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$C_{t-1,s} = b_{t-1}$  and  $C_{t-1,s+1} = c_{t-1}$ . (Naturally these values must imply the state  $q_a$  and the headposition  $s_t$  at time  $t$ .) Now player A is allowed to doubt one of these three claims, by playing the integer  $s' \in \{s-1, s, s+1\}$ , and player E has to justify his claim for  $C_{t-1,s'}$  by claiming values for  $C_{t-2,s'-1}$ ,  $C_{t-2,s'}$  and  $C_{t-2,s'+1}$  which imply his value for  $C_{t-1,s'}$  etc. Finally the value claimed for  $C_{0s''}$  is checked by comparison with the  $s''$ -th input symbol. If it is correct, then player E, otherwise player A wins.

If  $w$  is accepted by  $M$ , then the winning strategy for player E is to make always correct claims. If  $w$  is not accepted by  $M$ , then player A has a winning strategy. He always doubts one of the wrong claims of player E.

## 5. UPPER BOUNDS

PROPOSITION. 1. For all  $p \geq 0$ , the  $\exists^p \forall \exists^*$  class is logspace transformable to the monadic  $\exists \forall \exists^*$  class via length order  $n$ .

2. The  $\exists^* \forall \exists^*$  class is logspace transformable to the monadic  $\exists^* \forall \exists^*$  class via length order  $n^2 / \log n$ .

*Proof.* The main ideas of this proof are due to Lewis [27, Lemma 7.1] and Ackermann [2, Section VIII.1]. Given a formula  $F$  of the class  $\exists^p \forall \exists^q$  with prefix  $\exists x_1 \dots \exists x_p \forall y \exists z_1 \dots \exists z_q$  and matrix  $M$ , let  $S$  be the set of atomic formulas in  $M$ . We define the set  $S'$  by  $S' = S \cup \{A[y/x_i] \mid A \in S \text{ and } 1 \leq i \leq p\}$ .

Let  $S' = \{A_1, \dots, A_r\}$ .

Then  $|S'| = r \leq (p+1) |S|$ .

Now we change the matrix  $M$  of  $F$  to get the formula  $F'$  with matrix  $M'$  by replacing (for  $j = 1, \dots, r$ ) all occurrences of the atomic formula  $A_j$  by  $P_j(y)$  (for a new monadic predicate symbol  $P_j$ ) and by adding — as a conjunct to  $M$  — a set  $B$  of biconditionals.

The set  $B$  is constructed to ensure that every Herbrand model  $\alpha'$  of the functional form of the formula  $F'$  (with matrix  $M'$ ) defines immediately a model  $\alpha$  of the functional form of  $F$  by  $|\alpha| = |\alpha'|$ ,

$c_k^\alpha = c_k^{\alpha'} = c_k$ ,  $k = 1, \dots, p$  (where  $c_k$  is the replacement of  $x_k$  in the functional forms of  $F$  and  $F'$ ),

$f_k^\alpha = f_k^{\alpha'}$ ,  $k = 1, \dots, q$  (where  $f_k(y)$  is the replacement of  $z_k$  in the functional forms of  $F$  and  $F'$ ),

$P^\alpha(b_1, \dots, b_n) = P_j^{\alpha'}(b)$ , if  $A_j \in S'$ ,  $b \in |\alpha'|$ ,  $b_1, \dots, b_n \in |\alpha|$  and there exist variables  $v_1, \dots, v_n$  fulfilling for all  $i, k$  the following properties:

- a)  $A_j = P(v_1, \dots, v_n)$ ,
- b) if  $v_i = x_k$  then  $b_i = c_k^\alpha$ ,
- c) if  $v_i = y$  then  $b_i = b$ ,
- d) if  $v_i = z_k$  then  $b_i = f_k^\alpha(b)$ .

$P^\alpha(b_1, \dots, b_n)$  is defined arbitrarily (e.g. false) if no such  $A_j$  and  $b$  exist. There might exist several  $A_j$  and  $b$  having these properties. To ensure that in this case the definition of  $P^\alpha(b_1, \dots, b_n)$  is correct, i.e. independent of the particular choice of  $A_j$  and  $b$ , we conjoin the set  $B$  of biconditionals to the matrix  $M$ .

Take any  $n$ -tupel  $(b_1, \dots, b_n) \in |\alpha|^n$ . In the following cases, several  $A_j \in S'$  and  $b \in |\alpha|$  might satisfy the conditions a), b), c), d):

1.  $\{b_1, \dots, b_n\} \subseteq \{c_1^\alpha, \dots, c_p^\alpha\}$ .
2. There is a  $b'$  in  $\{c_1^\alpha, \dots, c_p^\alpha\}$  such that  $\{b_1, \dots, b_n\} \subseteq \{c_1^\alpha, \dots, c_p^\alpha, f_1^\alpha(b'), \dots, f_q^\alpha(b')\}$ .
3. There is a  $b''$  in  $\{b_1, \dots, b_n\}$ , such that  $\{b_1, \dots, b_n\} \subseteq \{c_1^\alpha, \dots, c_p^\alpha, b''\}$ .

