Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 27 (1981)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ALTERNATION AND THE ACKERMANN CASE OF THE DECISION PROBLEM

Autor: Fürer, Martin

Kapitel: 5. Upper bounds

DOI: https://doi.org/10.5169/seals-51744

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

$C_{t-1,s} = b_{t-1}$ and $C_{t-1,s+1} = c_{t-1}$. (Naturally these values must imply the state q_a and the headposition s_t at time t.) Now player A is allowed to doubt one of these three claims, by playing the integer $s' \in \{s-1, s, s+1\}$, and player E has to justify his claim for $C_{t-1,s'}$ by claiming values for $C_{t-2'\,s'-1}$, $C_{t-2,s'}$ and $C_{t-2,s'+1}$ which imply his value for $C_{t-1,s'}$ etc. Finally the value claimed for $C_{0s''}$ is checked by comparison with the s''-th input symbol. If it is correct, then player E, otherwise player A wins.

If w is accepted by M, then the winning strategy for player E is to make always correct claims. If w is not accepted by M, then player A has a winning strategy. He always doubts one of the wrong claims of player E.

5. Upper bounds

Proposition. 1. *For all* $p \geqslant 0$, *the* $\exists^p \forall \exists^*$ *class is logspace transformable to the monadic* $\exists \forall \exists^*$ *class via length order* n.

2. *The* $\exists^* \forall \exists^*$ *class is logspace transformable to the monadic* $\exists^* \forall \exists^*$ *class via length order* $n^2/\log n$.

Proof. The main ideas of this proof are due to Lewis [27, Lemma 7.1] and Ackermann [2, Section VIII.1]. Given a formula F of the class $\exists^p \forall \exists^q$ with prefix $\exists x_1 \ldots \exists x_p \forall y \exists z_1 \ldots \exists z_q$ and matrix M, let S be the set of atomic formulas in M. We define the set S' by $S' = S \cup \{A[y/x_i] \mid A \in S$ and $1 \leqslant i \leqslant p\}$.

Let $S' = \{A_1, \ldots, A_r\}$.
Then $|S'| = r \leqslant (p+1)|S|$.

Now we change the matrix M of F to get the formula F' with matrix M' by replacing (for $j = 1, \ldots, r$) all occurrences of the atomic formula A_j by $P_j(y)$ (for a new monadic predicate symbol P_j) and by adding — as a conjunct to M — a set B of biconditionals.

The set B is constructed to ensure that every Herbrand model α' of the functional form of the formula F' (with matrix M') defines immediately a model α of the functional form of F by $|\alpha| = |\alpha'|$,

$c_k^\alpha = c_k^{\alpha'} = c_k$, $k = 1, \ldots, p$ (where c_k is the replacement of x_k in the functional forms of F and F'),

$f_k^\alpha = f_k^{\alpha'}$, $k = 1, ..., q$ (where $f_k(y)$ is the replacement of z_k in the functional forms of F and F'),

$P^\alpha(b_1, ..., b_n) = P_j^{\alpha'}(b)$, if $A_j \in S'$, $b \in |\alpha'|$, $b_1, ..., b_n \in |\alpha|$ and there exist variables $v_1, ..., v_n$ fulfilling for all i, k the following properties:

a) $A_j = P(v_1, ..., v_n)$,

b) if $v_i = x_k$ then $b_i = c_k^\alpha$,

c) if $v_i = y$ then $b_i = b$,

d) if $v_i = z_k$ then $b_i = f_k^\alpha(b)$.

$P^\alpha(b_1, ..., b_n)$ is defined arbitrarily (e.g. false) if no such A_j and b exist. There might exist several A_j and b having these properties. To ensure that in this case the definition of $P^\alpha(b_1, ..., b_n)$ is correct, i.e. independent of the particular choice of A_j and b, we conjoin the set B of biconditionals to the matrix M.

