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Note finally that, if $\mathscr{L}$ is any line bundle on X and $D = \Sigma\, n_P P \in \operatorname{Div} X$, then $\mathscr{L} \otimes \mathcal{O}(D)$ can be identified with the sheaf of germs of meromorphic sections σ of $\mathscr{L}$ such that $\operatorname{ord}_P \sigma \geqslant -n_P$.

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let $f: X \to Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and $\mathscr{V}$ a vector bundle on X. Then the natural maps $H^i(Y, f_0(\mathscr{V})) \to H^i(X, \mathscr{V})$ are isomorphisms for all $i \geqslant 0$.*

Proof: If $\mathfrak{U}$ is a sufficiently fine open covering of Y, then it is clear that, for each $U \in \mathfrak{U}$, $f_0(\mathscr{V}) \mid U$ is $\mathcal{O}_Y$-free, and that $f^{-1}(U)$ is a finite disjoint union of coordinate open sets in X, restricted to each of which $\mathscr{V}$ is free. Since, for $i > 0$, $H^i(W, \mathcal{O}_W) = 0$ for any open $W \subset \mathbf{C}$, it follows that $\mathfrak{U}$ and $\mathfrak{U}' = \{f^{-1}(U) : U \in \mathfrak{U}\}$ are Leray coverings for $f_0(\mathscr{V})$ and $\mathscr{V}$ respectively. Now the natural maps $H^i(\mathfrak{U}, f_0(\mathscr{V})) \to H^i(\mathfrak{U}', \mathscr{V})$ are obviously bijective, q.e.d.

(2.17) *Remark.* Propositions (2.4) and (2.16) are valid (with the same proofs) even if X is not compact, provided we assume that f is *proper*.

(2.18) *Remark.* We know by (2.10) that any (compact) X admits a nonconstant meromorphic function, i.e. a nonconstant holomorphic map $f: X \to \mathbf{P}^1$. Since $\mathbf{P}^1$ is covered by *two* coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on $\mathbf{P}^1$, it follows by (2.16) that $H^i(X, \mathscr{V}) = 0$ for $i \geqslant 2$ for any compact Riemann surface X and any vector bundle $\mathscr{V}$ on it. This proof is valid in the algebraic situation also. This is the reason for including the case $i \geqslant 2$ in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for H^1; the fact that for a Leray covering $\mathfrak{U}$, $H^2(\mathfrak{U}, \mathscr{F}) \to H^2(X, \mathscr{F})$ is surjective (which is what was needed above) is also trivial if we use resolutions.

§ 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface X.

(3.1) *Notation—Definition.* For any vector bundle $\mathscr{V}$ on X, we set

$$h^i(\mathscr{V}) = \dim_{\mathbf{C}} H^i(X, \mathscr{V}),\ i = 0, 1 \text{ and } \chi(\mathscr{V}) = h^0(\mathscr{V}) - h^1(\mathscr{V}).$$

The *genus* g of X is $h^1(\mathcal{O}_X)$.

(3.2) *Remark.* If $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ is an exact sequence of vector bundles, then $\chi(\mathcal{V}) = \chi(\mathcal{V}') + \chi(\mathcal{V}'')$, as follows from the cohomology exact sequence (since $H^2 = 0$).

(3.3) *Definition.* The *degree* $\deg D$ of $D = \Sigma\, n(P)\, P \in \mathrm{Div}\, X$ is $\Sigma\, n(P)$.

(3.4) PROPOSITION. *For any* $D \in \mathrm{Div}\,(X)$,

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D = \deg D - g + 1\,.$$

Proof: (Serre [5], pp. 20-21). The assertion is a tautology for $D = 0$; hence we need only prove that it holds for $D \in \mathrm{Div}\,(X)$ iff it holds for a divisor of the form $D' = D + P, P \in X$. Now $\mathcal{O}(D)$ is a subsheaf of $\mathcal{O}(D')$, and the quotient sheaf $\mathcal{Q} = \mathcal{O}(D')/\mathcal{O}(D)$ is concentrated at P with stalk isomorphic to $\mathcal{O}_P/\mathfrak{m}_P$. Hence $h^0(\mathcal{Q}) = 1$, and $h^1(\mathcal{Q}) = 0$. Now the exact sequence

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D') \to \mathcal{Q} \to 0$$

yields the exact sequence

$$0 \to H^0(X, \mathcal{O}(D)) \to \dots \to H^0(X, \mathcal{Q}) \to H^1(X, \mathcal{O}(D)) \\ \to H^1(X, \mathcal{O}(D')) \to 0\,,$$

so that $\chi(\mathcal{O}(D')) - \chi(\mathcal{O}(D)) = 1$. Since $\deg D' - \deg D = 1$, the desired assertion follows, q.e.d.

(3.5) COROLLARY. $h^0(D) \geqslant \deg D - g + 1$.

(3.6) COROLLARY. *For any* $P \in X$, *there exists a nonconstant meromorphic function on* X, *holomorphic in* $X - P$, *with a pole of order* $\leqslant g + 1$ *at* P.

Proof: For $D = (g+1)P$, $h^0(D) \geqslant 2$ by (3.4), i.e. $H^0(X, \mathcal{O}(D))$ contains a nonconstant element.

(3.7) COROLLARY. *For any vector bundle* $\mathcal{V}$ *on* X, *and any* $P \in X$, $H^1(X - \{P\}, \mathcal{V}) = 0$.

Proof: By (3.6), there exists a holomorphic map $f: X \to \mathbf{P}^1$ with $P = f^{-1}(\infty)$. Now use (2.11), (2.12), (2.16) and (2.17).

(3.8) COROLLARY. $g = 0$ *iff* $X \approx \mathbf{P}^1$.

Proof: $g = 0$ for $X = \mathbf{P}^1$ by Laurent's theorem. Conversely, if $g = 0$, then there exists by (3.6) a meromorphic function f on X with just one *simple* pole and no other singularities. It is easy to see that $f: X \to \mathbf{P}^1$ is then an isomorphism.

(3.9) COROLLARY. *If* $D \sim D'$, *then* $\deg D = \deg D'$.

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle $\mathcal{L}$ is the degree of any $D \in \operatorname{Div} X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of $\mathcal{L}$.

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \operatorname{Pic} X \to \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle $\mathcal{V}$ is that of the line bundle $\det \mathcal{V} = \overset{r}{\wedge}_{\mathcal{O}_X} \mathcal{V}$, $r = \operatorname{rank} \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \operatorname{Hom}(\det \mathcal{V}, \mathcal{O}_X)$ at any $P \in X$ consists $\mathcal{O}_P$-multilinear alternate maps $\mathcal{V}_P \times \ldots \times \mathcal{V}_P$ (r times) $\to \mathcal{O}_P$.

(3.15) PROPOSITION. *If* $0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$ *is an exact sequence of vector bundles, then* $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle* $\mathcal{V}$ *on* X,

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \operatorname{rank} \mathcal{V} \,.\, \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on $\operatorname{rank} \mathcal{V}$.

§ 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) THEOREM. $\deg K = 2g - 2 = -2\chi(\mathcal{O})$.