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Artikel: THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES

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systems at P and Q, f is the map $Z \to Z^{e_p+1} = w$ of the unit disc $U \subset \mathbf{C}$ onto another copy W of it. Since $1, Z, \ldots, Z^{e_p}$ provide an $\mathcal{O}_W$-basis for $f_0(Q_U)$, the value of δ on a local generator of $\mathcal{L} \otimes \mathcal{L}$ is given by

$$\det\left(\tau(Z^{i+j})\right), \; 0 \leqslant i, j \leqslant e = e_p .$$

But

$$\tau(Z^{i+j}) = Z^{i+j}\left(1 + \zeta^{i+j} + (\zeta^{i+j})^2 + \ldots + (\zeta^{i+j})^e\right),$$

(ζ denoting a primitive $(e+1) - st$ root of unity), hence

$$\begin{aligned} \tau(Z^{i+j}) &= (e+1)\, Z^{i+j} \quad \text{if} \quad i+j = 0 \quad \text{or} \quad e+1, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence $\det\left(\tau(Z^{i+j})\right)$ is a (nonzero) constant multiple of $Z^{(e+1)e} = w^e$ as asserted.

If $f^{-1}(Q)$ consists of several points, the situation is a direct sum of those considered above, and δ is indeed as asserted. This proves Theorem (4.1).

(4.5) *Remark.* Let the notation be as above, and let $E(X)$ denote the topological Euler-Poincaré characteristic of X. Then, using the formula $E(X)$ = number of vertices − number of edges + number of faces in any triangulation of X, it is easy to see that $E(X) = r\,E(Y) - \deg R\,(Y = \mathbf{P}^1)$. Indeed, choose any triangulation of Y which contains all the images of the ramification points of f as vertices, and lift it to a triangulation of X. Then, while r edges or faces lie over each edge or face of Y, the ramification points reduce the number of vertices over certain vertices of Y, and one gets the formula asserted. Since $E(Y) = 2$, (4.2) yields:

(4.6) COROLLARY. $\deg K_X = -E(X) = 2g - 2$, *i.e.* g *is also the topological genus* $(1/2)\, b_1(X)$ *of the compact oriented surface* X.

§ 5. RIEMANN-ROCH THEOREM (FINAL FORM). SERRE DUALITY

(5.1) (RIEMANN-ROCH THEOREM). For any line bundle $\mathcal{L}$ on X,

$$h^0(\mathcal{L}) - h^0(K \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - g + 1 .$$

Proof: It is enough to prove

(5.2) for all $\mathscr{L}$, $h^0(\mathscr{L}) - h^0(K \otimes \mathscr{L}^{-1}) \geqslant \deg \mathscr{L} - g + 1$. For then, replacing $\mathscr{L}$ by $K \otimes \mathscr{L}^{-1}$ changes only the sign of the left side, and the *same is true of the right side by* (4.1) (cf. [4], p. 147).

Now (5.2) is true if $\deg \mathscr{L} > \deg K$, for then $h^0(K \otimes \mathscr{L}^{-1}) = 0$, and we can use (3.5). Thus, to prove (5.2), we may assume that $\mathscr{L} = \mathcal{O}(D)$ for some $D \in \operatorname{Div} X$, and that (5.2) holds for $\mathscr{L}' = \mathcal{O}(D+P_0)$, $P_0 \in X$. Now it is clear that $h^0(\mathscr{L}') \leqslant h^0(\mathscr{L}) + 1$, and similarly $h^0(K \otimes \mathscr{L}'^{-1}) \leqslant h^0(K \otimes \mathscr{L}') + 1$ (cf. the proof of (3.4)). So (5.2) fails for $\mathscr{L}$ if and only if (*) $h^0(\mathscr{L}') = h^0(\mathscr{L}) + 1$, *and* $h^0(K \otimes \mathscr{L}^{-1}) = h^0(K \otimes \mathscr{L}'^{-1}) + 1$. But if (*) holds, there exist

$$\sigma \in H^0(X, \mathscr{L}') - H^0(X, \mathscr{L})$$

and

$$\omega \in H^0(X, K \otimes \mathscr{L}^{-1}) - H^0(X, K \otimes \mathscr{L}'^{-1}),$$

and then

$$\sigma\omega = \sigma \otimes \omega \in H^0(X, K \otimes \mathcal{O}(P_0)) - H^0(X, K),$$

i.e. $\sigma\omega$ is a meromorphic form with precisely one simple pole at P_0. But this is impossible: if D is a disc around P_0 in some coordinate system centred at P_0, then $\int_{\partial D} \sigma\omega = - \int_{\partial (X-D)} \sigma\omega = 0$ by Stokes' theorem, while $\int_{\partial D} \sigma\omega \neq 0$ by the Residue theorem. Thus (*) cannot hold, and (5.2) is proved, q.e.d.

(5.3) COROLLARY. *For any line bundle $\mathscr{L}$ on X, $h^1(\mathscr{L}) = h^0(K \otimes \mathscr{L}^{-1})$.*

Proof: Compare (5.1) and (3.4).

(5.4) COROLLARY. $h^0(K) = g$ *and* $h^1(K) = 1$.