Take any n-tupel $(b_1, ..., b_n) \in |\alpha|^n$. In the following cases, several $A_j \in S'$ and $b \in |\alpha|$ might satisfy the conditions a), b), c), d):

1. $\{b_1, ..., b_n\} \subseteq \{c_1^\alpha, ..., c_p^\alpha\}$.
2. There is a b' in $\{c_1^\alpha, ..., c_p^\alpha\}$ such that $\{b_1. ..., b_n\} \subseteq \{c_1^\alpha, ..., c_p^\alpha, f_1^\alpha(b'), ..., f_q^\alpha(b')\}$.
3. There is a b'' in $\{b_1, ..., b_n\}$, such that $\{b_1, ..., b_n\} \subseteq \{c_1^\alpha, ..., c_p^\alpha, b''\}$.

To make the definition correct in case 1, we add to B the following biconditionals:

If there is an A_j in S' such that $A_j = P(v_1, ..., v_n)$ with $\{v_1, ..., v_n\} \subseteq \{x_1, ..., x_p\}$, we add

$$P_j(y) \leftrightarrow P_j(x_1)$$

If $A_j = P(v_1, ..., v_n)$ with $\{v_1, ..., v_n\} \subseteq \{x_1, ..., x_p, y\}$ and $A_j[y/x_i] = A_{j'}[y/x_k]$ (for $A_j \neq A_{j'}$), then we add

$$P_j(x_i) \leftrightarrow P_{j'}(x_k).$$

Note: Here the length of the monadic formula might grow quadratically in p.

To make the definition correct in the case when 2 but not 3 holds, we add to B for all j, j', i with $A_j[y/x_i] = A_{j'}[y/x_i]$ the formula

$$P_j(x_i) \leftrightarrow P_{j'}(x_i).$$

To make the definition correct, when 3. but not 2. holds, we add to B the following biconditionals.

For all j, j', k such that $A_j = P(v_1, ..., v_n)$ with

$$y \in \{v_1, ..., v_n\} \subseteq \{x_1, ..., x_p, y\}$$

and $A_j[y/z_k] = A_{j'}$, we add

$$P_j(z_k) \leftrightarrow P_{j'}(y)$$

If both 2. and 3. but not 1. hold, and if there are atomic formulas A_j and $A_{j'}$, such that A_j contains y but no variables of $\{z_1, ..., z_q\}$ and $A_j[y/z_k] = A_{j'}[y/x_i]$, we have to make sure that

$$P_j^{\alpha'}(f_k^{\alpha'}(c_i^{\alpha'})) = P_{j'}^{\alpha'}(c_i^{\alpha'}).$$

But in this case S' contains an $A_{j''}$ with

$$A_{j''} = A_j[y/z_k]$$

and we have added the formulas:

$$P_j(z_k) \leftrightarrow P_{j''}(y) \qquad \text{(case 3)}$$

and

$$P_{j''}(x_i) \leftrightarrow P_{j'}(x_i) \qquad \text{(case 2)}$$

Hence

$$P_j^{\alpha'}(f_k^{\alpha'}(c_i^{\alpha'})) = P_{j''}^{\alpha'}(c_i^{\alpha'}) = P_{j'}^{\alpha'}(c_i^{\alpha'})$$

It is not obvious that the transformation from formula F to formula F' can be done in logarithmic space, because F might contain variables or predicate symbols with excessively long indices. But then a simple trick solves the problem. Instead of writing such an index on a work tape, only a pointer (= position number) to its location on the input tape is stored on a work tape.

If $|F| = n$, then at most $O(n/\log n)$ different atomic formulas appear in F (i.e. $|S| = O(n/\log n)$). The number $|S'|$ of different atomic formulas in F' is then bounded by $c(p+1)|S|$. Hence the transformation from F to F' is via length order n for constant p and via length order $n^2/\log n$ in general (i.e. for $p = O(n/\log n)$). □

Problem. Is there an efficient transformation from the $\exists^* \forall \exists^*$ class to the monadic $\exists^* \forall \exists^*$ class via length order n ?

THEOREM (Upper bound). *The satisfiability of the monadic prefix class* $\exists^* \forall \exists^*$ *is decidable by an alternating Turing machine* M *in space*

$O(n/\log n)$. *Furthermore* M *enters no universal states for formulas of the subclass* $\exists^* \forall \exists$.

Proof. Let the input F be the monadic formula

$$\exists x_1 \ldots \exists x_p \forall y \exists z_1 \ldots \exists z_q F_0$$

with F_0 quantifier-free. It is easy to find out if the input has this form or not. Let F_0 contain m different atomic formulas. Then $m = O(n/\log n)$ for $n = |F|$.