To make the definition correct in case 1, we add to  $B$  the following biconditionals:

If there is an  $A_j$  in  $S'$  such that  $A_j = P(v_1, \dots, v_n)$  with  $\{v_1, \dots, v_n\} \subseteq \{x_1, \dots, x_p\}$ , we add

$$P_j(y) \leftrightarrow P_j(x_1)$$

If  $A_j = P(v_1, \dots, v_n)$  with  $\{v_1, \dots, v_n\} \subseteq \{x_1, \dots, x_p, y\}$  and  $A_j[y/x_i] = A_{j'}[y/x_k]$  (for  $A_j \neq A_{j'}$ ), then we add

$$P_j(x_i) \leftrightarrow P_{j'}(x_k).$$

*Note:* Here the length of the monadic formula might grow quadratically in  $p$ .

To make the definition correct in the case when 2 but not 3 holds, we add to  $B$  for all  $j, j', i$  with  $A_j[y/x_i] = A_{j'}[y/x_i]$  the formula

$$P_j(x_i) \leftrightarrow P_{j'}(x_i).$$

To make the definition correct, when 3. but not 2. holds, we add to  $B$  the following biconditionals.

For all  $j, j', k$  such that  $A_j = P(v_1, \dots, v_n)$  with

$$y \in \{v_1, \dots, v_n\} \subseteq \{x_1, \dots, x_p, y\}$$

and  $A_j [y/z_k] = A_{j'}$ , we add

$$P_j(z_k) \leftrightarrow P_{j'}(y)$$

If both 2. and 3. but not 1. hold, and if there are atomic formulas  $A_j$  and  $A_{j'}$ , such that  $A_j$  contains  $y$  but no variables of  $\{z_1, \dots, z_q\}$  and  $A_j [y/z_k] = A_{j'} [y/x_i]$ , we have to make sure that

$$P_j^{\alpha'}(f_k^{\alpha'}(c_i^{\alpha'})) = P_{j'}^{\alpha'}(c_i^{\alpha'}).$$

But in this case  $S'$  contains an  $A_{j''}$  with

$$A_{j''} = A_j [y/z_k]$$

and we have added the formulas:

$$P_j(z_k) \leftrightarrow P_{j''}(y) \quad (\text{case 3})$$

and

$$P_{j''}(x_i) \leftrightarrow P_{j'}(x_i) \quad (\text{case 2})$$

Hence

$$P_j^{\alpha'}(f_k^{\alpha'}(c_i^{\alpha'})) = P_{j''}^{\alpha'}(c_i^{\alpha'}) = P_{j'}^{\alpha'}(c_i^{\alpha'})$$

It is not obvious that the transformation from formula  $F$  to formula  $F'$  can be done in logarithmic space, because  $F$  might contain variables or predicate symbols with excessively long indices. But then a simple trick solves the problem. Instead of writing such an index on a work tape, only a pointer (= position number) to its location on the input tape is stored on a work tape.

If  $|F| = n$ , then at most  $O(n/\log n)$  different atomic formulas appear in  $F$  (i.e.  $|S| = O(n/\log n)$ ). The number  $|S'|$  of different atomic formulas in  $F'$  is then bounded by  $c(p+1)|S|$ . Hence the transformation from  $F$  to  $F'$  is via length order  $n$  for constant  $p$  and via length order  $n^2/\log n$  in general (i.e. for  $p = O(n/\log n)$ ).  $\square$

*Problem.* Is there an efficient transformation from the  $\exists^* \forall \exists^*$  class to the monadic  $\exists^* \forall \exists^*$  class via length order  $n$ ?

**THEOREM (Upper bound).** *The satisfiability of the monadic prefix class  $\exists^* \forall \exists^*$  is decidable by an alternating Turing machine  $M$  in space*

$O(n/\log n)$ . Furthermore  $M$  enters no universal states for formulas of the subclass  $\exists^* \forall \exists$ .

*Proof.* Let the input  $F$  be the monadic formula

$$\exists x_1 \dots \exists x_p \forall y \exists z_1 \dots \exists z_q F_0$$

with  $F_0$  quantifier-free. It is easy to find out if the input has this form or not. Let  $F_0$  contain  $m$  different atomic formulas. Then  $m = O(n/\log n)$  for  $n = |F|$ .

