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Hence, there are at most  $[G : H]$  free parameters in determining  $\chi \in V^H$  and clearly each choice gives an invariant. This finishes the proof.

**COROLLARY 5.6.**  $\dim(H_W^{W^\theta}) = [W : W_\theta] = |W^\theta|$  and the  $X_w$ ,  $w \in W^\theta$ , are an  $\mathbf{R}$ -basis for  $H_W^{W^\theta}$ .

*Proof.* Chevalley [8] has shown that  $S_W$ , hence  $H_W$ , is abstractly equivalent to the regular representation of  $W$ , as a  $W$ -module. Hence, (5.5) applies and the result follows.

It is now possible to "restrict" the Pieri formula (4.5) for  $H_W$  to  $H_W^{W^\theta}$ . We have

**THEOREM 5.7.** If  $w, w' \in W^\theta$  and in  $H_W$

$$\begin{aligned} X_w \cdot X_{w'} &= \sum_{w'' \in W} c(w, w', w'') X_{w''} \\ \text{then in } H_W^{W^\theta} \quad X_w \cdot X_{w'} &= \sum_{w'' \in W^\theta} c(w, w', w'') X_{w''} \end{aligned}$$

*Proof.* One need only observe that the vector space map  $r : H_W \rightarrow H_W^{W^\theta}$  given by

$$r(X_w) = \begin{cases} X_w & \text{if } w \in W^\theta \\ 0 & \text{otherwise} \end{cases}$$

is a retraction. Then, applying  $r$  to both sides of the first equation yields the second equation since the invariants form a subalgebra.

This result will be useful in the next section for computing inside the algebra of  $W_\theta$ -invariants. Notice that an appropriate Giambelli formula for  $H_W^{W^\theta}$  is not as easily obtained. This is because the Giambelli formula for  $H_W$  gives  $X_w$  as a polynomial in the  $X_{s_\alpha}$ 's and not all of these are in the algebra  $H_W^{W^\theta}$ , so this is not quite the right thing.

## 6. APPLICATION:

### THE COMBINATORICS OF THE CLASSICAL PIERI FORMULA

In the last section we saw that given a pair  $(W, W_\theta)$  of a Coxeter group and a parabolic subgroup, one could construct a formula to describe the multiplication of Schubert generators in the invariant subalgebra  $H_W^{W^\theta}$ . In this section, we examine the particular case  $(\Sigma_{n+k}, \Sigma_k \times \Sigma_n)$  where  $\Sigma_m$  denotes the symmetric group on  $m$  letters. Indeed,  $\Sigma_{n+k}$  is the Weyl group of the root system of type  $A_{n+k-1}$ , which we recall briefly here. Let  $V' = \mathbf{R}^{n+k}$

equipped with the usual inner product and let  $e_1, \dots, e_{n+k}$  denote the standard basis.  $\Sigma_{n+k}$  acts by permuting these basic elements. This action is effective on the  $(n+k-1)$ -dimensional subspace

$$V = \left\{ \sum_{i=1}^{n+k} \lambda_i e_i : \sum_{i=1}^{n+k} \lambda_i = 0 \right\}$$

and it is easy to see  $\Delta = \{e_i - e_j\}_{j \neq i}$  can be chosen as the corresponding root system. In addition, the simple roots  $\Sigma = \{e_i - e_{i+1}\}_{1 \leq i \leq n+k-1}$  and the positive roots  $\Delta^+ = \{e_i - e_j\}_{i < j}$ , induce the usual transpositions of the basis vectors.

The main result of this section is the identification of the Pieri formula for  $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n}$  with the classical Pieri formula (see [7, 16]).

We begin with a rapid review of Chern's Schubert calculus for the cohomology of a complex grassmannian [7]. Let  $G_k(\mathbb{C}^{n+k})$  denote the space of  $k$ -dimensional complex subspaces in  $\mathbb{C}^{n+k}$ . This is a compact, smooth manifold of dimension  $2nk$ . Ehresmann [14] described a cell-decomposition for  $G_k(\mathbb{C}^{n+k})$  (along with other algebraic homogeneous spaces) whose cells are identified by certain Schubert symbols  $(d_1, \dots, d_k)$ , where

$$1 \leq d_1 < \dots < d_k \leq n + k .$$

Each symbol yields a cohomology class  $\langle d_1 \dots d_k \rangle$  of dimension

$$2 \sum_{i=1}^k (d_i - 1) = \binom{k}{\sum_{i=1}^k d_i} - \frac{k(k+1)}{2}$$

Geometrically,  $\langle d_1, \dots, d_k \rangle$  is the cocycle dual to the cell

$$[d_1, \dots, d_k] = \{X \in G_k(\mathbb{C}^{n+k}) : \dim (X \cap \mathbf{R}^{d_i}) \geq i\} .$$