Before proceeding to Serre duality, we examine the notion of residue in greater detail. Thus let $U \subset X$ be open, and ω a meromorphic 1-form on U with a pole at $P \in U$. Then, in terms of a uniformising parameter t at P, $\omega = f\,dt$ near P, with f a meromorphic function at P. The *residue* of ω at P is $\frac{1}{2\pi i}$ times the coefficient of $1/t$ in the Laurent expansion of f in powers of t. The independence of $\operatorname{Res}_P(\ )$ on the choice of t can be proved either by direct computation or by identifying it with $1/2\pi i \int_\gamma \omega$, where γ is a suitable curve around P. By the argument already used above (Stokes' theorem), one gets

(5.5) (RESIDUE THEOREM). *The sum of the residues of any meromorphic 1-form on X is zero.*

(5.6) COROLLARY. *Given distinct $P, Q \in X$, there exists a meromorphic 1-form on X, holomorphic outside P and Q, and with simple poles at P, Q of residue 1 and -1 respectively.*

Proof: Let $\mathscr{L} = K \otimes \mathcal{O}(P+Q)$. Then $\deg K \otimes \mathscr{L}^{-1} < 0$, hence $h^0(\mathscr{L}) = g + 1$ by (5.1), i.e. there exists $\omega \in H^0(X, \mathscr{L}) - H^0(X, K)$. Then it is clear that the residues of ω at P and Q must be non-zero, while their sum is zero (by (5.5)), hence a suitable constant multiple of ω will have the desired properties.

(5.7) PROPOSITION. *There is a* canonical *isomorphism* res : $H^1(X, K) \to \mathbf{C}$.

Proof: Pick any $P \in X$, and a coordinate neighbourhood U of P. Let $\mathfrak{U}$ be the covering $\{U, X - P\}$ of X. Then, by taking residues at P, we get a map $\operatorname{res}_P : Z^1(\mathfrak{U}, K) \to \mathbf{C}$. This map is not zero, and induces a map $H^1(\mathfrak{U}, K) \to C$ (by the residue theorem). Since $h^1(K) = 1$, $\operatorname{res}_P : H^1(\mathfrak{U}, K) \to H^1(X, K) \to \mathbf{C}$ is in fact an isomorphism. That the map $\operatorname{res}_P : H^1(X, K) \to \mathbf{C}$ is independent of the choice of $P \in X$ is precisely the meaning of (5.6), and we get the asserted canonical isomorphism res.

(5.8) SERRE DUALITY. *For any line bundle $\mathscr{L}$ on X, the natural bilinear form*

$$\zeta : H^0(X, \mathscr{L}) \times H^1(X, K \otimes \mathscr{L}^{-1}) \to H^1(X, K) \xrightarrow{res} \mathbf{C}$$

is nondegenerate.

(5.9) *Remark.* For any covering $\mathfrak{U}$ of X, the natural map $\mathscr{L} \times (K \otimes \mathscr{L}^{-1}) \to K$ defines an obvious pairing

$$H^0(X, \mathscr{L}) \times Z^1(\mathfrak{U}, K \otimes \mathscr{L}^{-1}) \to Z^1(\mathfrak{U}, K)$$

which is easily seen to induce the pairing

$$H^0(X, \mathscr{L}) \times H^1(X, K \otimes \mathscr{L}^{-1}) \to H^1(X, K)$$

figuring in (5.8).

Proof of (5.8). Since we already know that

$$h^0(X, \mathscr{L}) = h^1(X, K \otimes \mathscr{L}^{-1}),$$

we need only show that, if $\sigma \in H^0(X, \mathscr{L})$ is such that $\zeta(\sigma \otimes \gamma) = 0$ for all $\gamma \in H^1(X, K \otimes \mathscr{L}^{-1})$, then $\sigma \equiv 0$. Now choose any $P \in X$, and a coordinate neighbourhood (U, z) of P centred at P such that $\mathscr{L} \mid U \approx \mathcal{O}_U$. Then the covering $\mathfrak{U} = \{U, X - P\}$ is a Leray covering for $\mathscr{L}, K$ and $K \otimes \mathscr{L}^{-1}$ ((3.7)). The $z^n\, dz, n \in \mathbf{Z}$, can all be regarded as elements of $Z^1(\mathfrak{U}, K \otimes \mathscr{L}^{-1})$; let γ_n denote their images in $H^1(X, K \otimes \mathscr{L}^{-1})$. Then clearly $\rho(\sigma \otimes \gamma_n) = 0$ for all n implies that all the coefficients of the Taylor expansion of σ at P with respect to vanish, hence $\sigma \equiv 0$, q.e.d.

(5.9) Serre duality for vector bundles. *For any vector bundle $\mathscr{V}$ on X, let $\mathscr{V}^* = \operatorname{Hom}_{\mathcal{O}_X}(\mathscr{V}, \mathcal{O}_X)$. Then the natural pairing*

$$\zeta : H^0(X, \mathscr{V}) \times H^1(X, K \otimes \mathscr{V}^*) \to H^1(X, K) \overset{res}{\tilde{\to}} \mathbf{C}$$

is non-degenerate.

Proof: Arguing as in the proof of (5.8) we see that the map $H^0(X, \mathscr{V}) \to (H^1(X, K \otimes \mathscr{V}^*))^*$ induced by ζ is injective, hence $h^0(X, \mathscr{V}) \leqslant h^1(X, K \otimes \mathscr{V}^*)$. Replacing $\mathscr{V}$ by $K \otimes \mathscr{V}^*$, we also get $h^0(K \otimes \mathscr{V}^*) \leqslant h^1(\mathscr{V})$. But, by induction on rank $\mathscr{V}$, we easily deduce from (5.3) that $\chi(K \otimes \mathscr{V}^*) = -\chi(\mathscr{V})$, hence $h^0(X, \mathscr{V}) = h^1(X, K \otimes \mathscr{V}^*)$. Thus ζ is non-degenerate as before.

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