

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	27 (1981)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	THE RIEMANN-ROCH THEOREM FOR COMPACT RIEMANN SURFACES
Autor:	Simha, R. R.
Kapitel:	§3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)
DOI:	https://doi.org/10.5169/seals-51747

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Note finally that, if \mathcal{L} is any line bundle on X and $D = \sum n_P P \in \text{Div } X$, then $\mathcal{L} \otimes \mathcal{O}(D)$ can be identified with the sheaf of germs of meromorphic sections σ of \mathcal{L} such that $\text{ord}_P \sigma \geq -n_P$.

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let $f: X \rightarrow Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and \mathcal{V} a vector bundle on X . Then the natural maps $H^i(Y, f_0(\mathcal{V})) \rightarrow H^i(X, \mathcal{V})$ are isomorphisms for all $i \geq 0$.*

Proof: If \mathfrak{U} is a sufficiently fine open covering of Y , then it is clear that, for each $U \in \mathfrak{U}$, $f_0(\mathcal{V})|_U$ is \mathcal{O}_Y -free, and that $f^{-1}(U)$ is a finite disjoint union of coordinate open sets in X , restricted to each of which \mathcal{V} is free. Since, for $i > 0$, $H^i(W, \mathcal{O}_W) = 0$ for any open $W \subset \mathbf{C}$, it follows that \mathfrak{U} and $\mathfrak{U}' = \{f^{-1}(U) : U \in \mathfrak{U}\}$ are Leray coverings for $f_0(\mathcal{V})$ and \mathcal{V} respectively. Now the natural maps $H^i(\mathfrak{U}, f_0(\mathcal{V})) \rightarrow H^i(\mathfrak{U}', \mathcal{V})$ are obviously bijective, q.e.d.

(2.17) *Remark.* Propositions (2.4) and (2.16) are valid (with the same proofs) even if X is not compact, provided we assume that f is *proper*.

(2.18) *Remark.* We know by (2.10) that any (compact) X admits a non-constant meromorphic function, i.e. a nonconstant holomorphic map $f: X \rightarrow \mathbf{P}^1$. Since \mathbf{P}^1 is covered by two coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on \mathbf{P}^1 , it follows by (2.16) that $H^i(X, \mathcal{V}) = 0$ for $i \geq 2$ for any compact Riemann surface X and any vector bundle \mathcal{V} on it. This proof is valid in the algebraic situation also. This is the reason for including the case $i \geq 2$ in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for H^1 ; the fact that for a Leray covering \mathfrak{U} , $H^2(\mathfrak{U}, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$ is surjective (which is what was needed above) is also trivial if we use resolutions.

§ 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface X .

(3.1) *Notation—Definition.* For any vector bundle \mathcal{V} on X , we set

$$h^i(\mathcal{V}) = \dim_{\mathbf{C}} H^i(X, \mathcal{V}), \quad i = 0, 1 \text{ and } \chi(\mathcal{V}) = h^0(\mathcal{V}) - h^1(\mathcal{V}).$$

The *genus* g of X is $h^1(\mathcal{O}_X)$.

(3.2) *Remark.* If $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ is an exact sequence of vector bundles, then $\chi(\mathcal{V}) = \chi(\mathcal{V}') + \chi(\mathcal{V}'')$, as follows from the cohomology exact sequence (since $H^2 = 0$).

(3.3) *Definition.* The *degree* $\deg D$ of $D = \sum n(P)P \in \text{Div } X$ is $\sum n(P)$.

(3.4) **PROPOSITION.** *For any $D \in \text{Div}(X)$,*

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D = \deg D - g + 1.$$

Proof: (Serre [5], pp. 20-21). The assertion is a tautology for $D = 0$; hence we need only prove that it holds for $D \in \text{Div}(X)$ iff it holds for a divisor of the form $D' = D + P, P \in X$. Now $\mathcal{O}(D)$ is a subsheaf of $\mathcal{O}(D')$, and the quotient sheaf $\mathcal{Q} = \mathcal{O}(D')/\mathcal{O}(D)$ is concentrated at P with stalk isomorphic to $\mathcal{O}_P/\mathfrak{m}_P$. Hence $h^0(\mathcal{Q}) = 1$, and $h^1(\mathcal{Q}) = 0$. Now the exact sequence

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D') \rightarrow \mathcal{Q} \rightarrow 0$$

yields the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}(D)) &\rightarrow \dots \rightarrow H^0(X, \mathcal{Q}) \rightarrow H^1(X, \mathcal{O}(D)) \\ &\rightarrow H^1(X, \mathcal{O}(D')) \rightarrow 0, \end{aligned}$$

so that $\chi(\mathcal{O}(D')) - \chi(\mathcal{O}(D)) = 1$. Since $\deg D' - \deg D = 1$, the desired assertion follows, q.e.d.

(3.5) **COROLLARY.** $h^0(D) \geq \deg D - g + 1$.

(3.6) **COROLLARY.** *For any $P \in X$, there exists a nonconstant meromorphic function on X , holomorphic in $X - P$, with a pole of order $\leq g + 1$ at P .*

Proof: For $D = (g+1)P$, $h^0(D) \geq 2$ by (3.4), i.e. $H^0(X, \mathcal{O}(D))$ contains a nonconstant element.

(3.7) **COROLLARY.** *For any vector bundle \mathcal{V} on X , and any $P \in X$, $H^1(X - \{P\}, \mathcal{V}) = 0$.*

Proof: By (3.6), there exists a holomorphic map $f: X \rightarrow \mathbf{P}^1$ with $P = f^{-1}(\infty)$. Now use (2.11), (2.12), (2.16) and (2.17).

(3.8) **COROLLARY.** $g = 0$ iff $X \approx \mathbf{P}^1$.

Proof: $g = 0$ for $X = \mathbf{P}^1$ by Laurent's theorem. Conversely, if $g = 0$, then there exists by (3.6) a meromorphic function f on X with just one

simple pole and no other singularities. It is easy to see that $f : X \rightarrow \mathbf{P}^1$ is then an isomorphism.

(3.9) COROLLARY. *If $D \sim D'$, then $\deg D = \deg D'$.*

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle \mathcal{L} is the degree of any $D \in \text{Div } X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of \mathcal{L} .

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \text{Pic } X \rightarrow \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle \mathcal{V} is that of the line bundle $\det \mathcal{V} = \bigwedge^r \mathcal{O}_x \mathcal{V}, r = \text{rank } \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \text{Hom}(\det \mathcal{V}, \mathcal{O}_x)$ at any $P \in X$ consists \mathcal{O}_P -multilinear alternate maps $\mathcal{V}_P \times \dots \times \mathcal{V}_P$ (r times) $\rightarrow \mathcal{O}_P$.

(3.15) PROPOSITION. *If $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ is an exact sequence of vector bundles, then $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.*

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle \mathcal{V} on X ,*

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \text{rank } \mathcal{V} \cdot \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank \mathcal{V} .

§ 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) THEOREM. $\deg K = 2g - 2 = -2\chi(\mathcal{O})$.