H^0(X, \mathcal{L})$$

and

$$\omega \in H^0(X, K \otimes \mathcal{L}^{-1}) - H^0(X, K \otimes \mathcal{L}'^{-1}),$$

and then

$$\sigma\omega = \sigma \otimes \omega \in H^0(X, K \otimes \mathcal{O}(P_0)) - H^0(X, K),$$

i.e. $\sigma\omega$ is a meromorphic form with precisely one simple pole at P_0 . But this is impossible: if D is a disc around P_0 in some coordinate system centred at P_0 , then $\int_{\partial D} \sigma\omega = - \int_{\partial(X-D)} \sigma\omega = 0$ by Stokes' theorem, while $\int_{\partial D} \sigma\omega \neq 0$ by the Residue theorem. Thus (*) cannot hold, and (5.2) is proved, q.e.d.

(5.3) COROLLARY. For any line bundle \mathcal{L} on X , $h^1(\mathcal{L}) = h^0(K \otimes \mathcal{L}^{-1})$.

Proof: Compare (5.1) and (3.4).

(5.4) COROLLARY. $h^0(K) = g$ and $h^1(K) = 1$.

Before proceeding to Serre duality, we examine the notion of residue in greater detail. Thus let $U \subset X$ be open, and ω a meromorphic 1-form on U with a pole at $P \in U$. Then, in terms of a uniformising parameter t at P , $\omega = f dt$ near P , with f a meromorphic function at P . The residue of ω at P is $\frac{1}{2\pi i}$ times the coefficient of $1/t$ in the Laurent expansion of f in powers of t . The independence of $\text{Res}_P(\omega)$ on the choice of t can be proved either by direct computation or by identifying it with $1/2\pi i \int_{\gamma} \omega$, where γ is a suitable curve around P . By the argument already used above (Stokes' theorem), one gets

(5.5) (RESIDUE THEOREM). *The sum of the residues of any meromorphic 1-form on X is zero.*

(5.6) COROLLARY. *Given distinct $P, Q \in X$, there exists a meromorphic 1-form on X , holomorphic outside P and Q , and with simple poles at P, Q of residue 1 and -1 respectively.*

Proof: Let $\mathcal{L} = K \otimes \mathcal{O}(P+Q)$. Then $\deg K \otimes \mathcal{L}^{-1} < 0$, hence $h^0(\mathcal{L}) = g + 1$ by (5.1), i.e. there exists $\omega \in H^0(X, \mathcal{L}) - H^0(X, K)$. Then it is clear that the residues of ω at P and Q must be non-zero, while their sum is zero (by (5.5)), hence a suitable constant multiple of ω will have the desired properties.

(5.7) PROPOSITION. *There is a canonical isomorphism $\text{res} : H^1(X, K) \rightarrow \mathbf{C}$.*

Proof: Pick any $P \in X$, and a coordinate neighbourhood U of P . Let \mathcal{U} be the covering $\{U, X - P\}$ of X . Then, by taking residues at P , we get a map $\text{res}_P : Z^1(\mathcal{U}, K) \rightarrow \mathbf{C}$. This map is not zero, and induces a map $H^1(\mathcal{U}, K) \rightarrow \mathbf{C}$ (by the residue theorem). Since $h^1(K) = 1$, $\text{res}_P : H^1(\mathcal{U}, K) \rightarrow H^1(X, K) \rightarrow \mathbf{C}$ is in fact an isomorphism. That the map $\text{res}_P : H^1(X, K) \rightarrow \mathbf{C}$ is independent of the choice of $P \in X$ is precisely the meaning of (5.6), and we get the asserted canonical isomorphism res .

(5.8) SERRE DUALITY. *For any line bundle \mathcal{L} on X , the natural bilinear form*

$$\zeta : H^0(X, \mathcal{L}) \times H^1(X, K \otimes \mathcal{L}^{-1}) \rightarrow H^1(X, K) \xrightarrow{\text{res}} \mathbf{C}$$

is nondegenerate.