It is easy to see the  $d_i$ 's describe the "jump-points" in the sequence

$$0 \leq \dim (X \cap \mathbf{C}^1) \leq \dim (X \cap \mathbf{C}^2) \leq \dots \leq \dim (X \cap \mathbf{C}^{n+k-1}) \leq n + k$$

where  $0 \subseteq \mathbf{C}^1 \subseteq \mathbf{C}^2 \subseteq \dots \subseteq \mathbf{C}^{n+k}$  is the standard flag determined by the coordinate axes.

On the other hand,  $G_k(\mathbb{C}^{n+k})$  can also be profitably thought of as the homogeneous space  $G/P$  where  $G$  is the complex Lie group  $GL_{n+k}(\mathbb{C})$  and  $P$  is the maximal parabolic subgroup of the form

$$\left( \begin{array}{c|c} GL_k(\mathbb{C}) & * \\ \hline 0 & GL_n(\mathbb{C}) \end{array} \right)$$

If  $K$  denotes the maximal compact subgroup  $U_{n+k}$  of  $G$  then we also have the identification  $G_k(\mathbf{C}^{n+k}) = K/(U_k \times U_n)$ .

More generally, one can consider a complex semisimple Lie group  $G$  and a parabolic  $P_\theta$  in  $G$  corresponding to a subset  $\theta$  of the simple roots  $\Sigma$ . The homogeneous space  $G/P_\theta$  has been studied by various authors and we will assume known that

$$\begin{aligned} H^*(G/P_\theta; \mathbf{R}) &\cong H^*(G/B; \mathbf{R})^{W_\theta} \\ &\cong (S_W)^{W_\theta} . \end{aligned}$$

This will be the basic topological input [2].

Now we fix  $G$  to be the Lie group of type  $A_{n+k-1}$  and  $\theta = \Sigma - \{\alpha_k\}$  (where we write  $\alpha_j = e_j - e_{j+1}$  and  $s_j = s_{\alpha_j}$ ) so that  $G_k(\mathbf{C}^{n+k}) \cong G/P_\theta$ . We begin with some easy length computations.

LEMMA 6.1. *If  $w \in W$ , then*

$$l(ws_{ij}) - l(w) = p_{ij}(2|I_{i,j}| + 1)$$

where

$$p_{ij} = \begin{cases} +1 & w(i) < w(j) \\ -1 & w(i) > w(j) \end{cases}$$

and

$$I_{i,j} = \{i < z < j : w(z) \text{ is between } w(i) \text{ and } w(j)\} .$$

In particular,  $l(ws_{ij}) = l(w) + 1$  if and only if (i)  $w(i) < w(j)$  and (ii) there are no intermediate  $w$ -values, i.e.  $I_{i,j} = \phi$  (we often abbreviate this pair of conditions by  $w(i) << w(j)$ ).

*Proof.* Recall the length function on  $\sum_{n+k}$  is given by  $l(w) = \sum_{j=1}^{n-1} e_j(w)$ , where  $e_j = |\{i > j : w(i) < w(j)\}|$ , the number of inversions of  $w$ . Hence

$$l(ws_{ij}) - l(w) = (e'_i - e_i) + (e'_j - e_j) + \sum_{i < z < j} (e'_z - e_z)$$

where  $e_l = e_l(w)$  and  $e_l = e_l(ws_{ij})$ . Certainly, right multiplication by  $s_{ij}$  does not affect the values of  $e_l$  below  $i$  or above  $j$ . Also

$$\begin{aligned} e'_i &= e_j + |\{i \leq z < j : w(z) < w(j)\}| = e_j + e \\ e'_j &= e_i - |\{i < z \leq j : w(z) < w(i)\}| = e_j - \bar{e} \end{aligned}$$

so we get

$$\begin{aligned} (e'_i - e_i) + (e'_j - e_j) &= (e_j + e - e_i) + (e_i - \bar{e} - e_j) \\ &= e - \bar{e} = p_{i,j}(|I_{i,j}| + 1) \end{aligned}$$

It is easy to see

$$e_z' - e_z = \begin{cases} p_{i,j} & \text{if } z \in I_{i,j} \\ 0 & \text{otherwise} \end{cases}$$

putting this all together we get the result. The second assertion follows immediately.