Let  $(v_1, \dots, v_{p+q+1})$  be  $(x_1, \dots, x_p, y, z_1, \dots, z_q)$  and let  $A_1, \dots, A_m$  be the atomic formulas  $P_j(v_i)$  of  $F_0$  in lexicographical order according to  $(i, j)$ .

$T_1, \dots, T_m$  is a sequence of truth values for the atomic formulas. (The atomic formula  $A_k$  is interpreted to be true if  $T_k = \text{true}$ .)

The alternating Turing machine  $M$  executes the following satisfiability test:

### Program

1. begin

for all  $k$  such that the atomic formula  $A_k$  contains an  $x_i$ , choose existentially  $T_k$  to be true or false;

for  $r := 1$  to  $\max(1, p)$  do

begin

2. for all  $k, k', j$  such that  $A_k$  is  $P_j(y)$  and  $A_{k'}$  is  $P_j(x_r)$  do  $T_k := T_{k'}$ ;

3. for all  $k, j$  such that  $A_k$  is  $P_j(y)$  and  $P_j(x_r)$  does not appear in  $F$  do choose existentially a value of  $\{\text{true}, \text{false}\}$  for  $T_k$ ;

4. for counter  $:= 1$  to  $2^m$  do

begin

5. for all  $k$  such that  $A_k$  is a  $P_j(z_i)$  do choose existentially a truth value for  $T_k$ ; check that the interpretation of the atomic formulas  $A_k$  ( $k = 1, \dots, m$ ) by  $T_k$  gives the value true to the matrix  $F_0$ , otherwise stop rejecting;

7. if  $q = 0$  then goto  $E$ ;

if  $q = 1$  then  $s := 1$  (i.e.  $z_s = z_1$ );

if  $q > 1$  then choose universally a value from  $\{1, \dots, q\}$  for  $s$ ;

8. for all  $k, k', j$  such that  $A_k$  is  $P_j(y)$  and  $A_{k'}$  is  $P_j(z_s)$  do  $T_k := T_{k'}$ ;

9. for all  $k$  such that (for any  $j$ )  $A_k$  is  $P_j(y)$  and  $P_j(z_s)$  does not appear in  $F$  do choose existentially a truth value for  $T_k$ ;  
end;

E : end;

stop accepting;

end.

To execute this program, the alternating Turing machine  $M$  uses only space

$m$  to count to  $2^m$ ,

$m$  to store  $T_1, \dots, T_m$ ,

$\log p < \log m$  to store  $r$ ,

$c \log n$  for auxiliary storage, especially to store position numbers of certain information on the input tape, e.g. long indices, which are not copied to the work tapes.

Because  $m = O(n/\log n)$ , there is an upper bound  $O(n/\log n)$  (independent of  $p$  and  $q$ ) for the space used by  $M$ .

We have to show that the above program decides satisfiability of the formula  $F$  correctly.

Let  $F' = \forall y F'_0$  be the functional form of  $F = \exists x_1 \dots \exists x_p \forall y \exists z_1 \dots \exists z_q F_0$ , obtained by replacing  $x_i$  by  $c_i$  and  $z_i$  by  $f_i(y)$ .

- a) Let  $F'$  (and  $F$ ) be satisfiable and let  $\alpha$  be a model of  $F'$ .

We think the program of  $M$  extended by:

before 2.  $b := c_r^\alpha$

before 8.  $b := f_s^\alpha(b)$

Then good existential choices for the truth values  $T_k$  are

if  $A_k = P_j(x_i)$  then  $T_k := P_j^\alpha(c_i^\alpha)$

if  $A_k = P_j(y)$  then  $T_k := P_j^\alpha(b)$

if  $A_k = P_j(z_i)$  then  $T_k := P_j^\alpha(f_i^\alpha(b))$

The computation tree defined by these existential choices accepts the formula  $F$ .

- b) Assume the alternating Turing machine  $M$  accepts the formula  $F$ . Then each minimal accepting computation tree (without unnecessary branches) of  $M$  with input  $F$  can be used to construct a Herbrand model  $\alpha$  of  $F'$ .

Note that the Herbrand universe

$$|\alpha| = \{c_1, \dots, c_p, f_1(c_1), \dots, f_2(f_1(c_3)), \dots\}$$

(as a set of terms) and the functions  $f_1^\alpha, \dots, f_q^\alpha$  of a possible Herbrand model of  $F'$  are uniquely defined. We have to define the predicates  $P_1^\alpha, P_2^\alpha, \dots$ .