(5.9) Remark. For any covering \mathcal{U} of X , the natural map $\mathcal{L} \times (K \otimes \mathcal{L}^{-1}) \rightarrow K$ defines an obvious pairing

$$H^0(X, \mathcal{L}) \times Z^1(\mathcal{U}, K \otimes \mathcal{L}^{-1}) \rightarrow Z^1(\mathcal{U}, K)$$

which is easily seen to induce the pairing

$$H^0(X, \mathcal{L}) \times H^1(X, K \otimes \mathcal{L}^{-1}) \rightarrow H^1(X, K)$$

figuring in (5.8).

Proof of (5.8). Since we already know that

$$h^0(X, \mathcal{L}) = h^1(X, K \otimes \mathcal{L}^{-1}),$$

we need only show that, if $\sigma \in H^0(X, \mathcal{L})$ is such that $\zeta(\sigma \otimes \gamma) = 0$ for all $\gamma \in H^1(X, K \otimes \mathcal{L}^{-1})$, then $\sigma \equiv 0$. Now choose any $P \in X$, and a coordinate neighbourhood (U, z) of P centred at P such that $\mathcal{L}|_U \approx \mathcal{O}_U$. Then the covering $\mathfrak{U} = \{U, X - P\}$ is a Leray covering for \mathcal{L}, K and $K \otimes \mathcal{L}^{-1}$ ((3.7)). The $z^n dz, n \in \mathbf{Z}$, can all be regarded as elements of $Z^1(\mathfrak{U}, K \otimes \mathcal{L}^{-1})$; let γ_n denote their images in $H^1(X, K \otimes \mathcal{L}^{-1})$. Then clearly $\rho(\sigma \otimes \gamma_n) = 0$ for all n implies that all the coefficients of the Taylor expansion of σ at P with respect to vanish, hence $\sigma \equiv 0$, q.e.d.

(5.9) SERRE DUALITY FOR VECTOR BUNDLES. *For any vector bundle \mathcal{V} on X , let $\mathcal{V}^* = \text{Hom } \mathcal{O}_X(\mathcal{V}, \mathcal{O}_X)$. Then the natural pairing*

$$\zeta : H^0(X, \mathcal{V}) \times H^1(X, K \otimes \mathcal{V}^*) \rightarrow H^1(X, K) \xrightarrow{\text{res}} \mathbf{C}$$

is non-degenerate.

Proof: Arguing as in the proof of (5.8) we see that the map $H^0(X, \mathcal{V}) \rightarrow (H^1(X, K \otimes \mathcal{V}^*))^*$ induced by ζ is injective, hence $h^0(X, \mathcal{V}) \leq h^1(X, K \otimes \mathcal{V}^*)$. Replacing \mathcal{V} by $K \otimes \mathcal{V}^*$, we also get $h^0(K \otimes \mathcal{V}^*) \leq h^1(\mathcal{V})$. But, by induction on rank \mathcal{V} , we easily deduce from (5.3) that $\chi(K \otimes \mathcal{V}^*) = -\chi(\mathcal{V})$, hence $h^0(X, \mathcal{V}) = h^1(X, K \otimes \mathcal{V}^*)$. Thus ζ is non-degenerate as before.

REFERENCES

- [1] GRAUERT, H. and R. REMMERT. *Theory of Stein Spaces*. Springer-Verlag, 1979.
- [2] GUNNING, R. C. *Lectures on Riemann surfaces*. Princeton University Press.
- [3] GUNNING, R. C. and H. ROSSI. *Analytic Functions of Several Complex Variables*. Prentice Hall, 1965.
- [4] MUMFORD, D. *Algebraic Geometry I: Complex Projective Varieties*. Springer-Verlag, 1976.
- [5] SERRE, J.-P. *Groupes Algébriques et Corps de Classes*. Hermann, 1959.

(Reçu le 10 juillet 1980)

R. R. Simha

School of Mathematics
Tata Institute of Fundamental Research
Bombay 400 005
India