We can now write down (4.6) for  $H_W$ ,  $W = \sum_{n+k}$

**PROPOSITION 6.2.** *If  $w \in \sum_{n+k}$ ,  $1 \leq i \leq n+k-1$ , then in  $H_{\Sigma_{n+k}}$*

$$X_{s_i} \cdot X_w = \sum_{(b,t)} X_{ws_{bt}}$$

where  $(b, t)$  satisfies  $b \leq i < t$  and  $w(b) \ll w(t)$ .

*Proof.* By (4.6),  $X_{ws_{bt}}$  appears with coefficient  $((e_b - e_t)^v, \omega_i)$  if and only if  $l(ws_{bt}) = l(w) + 1$ . This is equivalent to the last condition by (6.1). Finally  $(e_b - e_t)^v = e_b - e_t = \alpha_b + \dots + \alpha_{t-1}$ , so that first condition is also needed and the coefficient is correct.

*Remark.* The Poincaré dual of this formula appears in [18, p. 265].

We now identify the set of left coset representatives  $W^\theta$ . If  $1 \leq d_1 < \dots < d_k \leq n+k$  and  $d_1' < \dots < d_n'$  is an ordered enumeration of the complement then we define  $(d_1, \dots, d_k) \in \sum_{n+k}$ , by

$$(d_1, \dots, d_k)(i) = \begin{cases} d_i & 1 \leq i \leq k \\ d_{i-k} & k+1 \leq i \leq k+n \end{cases} .$$

(We also write  $X(d_1, \dots, d_k)$  when it is convenient.)

**LEMMA 6.3.**

$$W^\theta = \{(d_1, \dots, d_k) : 1 \leq d_1 < \dots < d_k \leq n+k\}$$

$$\text{and } l(d_1, \dots, d_k) = \sum_{j=1}^k (d_j - j).$$

*Proof.* Clearly  $l((d_1, \dots, d_k) s_i) = l(d_1, \dots, d_k) + 1$ , for all  $i \neq k$  by (6.1), for example. Since  $|W^\theta| = |W| / |W_\theta| = \binom{n+k}{k}$  the first assertion follows. For the second, we need only observe

$$e_j(d_1, \dots, d_k) = \begin{cases} d_j - j & \text{if } j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

This lemma indicates how the Schubert notation arises from a group-theoretic point of view. That this notation is consistent with the geometry is a theorem of Demazure [12].

A Pieri formula should compute the product of  $X_w$  (a linear generator, by (5.6)) and an algebraic generator. Since the map

$$S(V)^{W_\theta} \xrightarrow{c} (H_W)^{W_\theta}$$

is onto we can find algebraic generators by computing the images of  $W_\theta$ -invariants. In general,  $W_\theta$  is a (reducible) Coxeter group, so we have the fundamental invariants [20]. In our case, we have simply

$$S(V)^{W_\theta} = \mathbf{Z}[\tau_1, \dots, \tau_k, \sigma_1, \dots, \sigma_n]$$

where  $\tau_i = s_i(e_1, \dots, e_k)$ ,  $1 \leq i \leq k$ , and  $\sigma_j = s_j(e_{k+1}, \dots, e_{k+n})$ ,  $1 \leq j \leq n$  and  $s_j$  denotes the  $j^{\text{th}}$  elementary symmetric function in an appropriate number of variables. One knows  $c(\sigma_j)$  suffices to generate  $H_W^{W_\theta}$ . So we compute

LEMMA 6.4.

$$c(\sigma_j) = (-1)^j X_{s_{k+j-1}, \dots, s_k} = (-1)^j X(1, 2, \dots, k-1, k+j) \quad .$$

*Proof.* By section 2

$$c(\sigma_j) = \sum_{l(w)=j} \Delta_w(\omega_j) X_w$$

If we write  $\Delta_t$  for  $\Delta_{\alpha_t}$ , then clearly  $\Delta_t(\sigma_j) = 0$ , if  $t \neq k$  and

$$\begin{aligned} \Delta_k(\sigma_j) &= \frac{s_j(e_{k+1}, \dots, e_{k+n}) - s_j(e_k, \dots, e_{k+n})}{e_k - e_{k+1}} \\ &= \frac{(e_{k+1} - e_k) s_{j-1}(e_{k+2}, \dots, e_{k+n})}{e_k - e_{k+1}} \\ &= (-1) s_{j-1}(e_{k+2}, \dots, e_{k+n}) \end{aligned}$$

We can continue by induction and get  $\Delta_{k+j-1} \dots \Delta_k(\sigma_j) = (-1)^j$ , while any other sequence of simple roots yields zero.