We look at the program extended by

$$b := c_r \quad (\text{before 2) and}$$

$$b := f_s^\alpha(b) \quad (\text{before 8) as in a).}$$

All elements of  $|\alpha|$  with nesting depth  $\leq 2^m$  are assigned to  $b$  somewhere in the accepting computation tree. The current values of the sequence  $T_1, \dots, T_m$  define some truth values of predicates in  $c_1^\alpha, \dots, c_p^\alpha, b, f_1^\alpha(b), \dots, f_q^\alpha(b)$  by

$$P_i^\alpha(c_k^\alpha) = T_j \quad \text{if} \quad A_j = P_i(x_k)$$

$$P_i^\alpha(b) = T_j \quad \text{if} \quad A_j = P_i(y)$$

$$P_i^\alpha(f_k^\alpha(b)) = T_j \quad \text{if} \quad A_j = P_i(z_k).$$

The other truth values of the predicates  $P_i^\alpha$  are defined arbitrarily. This method of defining predicates of  $b$  is used on each path in the tree  $(|\alpha|, f_1^\alpha, \dots, f_q^\alpha)$ , only until the first repetition of all truth values on that path. That happens on each path in a depth  $\leq 2^m$ . Let  $b'$  be the node on the path to  $b$  with the same truth values for all predicates as  $b$ . Then (inductively) the predicates are defined to have the same values on the subtree with root  $b$  as on the subtree with root  $b'$ . The so constructed structure  $\alpha$  is a model of  $F$ .  $\square$

**COROLLARY 1** (Lewis [27]). *The set of satisfiable formulas of the monadic  $\exists^* \forall \exists^*$  class is (for a constant  $c > 1$ ) in  $\text{DTIME}(c^{n/\log n})$ .*

*Proof.* The alternating Turing machine of the upper bound theorem can be simulated in deterministic time  $c^{n/\log n}$ .  $\square$

The direct construction of a deterministic  $c^{n/\log n}$  time decision procedure of Lewis [27] is easier. He starts with a big structure (with  $2^m$  elements, where  $m$  is the number of predicate symbols), and eliminates bad elements of this structure, to get either a model or the non-existence of a model.

We have chosen the decision procedure by an alternating Turing machine to get the following result for free.

**COROLLARY 2.** *The satisfiable formulas of the monadic  $\exists^* \forall \exists$  class are in  $NSPACE(n/\log n)$ .*

*Proof.* The universal states of the alternating Turing machine  $M$  which decides the monadic  $\exists^* \forall \exists^*$  class are not used for the subclass  $\exists^* \forall \exists$ . If we drop them, we get a nondeterministic Turing machine.  $\square$

By combining the proposition with the upper bound theorem we get immediately.

**COROLLARY 3.** *The satisfiable formulas of the  $\exists^* \forall \exists^*$  class are in  $DTIME(c^{(n/\log n)^2})$  for some  $c$ .*  $\square$

**COROLLARY 4.** *The satisfiable formulas of the  $\exists^* \forall \exists$  class are in  $NSPACE((n/\log n)^2)$ .*  $\square$

Lewis [27] claims the same time bound in Corollary 3 as for the monadic case. But this seems not to work. For example, if  $P(x_1, y), \dots, P(x_p, y)$  and  $P(y, x_1), \dots, P(y, x_p)$  appear in the formula, then  $p^2$  truth values for  $P^\alpha(c_i^\alpha, c_j^\alpha)$  ( $i, j = 1, \dots, p$ ) have to be guessed.

But these upper bounds are not very good, as e.g. in Corollary 3 the Turing machine could be replaced by one which works a short time ( $O((n/\log n)^2)$  steps) nondeterministically and then only  $c^{n/\log n}$  steps deterministically.

### *The $\exists^* \forall$ class*

Formulas of the  $\exists^* \forall$  class are transformed by our procedure in monadic formulas again of the  $\exists^* \forall$  class. For these formulas, the procedure of the upper bound theorem works in nondeterministic polynomial time. On the other hand the  $\exists^* \forall$  class is certainly more difficult than propositional calculus. Therefore the set of satisfiable formulas of the  $\exists^* \forall$  class is  $NP$ -complete. ( $NP$ -completeness is discussed in [15].)

In fact, as the Herbrand models of the satisfiable formulas of the  $\exists^p \forall^q$  class, have only  $\max(p, 1)$  elements, it is easy to see that the satisfiability problem for all the following classes in  $NP$ -complete:

- a)  $\exists^p \forall^q \quad p + q \geq 1$
- b)  $\exists^* \forall^q \quad q \geq 0$
- c)  $\forall^*$
- d)  $\exists \forall^*$

But the classes  $\exists \exists \forall^*$  and  $\exists^* \forall^*$  need  $NTIME c^{n/\log n}$  resp.  $c^n$ .