Let $(v_1, \ldots, v_{p+q+1})$ be $(x_1, \ldots, x_p, y, z_1, \ldots, z_q)$ and let $A_1, \ldots, A_m$ be the atomic formulas $P_j(v_i)$ of F_0 in lexicographical order according to (i, j).

$T_1, \ldots, T_m$ is a sequence of truth values for the atomic formulas. (The atomic formula A_k is interpreted to be true if T_k = true.)

The alternating Turing machine M executes the following satisfiability test:

Program

1. begin

   for all k such that the atomic formula A_k contains an x_i, choose existentially T_k to be true or false;
   for $r := 1$ to max $(1, p)$ do
   begin

2. for all k, k', j such that A_k is $P_j(y)$ and $A_{k'}$ is $P_j(x_r)$ do $T_k := T_{k'}$;

3. for all k, j such that A_k is $P_j(y)$ and $P_j(x_r)$ does not appear in F do choose existentially a value of $\{$true, false$\}$ for T_k;

4. for counter $:= 1$ to 2^m do
   begin

5. for all k such that A_k is a $P_j(z_i)$ do choose existentially a truth value for T_k; check that the interpretation of the atomic formulas A_k $(k = 1, \ldots, m)$ by T_k gives the value true to the matrix F_0, otherwise stop rejecting;

7. if $q = 0$ then goto E;
   if $q = 1$ then $s := 1$ (i.e. $z_s = z_1$);
   if $q > 1$ then choose universally a value from $\{1, \ldots, q\}$ for s;

8. for all k, k', j such that A_k is $P_j(y)$ and $A_{k'}$ is $P_j(z_s)$ do $T_k := T_{k'}$;

9. for all k such that (for any j) A_k is $P_j(y)$ and $P_j(z_s)$ does not appear in F do choose existentially a truth value for T_k;

end;

E : end;

stop accepting;

end.

To execute this program, the alternating Turing machine M uses only space

m to count to 2^m,

m to store $T_1, \ldots, T_m$,

$\log p < \log m$ to store r,

$c \log n$ for anxillary storage, especially to store position numbers of certain information on the input tape, e.g. long indices, which are not copied to the work tapes.

Because $m = O(n/\log n)$, there is an upper bound $O(n/\log n)$ (independent of p and q) for the space used by M.

We have to show that the above program decides satisfiability of the formula F correctly.

Let $F' = \forall y\, F_0'$ be the functional form of $F = \exists x_1 \ldots \exists x_p \forall y \exists z_1 \ldots \exists z_q\, F_0$, obtained by replacing x_i by c_i and z_i by $f_i(y)$.

a) Let F' (and F) be satisfiable and let α be a model of F'.
We think the program of M extended by:

before 2. $\quad b := c_r^\alpha$

before 8. $\quad b := f_s^\alpha(b)$

Then good existential choices for the truth values T_k are

if $A_k = P_j(x_i)$ then $T_k := P_j^\alpha(c_i^\alpha)$

if $A_k = P_j(y)$ then $T_k := P_j^\alpha(b)$

if $A_k = P_j(z_i)$ then $T_k := P_j^\alpha(f_i^\alpha(b))$

The computation tree defined by these existential choices accepts the formula F.

b) Assume the alternating Turing machine M accepts the formula F. Then each minimal accepting computation tree (without unnecessary branches) of M with input F can be used to construct a Herbrand model α of F'.

Note that the Herbrand universe

$$|\alpha| = \{c_1, \ldots, c_p, f_1(c_1), \ldots, f_2(f_1(c_3)), \ldots\}$$

(as a set of terms) and the functions $f_1^\alpha, \ldots, f_q^\alpha$ of a possible Herbrand model of F' are uniquely defined. We have to define the predicates $P_1^\alpha, P_2^\alpha, \ldots$.