We now proceed to a computation of

$$X(1, 2, \dots, k-1, k+j) X(d_1, \dots, d_k) \quad .$$

The case  $j = 1$  is easy.

PROPOSITION 6.5.

$$\begin{aligned}
 & X(1, 2, \dots, k-1, k+1) X(d_1, \dots, d_k) \\
 &= \sum_{d_{i+1} < d_{i+1}} X(d_1, \dots, d_i + 1, \dots, d_k) .
 \end{aligned}$$

*Proof.* Since  $(1, 2, \dots, k-1, k+1) = s_k$ , we apply the case  $i = k$  of (6.2) and observe  $w(b) \ll w(t)$  if and only if  $w(t) = w(b) + 1$ .

We now observe

LEMMA 6.6.

$$c(\sigma_j) = s_j(X_{s_{k+1}} - X_{s_k}, X_{s_{k+2}} - X_{s_{k+1}}, \dots, -X_{s_{n+k-1}})$$

*Proof.* By the tables of [6], the  $i$ -th fundamental weight is

$$\omega_i = e_1 + \dots + e_i - \left(\frac{i}{n+k}\right) \sigma_1(e_1, \dots, e_{n+k}) .$$

Hence  $\omega_i \equiv e_1 + \dots + e_i \pmod{I_w}$  and we get

$$\begin{aligned}
 c(\sigma_j) &= c(s_j(e_{k+1}, \dots, e_{k+n})) \\
 &= c(s_j(\omega_{k+1} - \omega_k, \dots, -\omega_{n+k-1})) \\
 &= s_j(X_{s_{k+1}} - X_{s_k}, \dots, -X_{s_{n+k-1}})
 \end{aligned}$$

since  $c$  kills  $I_w$  and (3.4 ii).

This suggests the following computation.

LEMMA 6.7. For all  $i, k+1 \leq i \leq k+n, w \in W$ ; in  $H_W$

$$\begin{aligned}
 (X_{s_i} - X_{s_{i+1}}) X_w &= \sum_{\substack{i < t \\ w(i) \ll w(t)}} X_{wsit} - \sum_{\substack{k < b < i \\ w(b) \ll w(i)}} X_{wsbi} \\
 &\quad - \sum_{\substack{b \leq k \\ w(b) \ll w(i)}} X_{wsbi}
 \end{aligned}$$

*Proof.* Computing with (6.2), we get

$$X_{s_i} X_w = \sum_{\substack{b \leq i-1 \\ i < t \\ w(b) \ll w(t)}} X_{wsbt} + \sum_{\substack{i < t \\ w(i) \ll w(t)}} X_{wsit}$$

and

$$X_{s_{i-1}} X_w = \sum_{\substack{b \leq i-1 \\ i < t \\ w(b) \ll w(t)}} X_{ws_{bt}} + \sum_{\substack{b < i \\ w(b) \ll w(i)}} X_{ws_{bi}}$$

Upon subtracting and breaking up the second term the desired expression follows.

**THEOREM 6.8.** *In*

$$s_j (X_{s_{k+1}} - X_{s_k}, \dots, -X_{s_{n+k-1}}) X(d_1, \dots, d_k) = (-1)^j \Sigma X(e_1, \dots, e_k)$$

where the summation ranges over  $(e_1, \dots, e_k)$  satisfying  $d_i \leq e_i \leq d_{i+1}$

and  $\sum_{i=1}^k e_i = j + \sum_{i=1}^k d_i$ .

*Proof.* Of course

$$s_j = \sum_{k+1 \leq t_1 < \dots < t_j \leq k+n} (X_{s_{t_j}} - X_{s_{t_j-1}}) \dots (X_{s_{t_1}} - X_{s_{t_1-1}})$$

where we set  $X_{s_{k+n}} = 0$ . It is not difficult to check that the third term of (6.7) alone yields the right-hand side. Hence it remains to show that the contributions arising whenever either of the first two terms of (2.7) are involved cancel in the final summation. To do this it suffices to show that the resulting subscripts in  $W$  do not lie in  $W^\theta$ . (Then they must have coefficient zero since  $H_{W^\theta}^W$  is a subalgebra of  $H_w$ .)