We look at the program extended by

$$b := c_r^\alpha \qquad \text{(before 2) and}$$
$$b := f_s^\alpha(b) \qquad \text{(before 8) as in a).}$$

All elements of $|\alpha|$ with nesting depth $\leqslant 2^m$ are assigned to b somewhere in the accepting computation tree. The current values of the sequence $T_1, \ldots, T_m$ define some truth values of predicates in $c_1^\alpha, \ldots, c_p^\alpha, b, f_1^\alpha(b), \ldots, f_q^\alpha(b)$ by

$$\begin{aligned} P_i^\alpha(c_k^\alpha) &= T_j \quad \text{if} \quad A_j = P_i(x_k) \\ P_i^\alpha(b) &= T_j \quad \text{if} \quad A_j = P_i(y) \\ P_i^\alpha(f_k^\alpha(b)) &= T_j \quad \text{if} \quad A_j = P_i(z_k)\,. \end{aligned}$$

The other truth values of the predicates P_i^α are defined arbitrarily. This method of defining predicates of b is used on each path in the tree $(|\alpha|, f_1^\alpha, \ldots, f_q^\alpha)$, only until the first repetition of all truth values on that path. That happens on each path in a depth $\leqslant 2^m$. Let b' be the node on the path to b with the same truth values for all predicates as b. Then (inductively) the predicates are defined to have the same values on the subtree with root b as on the subtree with root b'. The so constructed structure α is a model of F. □

COROLLARY 1 (Lewis [27]). *The set of satisfiable formulas of the monadic* $\exists^* \forall \exists^*$ *class is (for a constant* $c > 1$*) in* $DTIME(c^{n/\log n})$.

Proof. The alternating Turing machine of the upper bound theorem can be simulated in deterministic time $c^{n/\log n}$. □

The direct construction of a deterministic $c^{n/\log n}$ time decision procedure of Lewis [27] is easier. He starts with a big structure (with 2^m elements, where m is the number of predicate symbols), and eliminates bad elements of this structure, to get either a model or the non-existence of a model.

We have chosen the decision procedure by an alternating Turing machine to get the following result for free.

COROLLARY 2. *The satisfiable formulas of the monadic* $\exists^* \forall \exists$ *class are in* $NSPACE\,(n/\log n)$.

Proof. The universal states of the alternating Turing machine M which decides the monadic $\exists^* \forall \exists^*$ class are not used for the subclass $\exists^* \forall \exists$. If we drop them, we get a nondeterministic Turing machine. □

By combining the proposition with the upper bound theorem we get immediately.

COROLLARY 3. *The satisfiable formulas of the* $\exists^* \forall \exists^*$ *class are in* $DTIME\,(c^{(n/\log n)^2})$ *for some* c. □

COROLLARY 4. *The satisfiable formulas of the* $\exists^* \forall \exists$ *class are in* $NSPACE\,((n/\log n)^2)$. □

Lewis [27] claims the same time bound in Corollary 3 as for the monadic case. But this seems not to work. For example, if $P(x_1, y), \ldots, P(x_p, y)$ and $P(y, x_1), \ldots, P(y, x_p)$ appear in the formula, then p^2 truth values for $P^\alpha(c_i^\alpha, c_j^\alpha)$ $(i, j = 1, \ldots, p)$ have to be guessed.

But these upper bounds are not very good, as e.g. in Corollary 3 the Turing machine could be replaced by one which works a short time $(O((n/\log n)^2)$ steps) nondeterministically and then only $c^{n/\log n}$ steps deterministically.

The $\exists^* \forall$ *class*

Formulas of the $\exists^* \forall$ class are transformed by our procedure in monadic formulas again of the $\exists^* \forall$ class. For these formulas, the procedure of the upper bound theorem works in nondeterministic polynomial time. On the other hand the $\exists^* \forall$ class is certainly more difficult than propositional calculus. Therefore the set of satisfiable formulas of the $\exists^* \forall$ class is NP-complete. (NP-completeness is discussed in [15].)

In fact, as the Herbrand models of the satisfiable formulas of the $\exists^p \forall^q$ class, have only max $(p, 1)$ elements, it is easy to see that the satisfiability problem for all the following classes in NP-complete:

a) $\exists^p \forall^q \quad p + q \geqslant 1$

b) $\exists^* \forall^q \quad q \geqslant 0$

c) $\forall^*$

d) $\exists \forall^*$

But the classes $\exists\exists\forall^*$ and $\exists^* \forall^*$ need $NTIME\ c^{n/\log n}$ resp. c^n.