Now the first two terms of (6.7) always give a transposition above  $k+1$  and it must be elementary one by (6.1), say  $s_i, i \geq k$ . Such a transposition will send an element of  $W^\theta$  out of  $W^\theta$ . We claim no further transposition  $s_{bt}$ , with either  $b \geq i$  or  $t \geq i$ , will put the subscript back in  $W^\theta$ . Both cases are easy to check and the proof is complete. Finally, by a substitution from (6.4) we get

**COROLLARY 6.9.** (Pieri formula). *In*  $H_{\Sigma_{n+k}}^{\Sigma_k \times \Sigma_n} = H^*(G_k(\mathbb{C}^{n+k}))$

$$X(1, 2, \dots, k-1, k+j) X(d_1, \dots, d_k) = \Sigma X(e_1, \dots, e_k)$$

where the summation is as in (6.8).

*Remarks 1.* The above formal arguments are equally valid for the Chow ring of the grassmann variety over an arbitrary algebraically closed field.

2. One can hope to mimick the above strategy for the homogeneous space  $SO_{2n+1}/U_n$ . The group  $G$  is of type  $B_n$  and the maximal parabolic is determined by the "right-end" root. It is not difficult to write out the Pieri formula for the flag manifold of type  $B_n$  (see the author's "Pieri formulae for classical groups", preprint). In addition,  $W^\theta$ , for this case, can be identified with the power set of  $\{1, 2, \dots, n\}$  and one can compute  $c(\sigma_j) = 2X_{\{j\}}$ . (The 2 occurs because  $c$  is not onto.) Still the problem is complicated by multiplicities. We hope to return to this elsewhere.

#### REFERENCES

- [1] ARTIN, E. *Galois Theory*. Notre Dame Math. Lect., Notre Dame, 1958.
- [2] BERNSTEIN, I. N., I. M. GELFAND and S. I. GELFAND. Schubert cells and the cohomology of the spaces  $G/P$ . *Russian Math. Surveys* 28 (1973), 1-26.
- [3] BOREL, A. Sur la cohomologie des espaces fibres principaux et des espaces homogenes de groupes de Lie compacts. *Ann. Math.* 57 (1953), 115-207.
- [4] BOREL A. and F. HIRZEBRUCH. Characteristic classes and homogeneous spaces. *Amer. J. Math.* 81 (1959), 315-382.
- [5] BOREL, A. Sous groupes commutatifs et torsion des groupes de Lie compacts connexes. *Tohoku Math. J.* 13 (1961), 216-240.
- [6] BOURBAKI, N. *Groupes et algèbres de Lie. Chapitres IV, V et VI*. Paris, Hermann, 1968.
- [7] CHERN, S. On the multiplication in the characteristic ring of a sphere bundle. *Ann. Math.* 49 (1948), 362-372.
- [8] CHEVALLEY, C. Invariants of finite groups generated by reflections. *Amer. J. Math.* 77 (1955), 778-782.
- [9] COXETER, H. S. M. Discrete groups generated by reflections. *Ann. Math.* 35 (1934), 588-621.
- [10] ——— *Regular Polytopes*. MacMillan, New York, 1963.
- [11] DEMAZURE, M. Invariants symétriques entiers des groupes de Weyl et torsion. *Inv. Math.* 21 (1973), 287-301.
- [12] ——— Désingularisation des variétés de Schubert généralisées. *Ann. Scient. Ec. Norm. Sup., 4<sup>e</sup> Serie, t. 7* (1974), 53-88.
- [13] DEODHAR, V. Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function. *Inv. Math.* 39 (1977), 187-198.
- [14] EHRESMANN, C. Sur la topologie de certains espaces homogènes. *Ann. Math.* 35 (1934), 396-443.
- [15] GARLAND, H. and M. RAGHUNATHAN. A Bruhat decomposition for the loop space of a compact group: a new approach to results of Bott. *Proc. Nat. Acad. Sci., U.S.A.* 72 (1975), 4716-4717
- [16] KLEIMAN, S. Problem 15. *Rigorous foundations of Schubert's enumerative calculus*. Proc. Symp. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976.
- [17] KOSTANT, B. Lie algebra cohomology and generalized Schubert cells. *Ann. Math.* 77 (1963), 72-144.
- [18] MONK, D. The geometry of flag manifolds. *Proc. London Math. Soc.* 9 (1959), 253-286.